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DISTANCE SETS, RATIO SETS AND CERTAIN TRANSFORMATIONS OF SETS OF REAL NUMBERS

TIBOR NEUBRUNN and TIBOR ŠALÁT, Bratislava

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For a non-empty set A of real numbers let $D(A)$ denote the set of all numbers $|x - y|$ where $x, y \in A$. $D(A)$ is said to be the distance set of the set A. The symbol $R(A)$ stands for the set of all x/y where $x, y \in A$, $y \neq 0$. $R(A)$ is said to be the ratio set of the set A.

Many papers are devoted to the study of the sets $D(A)$ and $R(A)$ for various A. A survey of fundamental results on the sets $D(A)$ is given in the monography [1] and in the survey paper [2]. The paper [3] transfers some fundamental results on the sets $D(A)$, especially those of metrical character, to the sets $R(A)$. The present paper is a contribution to the study of properties both of the sets $D(A)$ and $R(A)$.

In the first part certain transformations of the sets of real numbers will be studied. These transformations are a generalization of those leading to the sets $D(A)$ and $R(A)$. In the second part some new results on $D(A)$ and $R(A)$ are given and some strengthening of known results concerning this problem is done.

1. CERTAIN TRANSFORMATIONS OF THE SETS OF REAL NUMBERS

In all what follows, E_1 denotes the set of all real numbers. $\mathscr{L}$ denotes the family of all Lebesgue measurable subsets of the set E_1. If $A \in \mathscr{L}$ then $|A|$ stands for the Lebesgue measure of the set A. Suppose that with each element ω belonging to a metric space Ω certain transformation T_ω of the system $\mathscr{L}$ into $\mathscr{L}$ is associated.

Let the following assumptions be satisfied:

(i) There exists $\omega_0 \in \Omega$ such that for every interval $\langle a, b\rangle \subset E_1$ and every sequence $\{\omega_n\}_{n=1}^{\infty}$ of elements belonging to Ω and converging to ω_0,

$$\lim_{n\to\infty} (\inf T_{\omega_n}(\langle a, b\rangle)) = a\,, \quad \lim_{n\to\infty} (\sup T_{\omega_n}(\langle a, b\rangle)) = b$$

holds;

(ii) If $E, F \in \mathscr{L}$ and $E \subset F$ then for every $\omega \in \Omega$, $T_\omega(E) \subset T_\omega(F)$;

(iii) If $E \in \mathscr{L}$ and $\omega_n \to \omega_0$ (in Ω), then

$$\lim_{n\to\infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|\,.$$

Example 1,1. Put $\Omega = E_1$ (E_1 is supposed to be the metric space with the Euclidean metric). If $E \in \mathscr{L}$ then let $T_\omega(E) = E + \omega$ (i.e. the set of all numbers of the form $x + \omega$, $x \in E$). Taking 0 as ω_0 one can check easily that properties $(i)-(iii)$ are satisfied.

Example 1,2. Put $\Omega = (0,1\rangle$ ($(0,1\rangle$ is supposed to be the metric space with the Euclidean metric). If $E \in \mathscr{L}$, then for $\omega \in (0,1\rangle$ $T_\omega(E) = \omega E$ (i.e. the set of all numbers of the form ωx, $x \in E$). If we put $\omega_0 = 1$ then properties $(i)-(iii)$ are satisfied.

Further examples may be constructed putting a set $E + f(\omega)$ or $f(\omega)$. E instead of $T_\omega(E)$ $(\Omega \subset E_1)$ where f is a given real function fulfilling certain conditions.

The known Steinhaus theorem states that $D(A)$ contains an interval $(0, \eta)$, $\eta > 0$, if $|A| > 0$ (see [2], [4]). An analogical results for the sets $R(A)$ is valid. If $|A| > 0$, then $R(A)$ contains an interval of the form $(\eta, 1)$, $0 < \eta < 1$ (see [3]).

The following theorem gives a unified view on both above results. They both follow as special cases from Theorem 1,1 (see corollaries 1,2 and 1,3).

Theorem 1,1. *Let Ω and $T_\omega(\omega \in \Omega)$ have the same meaning as above and let conditions* (i), (ii), (iii) *be satisfied. Let $A \in \mathscr{L}$, $|A| > 0$ and $\omega_n \to \omega_0$ (in Ω). Then there exists a natural number n_0 such that for $n \geqq n_0$, $A \cap T_{\omega_n}(A) \neq \emptyset$ holds.*

Proof. Let $A \in \mathscr{L}$, $|A| > 0$. Then there is an interval I with the end points a, b such that

$$|A \cap I| > \tfrac{3}{4}\delta\,, \quad \delta = |I| \tag{1}$$

(see [5] p. 72). Let $\omega_n \in \Omega$, $\omega_n \to \omega_0$. Put

$$a_n = \inf T_{\omega_n}(I)\,, \quad b_n = \sup T_{\omega_n}(I) \quad (n = 1, 2, \ldots)\,.$$

On account of condition (i) there exists n_1 such that for $n \geqq n_1$ we have

$$|a_n - a| < \frac{\delta}{8}\,, \quad |b_n - b| < \frac{\delta}{8}\,. \tag{2}$$

Further, according to (iii) let n_2 be such that if $n \geqq n_2$ then

$$\big||T_{\omega_n}(A \cap I)| - |A \cap I|\big| < \frac{\delta}{4}\,. \tag{3}$$

