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PARALLEL DISPLACEMENT OF VECTORS IN A RHEONOMOUS RIEMANNIAN SPACE

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In this paper we shall study some generalizations of the parallel displacement of vectors in a Riemannian space for the case of a rheonomous Riemannian space. A simple physical application will be shown at the end of this paper. We shall follow mostly the terminology and notations of [4]. Latin indices always take all positive integral values from 1 to m, $m > 1$. The symbol R_m denotes the arithmetical space of ordered sets of m real numbers, with natural topology. We shall denote $\{x^a\}$ or $\{x^a, t\}$ a current point of the space R_m or R_{m+1} respectively.

Let W_{m+1} be a differentiable variety of $(m + 1)$ dimensions. Let us denote $[x^a, t]$ a current point of this variety the coordinates of which are x^a, t. For the sake of simplicity we shall suppose that there exists a one-to-one mapping $[x^a, t] \to \{x^a, t\}$ of the variety W_{m+1} on some domain $\Omega \subset R_{m+1}$ where $\Omega = O \times I$, $O \subset R_m$, $I \subset R_1$.

Definition. A variety W_{m+1} is said to be a *rheonomous Riemannian space* $r - V_m(t)$ whenever the following suppositions are fulfilled:

1. All admissible transformations of the parameters x^a, t of the variety W_{m+1} are described by all possible functions of the third class

(1) $$\bar{x}^a = \bar{x}^a(x^b)\,, \quad x^b \in O\,,$$

(2) $$\bar{t} = t + C\,, \quad t \in I\,, \quad C = \text{const.}\,,$$

which realize a one-to-one mapping of the domain O or the interval I on a domain from R_m or an interval from R_1 respectively.

2. There are given m^2 functions of the second class

(3) $$g_{ij} = g_{ij}(x^a, t)\,, \quad \{x^a, t\} \in \Omega\,,$$

which define, at every point of the variety $V_m(t_0) \subset W_{m+1}$, $t_0 \in I$, described by the equation $t = t_0 = \text{const.}$, the covariant coordinates of the positively definite metric tensor.

Remark. Let V_m be a Riemannian space and O the domain of its parameters. According to the preceding definition we may consider the cartesian product $V_m \times I$ as a rheonomous Riemannian space. We shall call it a *stationary* space $r - V_m(t)$.

Let us agree that a tangent space or a tensor or a connection defined at the point $[x^a, t]$ of a space $V_m(t)$ will be said to be the tangent space or tensor or the connection at the point $[x^a, t]$ of the rheonomous Riemannian space $r - V_m(t)$ respectively. Similarly, we may consider a tensor field of a rheonomous space $r - V_m(t)$ and define the covariant derivative of this field. If, for example, the functions of the first class

$$v^a = v^a(x^b, t), \quad \{x^b, t\} \in \Omega, \tag{4}$$

define a vector field in $r - V_m(t)$, then the covariant derivative of this field is defined by the relation

$$D_c v^a = \frac{\partial v^a}{\partial x^c} + \left\{ {a \atop c\ k} \right\} v^k, \tag{5}$$

where $\left\{ {a \atop c\ k} \right\}$ are the so-called Christoffel symbols.

Definition. A curve in the rheonomous space $r - V_m(t)$ is said to be a *trajectory* whenever its parametric equations may be written in the form

$$x^a = x^a(T), \quad t = T, \quad T \in J \subset I, \tag{6}$$

where $x^a(T)$ are functions of the first class and J an open interval. A trajectory described by the parametric equationes

$$x^a = x_0^a = \text{const.}, \quad t = T, \quad T \in I$$

is called a *parametric t-curve*.

The notion of the parametric t-curve is evidently invariant with repsect to the admissible transformations (1), (2). Likewise, the length s of the trajectory (6) between its two points $[x^a(T_1), T_1]$, $[x^a(T_2), T_2]$, defined by the relation

$$s = \int_{T_1}^{T_2} \sqrt{\left(g_{ij} \frac{dx^i}{dT} \frac{dx^j}{dT}\right)} dT, \tag{7}$$

is an invariant notion with respect to these transformations. If the trajectory (6) is a parametric t-curve then $s = 0$. The *tangent vector* of the trajectory (6) at its point $[x^a(T), T]$ is meant to be the vector with the contravariant coordinates $dx^a(T)/dT$. If this vector is non-zero for all $T \in J$ then the trajectory is said to be *regular*.

We may define the *absolute derivative* of the tensor field along the trajectory (6) in the usual way. In the case of the vector field (4) this derivative is defined by the relation

$$\mathrm{D}_T v^a = \frac{\mathrm{d}v^a}{\mathrm{d}T} + \begin{Bmatrix} a \\ c\ k \end{Bmatrix} v^k \frac{\mathrm{d}x^c}{\mathrm{d}t}. \tag{8}$$

We shall denote by $\mathrm{D}_t v^a$ the absolute derivative of the vector field (6) along the parametric t-curve. Evidently, we may write

$$\mathrm{D}_T v^a = \mathrm{D}_c v^a \frac{\mathrm{d}x^c}{\mathrm{d}T} + \mathrm{D}_t v^a.$$

We may show that the absolute derivative of the sum or the difference or the product of two tensor fields are given by the same rule as in ordinary differentiation