Put $n_0 = \max(n_1, n_2)$. We shall show that if $n \geqq n_0$ the sets $A \cap I$ and $T_{\omega_n}(A \cap I)$ are not disjoint. If they were disjoint then according to (1) and (3)

$$(4) \qquad v = |(A \cap I) \cup T_{\omega_n}(A \cap I)| =$$
$$= |A \cap I| + |T_{\omega_n}(A \cap I)| > \frac{3}{4}\delta + \frac{3}{4}\delta - \frac{\delta}{4} = \frac{5}{4}\delta\,.$$

On the other side, (2) and (ii) imply that both these sets are contained in the interval $\langle a - \delta/8,\ b + \delta/8\rangle$, hence $v \leqq (b - a) + \delta/4 = \frac{5}{4}\delta$. This is a contradiction to (4). So we have $(A \cap I) \cap T_{\omega_n}(A \cap I) \neq \emptyset$ for $n \geqq n_0$ and according to (ii) $A \cap T_{\omega_n}(A) \neq \emptyset$ if $n \geqq n_0$. The proof is finished.

Corollary 1,1. *Let a mapping g_ω of the set E_1 into E_1 be associated to every $\omega \in \Omega$ such that the image of any set belonging to $\mathscr{L}$ belongs to $\mathscr{L}$. If $A \in \mathscr{L}$ we put $T_\omega(A) = g_\omega(A)$. Then condition* (ii) *for T_ω is evidently fulfilled. Assume that* (i) *and* (iii) *are also satisfied. Let $\omega_n \to \omega_0$, $|A| > 0$. Then by the foregoing Theorem there is n_0 such that if $n \geqq n_0$ then $x_n \in A$ exists such that $g_{\omega_n}(x_n) \in A$.*

Corollary 1,2. *If $|A| > 0$, then $D(A)$ contains an interval $(0, \eta)$, $\eta > 0$.*

Proof. If $\omega \in E_1$ then it is sufficient to put $g_\omega(x) = x + \omega$ and to use the foregoing result.

Corollary 1,3. *If $|A| > 0$ then $R(A)$ contains an interval $(\eta, 1)$, $0 < \eta < 1$.*

Proof. It is sufficient to put $\Omega = (0, 1\rangle$, $\omega_0 = 1$ and $g_\omega(x) = \omega x$ for $\omega \in (0, 1\rangle$. Then we apply Corollary 1,1 to the set A.

In the following theorem we restrict ourselves to the transformations $T_\omega(\omega \in \Omega)$ from $\mathscr{L}$ to $\mathscr{L}$ which are induced by mappings g_ω of the set E_1 into E_1 (see Corollary 1,1). As we have seen condition (ii) is satisfied in this case. We shall assume that (i) and (iii) are satisfied as well. Instead of $T_\omega(\{x\}) = \{y\}$ we shall write $T_\omega(x) = y$. The following condition will be supposed to be valid:

(iv) If $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$ then for every $x \in E_1$, $T_{\omega_1}(x) \neq T_{\omega_2}(x)$.

Further we shall say that (v) is satisfied on a set $G \subset \Omega$ if

(v) Either $T_\omega(x) > x$ for all $\omega \in G$, $\omega \neq \omega_0$ and all $x \in E_1$ or $T_\omega(x) < x$ holds for all $\omega \in G$, $\omega \neq \omega_0$ and all $x \in E_1$.

The following theorem gives a unified view on two results of the papers [3], [4].

In what follows $\overline{M}$ for $M \subset \Omega$ denotes the closure of M in Ω.

Theorem 1,2. *Let Ω, T_ω, ω_0 have the above meaning while $\{T_\omega\}$, $\omega \in \Omega$ forms a group (under the usual composition of transformations). Let conditions* (i), (iii), (iv) *be satisfied. Let G be a countable subset of Ω such that $\omega_0 \in \overline{G - \{\omega_0\}}$ and for every $\omega_1, \omega_2 \in G$ there is $\bar{\omega} \in G$ such that $T_{\omega_1}T_{\omega_2} = T_{\bar{\omega}}$. Let* (v) *be satisfied on G. Let $E \in \mathscr{L}$, $|E| > 0$. Then, given $n \geqq 2$ there exists an n-tuple $(x_1, x_2, \ldots, x_n)$ of points* *belonging to E such that $x_i \neq x_j$ for $i \neq j$ $(i, j = 1, 2, \ldots, n)$ and for each two indices $i, j \leqq n$, $i \neq j$ there exists $\omega = \omega(i, j) \in G$ such that $T_\omega(x_i) = x_j$ or $T_\omega(x_j) = x_i$.*

Proof. The proof will be given for the case when $T_\omega(x) > x$ for all $\omega \in G$ and all $x \in E_1$. We shall apply the mathematical induction. For $n = 2$, Theorem follows from Corollary 1,1. Suppose that it is true for some $n \geqq 2$. It means that in every set $A \in \mathscr{L}$ of a positive measure there exist n points $x_1, x_2, \ldots, x_n$ with the properties stated in Theorem. We may suppose $x_1 < x_2 < \ldots < x_n$. In this case (owing to the fact that $T_\omega(x) > x$) choosing $i < j \leqq n$ there is $\omega = \omega(i, j) \in G$ such that $x_j = T_\omega(x_i)$ (the case $T_\omega(x_j) = x_i$ is obviously impossible). Now we shall prove the assertion for $n + 1$. Let $|E| > 0$. Denoting by F the set of all those $x \in E$ which appear as the last (the largest one) coordinate in some n-tuple mentioned above (see the assertion of the theorem), then

$$F = \bigcup E \cap T_{\omega_1}(E) \cap \ldots \cap T_{\omega_{n-1}}(E) \tag{5}$$

where the union is taken with respect to all $(n - 1)$-tuples $(\omega_1, \omega_2, \ldots, \omega_{n-1})$ for which $\omega_i \in G$, $\omega_i \neq \omega_0$ $(i = 1, 2, \ldots, n - 1)$ and $\omega_i \neq \omega_j$ if $i, j \leqq n - 1$, $i \neq j$.

We shall verify (5). Let $x \in F$. Then there is some n-tuple $(x_1, \ldots, x_{n-1}, x)$ such that $x_i \in E$ $(i = 1, \ldots, n - 1)$, $x_i < x_j$ if $i < j \leqq n - 1$ and to every $i \leqq n - 1$ there exists $\omega_i \in G$ such that $x = T_{\omega_i}(x_i)$. We have $\omega_i \neq \omega_0$, $\omega_i \neq \omega_j$ if $i \neq j$, $i, j \leqq n - 1$ owing to the fact that $x_i \neq x_j$ if $i \neq j$, $i, j \leqq n - 1$ and $x_i = T_{\omega_i}^{-1}(x)$ where $T_{\omega_i}^{-1}$ is the inverse transformation to T_{ω_i}. Hence x belongs to the right-hand side of (5).

Let x belong to the right-hand side of (5). Then $x \in E$ and

$$x = T_{\omega_i}(x_i), \tag{6}$$

where $x_i \in E (i = 1, \ldots, n - 1)$. Since $x_i < T_{\omega_i}(x_i) = x$, we conclude, that x is the largest among the numbers $x_1, \ldots, x_{n-1}, x$. Since $\omega_i \neq \omega_j$ if $i \neq j$ we have, in view of (iv) and (6), $x_i \neq x_j$. Hence $x \in F$.

We shall show that $|E - F| = 0$. In fact, if $|E - F| > 0$ then by the induction hypothesis there exists an n-tuple $(x_1, x_2, \ldots, x_n)$ of numbers $x_i \in E - F$ $(i = 1, 2, \ldots, n)$ such that $x_i \neq x_j$ if $i \neq j$ and for $i < j$, $T_\omega(x_i) = x_j$. Then by the definition of F, $x_n \in E - F$. But this is not possible.

Hence $|E| = |F| > 0$. From the first induction step there exist in F two numbers y, z, $y < z$ such that $T_\omega(y) = z$ where $\omega \in G$. Since $y \in F$, there exist x_i $(i = 1, \ldots, n - 1)$ in E such that $(x_1, \ldots, x_{n-1}, y)$ is an n-tuple with the properties described in Theorem.

Now by the assumption of Theorem we have

$$z = T_\omega(y) = T_\omega T_{\omega_i}(x_i) = T_{\bar{\omega}_i}(x_i) \ (\bar{\omega}_i \in G)$$

for every $i \leqq n - 1$. From here there follows that the $(n + 1)$-tuple $(x_1, \ldots, x_n, x_{n+1})$ where $x_n = y$, $x_{n+1} = z$ consists of mutually distinct numbers and for every two x_i, x_j $(i < j,\ i, j \leqq n > 1)$ there exists $\omega = \omega(i, j) \in G$ such that $x_j = T_\omega(x_i)$. The proof is finished.

Corollary 1,3. *Any linear set of positive measure contains for every $n \geqq 2$, n mutually distinct points such that the distance of any two of them is rational.*

Proof. It is sufficient to put in the preceding theorem $G = R^+$ (R^+ is the set of all positive rational numbers), $\Omega = E_1$, $\omega_0 = 0$, $T_\omega(x) = x + \omega$.

Note. The assumption that $\{T_\omega\}$ is a set of transformations from E_1 into E_1 is not substantial. We may consider transformations from I to I where I is a fixed interval on the real line. Then, if the other assumptions are satisfied, Theorem 1,2 is obviously true.

In view of this note we obtain the following result using Theorem 1,2.

Corollary 1,4. *Any linear set of positive measure contains for any $n \geqq 2$, n mutually distinct numbers such that the ratio of any two of them is rational.*

Proof. Evidently it is sufficient to prove the corollary for the sets of positive numbers. For this case let us consider the transformations $T_\omega(x) = \omega x$ where $x \in \in (0, +\infty)$ while $\Omega = (0, +\infty)$, $G = R^+ \cap (0, 1\rangle$, $\omega_0 = 1$.

By means of Corollary 1,3 a result is proved in [4] asserting that any linear set of positive measure contains a sequence of mutually distinct points such that the distance of any two of them is rational. A similar result for the ratio set is given in [3]. Here we are giving without proof a generalizing Theorem which may be obtained from the above result.

Theorem 1,3. *Under the assumptions of Theorem 1,2 there exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of mutually distinct elements of the set E such that for any two natural numbers $i \neq j$ there is $\omega = \omega(i, j) \in G$ such that either $x_j = T_\omega(x_i)$ or $x_i = T_\omega(x_j)$.*

In paper [2] there is proved that if A is a set with the Baire property (in E_1) and of the second category (in E_1) then $D(A)$ contains an interval $(0, \eta)$, $\eta > 0$. This fact will now be proved for a certain set of transformations of subsets of E_1. This result will give a generalization of that mentioned above and it will include at the same time the proof of an analogical result for the ratio sets (see Theorem 1,5).

In what follows Ω has the same meaning as in the beginning of the paper. Let to each $\omega \in \Omega$ a transformation F_ω of subsets of E_1 be associated, induced by some mapping from E_1 to E_1. A mapping F from 2^{E_1} to 2^{E_1} is said to be category preserving if $F(A)$ is of the first category in E_1 if and only if A is of the first category in E_1. For the sake of brevity we shall write again $y = F_\omega(x)$ instead of $\{y\} = F_\omega(\{x\})$.

Theorem 1,4. *Let Ω be a metric space. Let to each $\omega \in \Omega$ a mapping F_ω from 2^{E_1} into 2^{E_1} (induced by some mapping g_ω from E_1 to E_1) be associated. Suppose that for every $\omega \in \Omega$ the mapping F_ω is category preserving and if I is an interval then $F_\omega(I)$ is also interval. Let there be an $\omega_0 \in \Omega$ such that for every sequence $\omega_n \in \Omega$, $\omega_n \to \omega_0$ and for every two real numbers a, b; $a < b$ the following holds: If $F_{\omega_n}(\langle a, b\rangle)$ has the end points a_n, b_n, then $a_n \to a$, $b_n \to b$. Let $E \subset E_1$ have the Baire property and let it be of the second category in E_1. Then for every sequence $\omega_n \to \omega_0$ there exists n_0 such that if $n \geqq n_0$ then $E \cap F_{\omega_n}(E) \neq \emptyset$.*

Proof. Since E is of the second category in E_1, there exists an interval $I = \langle a, b\rangle$ such that $E \cap I$ is of the second category in E_1. Since E has the Baire property it may be expressed in the form $E = (G - P) \cup Q$ where P, Q are of the first category in E_1 and G is open in E_1. This representation implies that

$$E \cap I = (G \cap I - P_1) \cup Q_1\,, \quad P_1 = P \cap I\,, \quad Q_1 = Q \cap I\,,$$

P_1, Q_1 are of the first category in E_1. Since $E \cap I$ is of the second category in E_1 we have $G \cap I \neq \emptyset$. Consequently there is an interval $I_1 = \langle a_1, b_1\rangle$ contained in $G \cap I$ such that

$$a < a_1\,, \quad b_1 < b\,. \tag{7}$$

Then evidently

$$I_1 - E = I_1 - E \cap I = (I_1 - (G \cap I - P_1)) \cap (I_1 - Q_1)\,.$$

But $G \cap I \supset I_1$, hence $I_1 - (G \cap I - P_1) \subset I_1 - (I_1 - P_1) \subset P_1$ and so $I_1 - (G \cap I - P_1)$ is of the first category in E_1; therefore $I_1 - E$ is also of the first category in E_1. As a consequence the set $E \cap I_1$ is of the second category in E_1.

Now let $\omega_n \to \omega_0$. Then with regard to both the assumptions of the theorem and to inequalities (7) there is n_1 such that $F_{\omega_n}(I_1) \subset I$ holds for $n \geqq n_1$. Let $n \geqq n_1$. Then

$$F_{\omega_n}(I_1 - E) \supset F_{\omega_n}(I_1) - F_{\omega_n}(E)\,. \tag{8}$$

Since F_{ω_n} is category preserving, the set $F_{\omega_n}(I_1 - E)$ is of the first category. From the assumption of Theorem and on account of the fact that $E \cap I_1$ is of the second category, the existence of an interval $I_1^* \subset I_1$ follows so that for sufficiently large n (if $n \geqq n_0 \geqq n_1$) we have $F_{\omega_n}(I_1) \supset I_1^*$ and $E \cap I_1^*$ is of the second category in E_1. Then (8) implies that $I_1^* - F_{\omega_n}(E)$ is a set of the first category in E_1 and then the set

$$I_1^* - \bigcap_{n=n_0}^{\infty} F_{\omega_n}(E) = \bigcup_{n=n_0}^{\infty} (I_1^* - F_{\omega_n}(E))$$

is of the first category in E_1 as well. Hence also $E \cap I_1^* - E \cap \bigcap_{n=n_0}^{\infty} F_{\omega_n}(E)$ is a set of the first category in E_1. But $E \cap I_1^*$ is a set of the second category in E_1, hence in view of the foregoing consideration $E \cap \bigcap_{n=n_0}^{\infty} F_{\omega_n}(E) \neq \emptyset$. Consequently $E \cap F_{\omega_n}(E) \neq \emptyset$ (for $n \geqq n_0$). The proof is finished.

Note. If we choose particularly $\Omega = E_1$, $\omega_0 = 0$, $F_\omega(x) = x + \omega$ we get from the foregoing theorem a result analogical to the mentioned result of paper [2].

Theorem 1,5. *Let $E(E \subset E_1)$ have the Baire property and let E be of the second category in E_1. Then the set $R(E)$ contains an interval $(\eta, 1)$, $0 < \eta < 1$.*

Proof. Put $\Omega = (0, 1\rangle$, $\omega_0 = 1$, $g_\omega(x) = F_\omega(x) = \omega x$. The assumptions of Theorem 1,4 are satisfied. In fact, g_ω is a homeomorphism from E_1 onto E_1 and hence it is category preserving. Further $F_\omega(I)$ is evidently an interval if I is an interval. If $\omega_n \to \omega_0$ $(=1)$, then the end points of $F_{\omega_n}(I)$ (I is an interval) converge to the end points of I. So the present theorem is an easy corollary of Theorem 1,4.

2. SETS OF DISTANCES AND RATIO SETS OF SPECIAL LINEAR SETS

In paper [3] a theorem, is proved which we have mentioned above. According to it the set $R(A)$ contains an interval $(\eta, 1)$, $0 < \eta < 1$, if $|A| > 0$. The proof of this theorem is given in [3] in an analogous way as that of the mentioned Steinhaus theorem in paper [4]. Both the proofs are based on the following fundamental result on the sets of positive measure: If $|A| > 0$ then for every $\alpha \in (0, 1)$ there is an interval I such that $|A \cap I| > \alpha |I|$ (see [5], p. 72). This result was used also by us in the proof of Theorem 1,1.

Now we shall show that the mentioned result of paper [3] may be also derived as an easy consequence of the Steinhaus theorem. We shall give a new proof of the following theorem, which is evidently equivalent to the mentioned result of paper [3].

Theorem 2,1. *Let $|A| > 0$. Then $R(A)$ contains an interval $(1, \eta)$, $\eta > 1$.*

Proof. Let $|A| > 0$. Put $A_1 = A \cap (-\infty, 0)$, $A_2 = A \cap (0, +\infty)$. If $|A_1| = 0$ then $|A_2| > 0$, $A_2 \subset (0, +\infty)$ and $R(A_2) \subset R(A)$. If $|A_1| > 0$, put $A_1^* = \{x \in E_1; -x \in A_1\}$. Then evidently $|A_1^*| > 0$, $A_1^* \subset (0, +\infty)$ and $R(A_1^*) \subset R(A)$. In both cases there exists a set $M \subset (0, +\infty)$, $|M| > 0$ such that $R(M) \subset R(A)$. Since $|M| > 0$ there exist numbers a, b, $0 < a < b$ such that $|M \cap \langle a, b\rangle| > 0$. Put $M_1 = M \cap \langle a, b\rangle$. Evidently $R(M_1) \subset R(M)$. The function $\varphi(t) = \log t$ is absolutely continuous on $\langle a, b\rangle$. Thus we have for the measure $|\varphi(M_1)|$ of the set $\varphi(M_1)$ (see [6], p. 281–282)

$$|\varphi(M_1)| = \int_{M_1} \varphi'(t)\, dt = \int_{M_1} \frac{1}{t}\, dt \geqq \int_{M_1} \frac{1}{b}\, dt = \frac{1}{b}|M_1| > 0\,.$$

According to Steinhaus theorem, $D(\varphi(M_1))$ contains an interval $(0, \eta_1)$, $\eta_1 > 0$. Put $\eta = \exp\{\eta_1\} > 1$ and choose $d \in (1, \eta)$. Then $\log d \in (0, \eta_1)$ and as a consequence of

Steinhaus theorem there exist $x, y \in \varphi(M_1)$, $x > y$ such that $x - y = \log d$. Further $x = \varphi(t_1) = \log t_1$, $y = \varphi(t_2) = \log t_2$; $t_1, t_2 \in M_1$ and consequently $t_1/t_2 = d$. Hence $R(M_1) \supset (1, \eta)$ and evidently $R(A) \supset R(M_1)$. The proof is finished.

Also sets of zero measure may have the property that their distance set contains an interval $(0, \eta)$, $\eta > 0$. STEINHAUS has proved that for the Cantor discontinuum C, $D(C) = \langle 0, 1 \rangle$ holds (see [7]). In paper [8], Author proves that for almost all $d \in \langle 0, 1 \rangle$ the following assertion is true: For the number d there is an uncountable of the power of continuum set of pairs $(x, y) \in C \times C$ such that $|x - y| = d$. In connection with this the question arises what is the situation for the sets of positive measure. In particular: Does there exist for a set A of positive measure an interval $(0, \delta)$, $\delta > 0$ such that for every $d \in (0, \delta)$ the equality $|x - y| = d$ holds for an uncountable (of the power of continuum) pairs $(x, y) \in A \times A$? The following theorem gives the affirmative answer to the question.

Theorem 2,2. *Let $|E| > 0$. Then there exists a $\delta > 0$ such that $(0, \delta) \subset D(E)$ and for each $d \in (0, \delta)$ there holds: the set of all $x \in E$ for which there exist $y \in E$ with $|x - y| = d$ has positive measure.*

Proof. Let $|E| > 0$. Choose an interval I such that $|E \cap I| > \frac{3}{4}|I|$. Put $\delta = |I|/2$.

At first we shall show that $(0, \delta) \subset D(E)$. The sets $E \cap I$ and $(E \cap I) + d$ are included in an interval whose length is at most 3δ. These sets are not disjoint since if they were, the measure of their union would be equal to $2|E \cap I| - \frac{3}{2}|I| = 3\delta$. Hence the sets $E \cap I$, $(E \cap I) + d$ must have a non-empty intersection. Consequently there exists $y \in (E \cap I) \cap ((E \cap I) + d)$. This implies that y is of the form $y = x + d$, $x \in E \cap I$, hence $y - x = d$, $x, y \in E$.

Let us denote by F_d the set of all $x \in E$ for which there is a $y \in E$ such that $|x - y| = d$. Evidently

$$F_d = [E \cap (E + d)] \cup [E \cap (E - d)]\,.$$

We shall show that $|F_d| > 0$. Conversely, let $|F_d| = 0$. Put $H = E - F_d$. Then $|H| = |E| > 0$. Let I have the same meaning as above. Then $|H \cap I| = |E \cap I| > \frac{3}{4}|I|$. By a similar reasoning as was applied before to the set E it may by shown now that H and $H + d$ have a non-empty intersection. Hence there exist $x, y \in H$ such that $|x - y| = d$. But then $x \in F_d$ and we have $F_d \cap H \neq \emptyset$. This is a contradiction to the definition of H.

Now we shall show that taking δ sufficiently small we can achieve the result that for every $d \in (0, \delta)$ the number $|F_d|$ will be bounded from bellow by a positive constant which does not depend on d.

Theorem 2,3. *Let $|E| > 0$. Then there exist positive numbers δ and c such that $(0, \delta) \subset D(E)$ and for every $d \in (0, \delta)$ we have $|F_d| > c$.*

Proof. Choose an interval I such that $|E \cap I| > \frac{3}{4}|I|$. Put $\delta = |I|/4$, $c = \delta/2$. Let $d \in (0, \delta)$. We shall prove $|F_d| > c$. Let the converse hold. Then there exists a number d, $0 < d < \delta$ such that $|F_d| \leqq c$. Then

$$|(E - F_d) \cap I| = |E \cap I| - |F_d \cap I| > \tfrac{3}{4}|I| - \tfrac{1}{8}|I| = \tfrac{5}{8}|I| .$$

The sets $(E - F_d) \cap I$, $[(E - F_d) \cap I] + d$ are included in an interval having the length less or equal 5δ. If these sets were disjoint then the measure of their union would be greater than $\frac{5}{4}|I| = 5\delta$, but this is impossible. Hence the intersection of these sets contains a point y. We have $y \in E - F_d$ and simultaneously $y = x + d$, $x \in E - F_d$. Hence $|x - y| = d$ and by the definition of the set F_d the element x must belong to F_d. But this is a contradiction to the fact that $x \in E - F_d$.

From Theorem 2,3 in a simple way follows this known result (see [4], Theorem 1,3 and the text before the theorem).

Theorem 2,4. *Let $|E| > 0$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of mutually distinct elements belonging to E such that for every $m, n = 1, 2, \ldots$ $|x_m - x_n|$ is rational.*

Proof. Let δ have the same meaning as in the above Theorem. Choose a sequence $\{d_n\}_{n=1}^{\infty}$ of distinct rational numbers belonging to $(0, \delta)$. By Theorem 2,3 there exists $c > 0$ such that $|F_{d_n}| > c$ $(n = 1, 2, \ldots)$. Since

$$\left|\limsup_{n\to\infty} F_{d_n}\right| = \lim_{n\to\infty} \left|\bigcup_{k=n}^{\infty} F_{d_k}\right| \geqq \limsup_{n\to\infty} |F_{d_n}| \geqq c ,$$

we have $\limsup_{n\to\infty} F_{d_n} \neq \emptyset$. The last inequality implies that a number x belonging to infinitely many F_{d_n} exists. Let $x \in F_{d_{k_n}}$ $(n = 1, 2, \ldots)$. By the definition of the sets $F_{d_{k_n}}$ the existence of a sequence $\{x_n\}_{n=1}^{\infty}$ of elements belonging to E follows such that $|x_n - x| = d_{k_n}$ $(n = 1, 2, \ldots)$. Since $x_m - x_n = (x_m - x) + (x - x_n)$ and $x_m - x$, $x - x_n$ are rational numbers, the number $x_m - x_n$ is a rational number different from zero for $m \neq n$ on account of the fact that $\{d_{k_n}\}_{n=1}^{\infty}$ is a one-to-one sequence.

In paper [9] there is proved that if $A \subset (0, +\infty)$, $|A| > 0$ and $R_1(A)$ denotes the set of all d for which there exists an infinite number of pairs $(x, y) \in A \times A$ with $x/y = d$, then $R_1(A)$ contains an interval $(1, \eta)$, $\eta > 1$. But from the proof it may be seen that the author of [9] has proved if fact this more general assertion which is analogous to our Theorem 2,2.

Theorem 2,5. *Let $|A| > 0$. Then there is an interval $(1, \eta)$, $\eta > 1$ such that for every $d \in (1, \eta)$ the following assertion is true: if P_d denotes the set of all $x \in A$ for which there exist $y \in A$ such that $x/y = d$, then $|P_d| > 0$.*

Note that Theorem 2,5 may be derived from Theorem 2,2 by a procedure analogous to that applied in the proof of Theorem 2,1. A similar note concerns also the formulations and the proofs of those theorems (for the sets $R(A)$) which are analogous to Theorems 2,3 and 2,4. The reader himself will be able to complete the proofs in an easy way.

Now let us come back to the mentioned result of R. P. BOAS according to which there holds for almost all $d \in \langle 0, 1\rangle$: to the number d there exists an uncountable of the power of continuum set of pairs $(x, y) \in C \times C$ such that $|x - y| = d$. In what follows a strengthening of this result will be given. The strengthening will be done in detail for the discontinua W (see [10], [11]). For C it may be done in an analogous way.

In papers [10], [11] the following result is proved:

Given $A = \sum_{n=1}^{\infty} a_n < +\infty$, $0 < a_n \leqq 2R_n = 2\sum_{k=1}^{\infty} a_{n+k}$ $(n = 1, 2, \ldots)$, *let* W *be the set of all* x *of the form*

$$(8a) \qquad x = \sum_{n=1}^{\infty} \varepsilon_n a_n, \quad \varepsilon_n = 1 \quad \text{or} \quad -1 \quad (n = 1, 2, \ldots).$$

Then $D(W) = \langle 0, 2A\rangle$.

If moreover $|W| = 0$ *and* $a_n > R_n$ $(n = 1, 2, \ldots)$, *then for almost every* $d \in \langle 0, 2A\rangle$ *there is an uncountable of the power of continuum set* $(x', x'') \in W \times W$ *such that* $x'' - x' = d$.

We shall both strengthen the above result and show that the set of those $d \in \langle 0, 2A\rangle$ for which the existence of uncountable set of pairs $(x', x'') \in W \times W$ with $x'' - x' = d$ is not guaranteed, may be better estimated.

In what follows dim M denotes the Hausdorff dimension of M with respect to the system of measure functions

$$\mu^{(\alpha)}(t) = t^{\alpha}, \quad t \in (0, +\infty), \quad \alpha \in (0, 1)$$

(see [12]).

Theorem 2,6 a) *Let* $A = \sum_{n=1}^{\infty} a_n < +\infty$, $a_n > 0$ $(n = 1, 2, \ldots)$. *Let* W *have the same meaning as above. Let*

$$(9) \qquad R_n < a_n \quad (n = 1, 2, \ldots).$$

Then for every $d \in W + A$ *except a countable set there holds: To the number* d *there is an uncountable of the power of continuum set of pairs* $(x', x'') \in W \times W$ *such that* $x'' - x' = d$.

b) *If* $|W| = 0$ *and* $R_n < a_n \leqq 2R_n$ $(n = 1, 2, \ldots)$, *then there exists a set* $M \subset \langle 0, 2A\rangle$ *such that* $|M| = 0$, $\dim M = \dim W$ *and for every* $d \in \langle 0, 2A\rangle - M$ *there exists an uncountable of the power of continuum set of pairs* $(x', x'') \in W \times W$ *such that* $x'' - x' = d$.

First, the following lemma will be proved.

Lemma 2,1. *Let* $H \subset E_1$, $\gamma \neq 0$. *Then*

$$\dim(\gamma H) = \dim H.$$

Proof. Let $A \subset E_1$, $\lambda \neq 0$. We shall show that

$$\dim(\lambda A) \leqq \dim A\,. \tag{9a}$$

Let $\alpha > \dim A$, $\alpha \in (0,1)$. Then $\mu^{(\alpha)}\{A\} = 0$ where

$$\mu^{(\alpha)}\{A\} = \lim_{\eta \to 0+} \mu^{(\alpha)}_\eta\{A\}\,, \quad \mu^{(\alpha)}_\eta\{A\} = \inf_{V \in \mathscr{U}(\eta, A)} \sum_{i \in V} |i|^\alpha\,,$$

$\mathscr{U}(\eta, A)$ is the set of all η-covers of A *) (see [12]). Let $\varepsilon > 0$. Put $\varepsilon' = \varepsilon/|\lambda|^\alpha > 0$. By the preceding reasoning for $\varepsilon' > 0$ there is $\eta_0 > 0$ such that if $0 < \eta \leqq \eta_0$ then $\mu^{(\alpha)}_\eta\{A\} < \varepsilon'$ holds. This implies by the definition of $\mu^{(\alpha)}_\eta\{A\}$ that for any η, $0 < \eta \leqq \eta_0$ there is an η-cover $V_\eta = \{i\}$ of the set A such that

$$\sum_{i \in V_\eta} |i|^\alpha < \varepsilon'\,. \tag{9b}$$

Now let η' be any positive number fulfilling the inequality $\eta' \leqq |\lambda|\,\eta_0$. Put $V' = \{\lambda i\}$, $i \in V_\eta$, $\eta = \eta'/|\lambda|$. Then V' is evidently an η'-cover of the set λA. Further, in view of (9b) and by the definition of ε', $\sum_{i \in V'} |i|^\alpha = |\lambda|^\alpha \sum_{i \in V_\eta} |i|^\alpha < \varepsilon$ holds. Hence

$$\mu^{(\alpha)}\{\lambda A\} = \lim_{\eta' \to 0+} \mu^{(\alpha)}_{\eta'}\{\lambda A\} = 0\,.$$

Hence (9a) holds.

Now put $\lambda = \gamma$, $A = H$. Then according to (9a) we have $\dim(\gamma H) \leqq \dim H$. Further put $\lambda = 1/\gamma$, $A = \gamma H$, then again according to (9a) $\dim H \leqq \dim(\gamma H)$. Thus we have $\dim H = \dim(\gamma H)$.

Proof of Theorem 2,6. a) From condition (9), the unicity of the expansions (8a) of elements of the set W follows in an easy way. Let us denote by Y the set of all numbers of the form

$$y = \sum_{n=1}^{\infty} \vartheta_n a_n\,, \quad \vartheta_n = 0 \quad \text{or} \quad 2 \quad (n = 1, 2, \ldots)\,. \tag{10}$$

Hence $Y = W + A$. Obviously all the elements of the set Y except elements of a countable set have such property that their expansions (10) contain an infinite number of factors ϑ_n which are equal to zero. If y (see (10)) has such an expansion in which $\vartheta_n = 0$ for an infinite number of indices n's, then choose $x' = \sum_{n=1}^{\infty} \varepsilon'_n a_n \in W$, $x'' = \sum_{n=1}^{\infty} \varepsilon''_n a_n \in W$ such that

$$\varepsilon''_n - \varepsilon'_n = \vartheta_n \quad (n = 1, 2, \ldots)\,. \tag{11}$$

This is evidently possible and if $\vartheta_n = 0$ we can choose ε'_n, ε''_n in two different ways so that (11) is true (we can put either $\varepsilon'_n = \varepsilon''_n = 1$, or $\varepsilon'_n = \varepsilon''_n = -1$). From the unicity

*) η-cover of a set A is any countable system of intervals which is a cover of A, the length of every interval of the system not exceeding η.

of the expansions of elements belonging to W, validity of a) immediately follows.

Now b) will be proved. In [10] a proof was given according to which every $x \in \in \langle -A, A \rangle$ may be written in the form

$$x = \sum_{n=1}^{\infty} \eta_n a_n, \quad \eta_n = 0, 1 \quad \text{or} \quad -1 \quad (n = 1, 2, \ldots). \tag{12}$$

Denote by M_k the set of all those $x \in \langle -A, A \rangle$ for which the following is true: There is at least one expansion of the form (12) of the element x, such that 0 appears in the sequence $\{\eta_n\}_{n=1}^{\infty}$ exactly k-times. Then $M_0 = W$, $\dim M_0 = \dim W$ and $|M_0| = 0$. Further $M_1 = \bigcup_{l=1}^{\infty} N_l$ where N_l is the set of all those x for which the following is true: There is at least one expansion of the form (12) of the element x, such that $\eta_l = 0$ and $\eta_n = 1$ or -1 for every $n \neq l$. Obviously

$$(N_l + a_l) \cup (N_l + (-a_l)) = W. \tag{13}$$

Hence according to the invariancy with respect to a translation of the Hausdorff dimension, we have

$$\dim N_l \leqq \dim W \quad (l = 1, 2, \ldots). \tag{14}$$

If $B \subset \bigcup_{n=1}^{\infty} B_n$, then as it is known (see e.g. Lemma 4 in [12]) $\dim B \leqq \sup_{n=1,2,\ldots} \dim B_n$. Put as a special case $B_1 = N_l + a_l$, $B_2 = N_l + (-a_l)$, $B_k = \emptyset$ for $k > 2$, $B = W$. Then according to the above inequality,

$$\dim W \leqq \dim N_l \quad (l = 1, 2, \ldots). \tag{15}$$

(14) and (15) yields $\dim N_l = \dim W (l = 1, 2, \ldots)$ and using again Lemma 4, [2] we get $\dim M_1 = \dim W$.

Further we shall show that also $\dim M_2 = \dim W$ holds. Evidently

$$M_2 = \bigcup_{m,n=1}^{\infty} N_{m,n} \tag{16}$$

where $N_{m,n}$ is the set of all those $x \in \langle -A, A \rangle$ for which there holds: There is at least one expansion of the form (12) of the element x such that $\eta_l = 0$ if $l = m$ or $l = n$ and $\eta_l = 1$ or -1 for every $l \neq m, n$. Evidently

$$[N_{m,n} + (a_m + a_n)] \cup [N_{m,n} + (a_m - a_n)] \cup$$
$$\cup [N_{m,n} + (-a_m + a_n)] \cup [N_{m,n} + (-a_m - a_n)] = W$$

and by the procedure used above we get

$$\dim N_{m,n} = \dim W \quad (m, n = 1, 2, \ldots).$$

Then (16) implies $\dim M_2 = \dim W$. By the same method it is possible to show that $\dim M_k = \dim W$ also if $k > 2$.

Put $M = \langle 0, 2A \rangle \cap \bigcup_{k=0}^{\infty} 2M_k$. Notice that $M_0 \cap \langle 0, 2A \rangle = W \cap \langle 0, 2A \rangle$. Since W is symmetric with respect to the point 0, we have according to the foregoing facts $\dim M \geqq \dim W$. Lemma 2,4 and Lemma 4, [12] give $\dim M \leqq \dim W$. Hence $\dim M = \dim W$.

Let $d \notin M$, $d \in \langle 0, 2A \rangle$. Then $d/2 \notin M_k$ $(k = 0, 1, \ldots)$ and thus a sequence $\{\eta_n\}_{n=1}^{\infty}$ of numbers $0, 1, -1$ containing an infinite number of zeros exists such that $d = \sum_{n=1}^{\infty} 2\eta_n a_n$. Now the proof may be completed in a similar way as that of the case a).

To complete the proof of the Theorem it is sufficient to prove that $|M| = 0$. To prove this it is sufficient to prove that $|M_k| = 0$ $(k = 0, 1, \ldots)$. We have already seen that $|M_0| = |W| = 0$. Further, (13) implies $|N_l| = 0$ $(l = 1, 2, \ldots)$, hence $|M_1| = 0$. It may be shown in a similar way that $|M_k| = 0$ for $k > 1$.

In an analogous way the following theorem which is a strengthening of the mentioned result of R. P. Boas, may be proved.

Theorem 2,7. *There exists a set $M \subset \langle 0,1 \rangle$ such that*

$$\dim M = \dim C = \frac{\log 2}{\log 3} < 1$$

and for every $d \in \langle 0,1 \rangle - M$ there is an infinite of the power of continuum set of pairs $(x', x'') \in C \times C$ such that $x'' - x' = d$.

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Authors' address: Šmeralova 2b, Bratislava (Prírodovedecká fakulta UK).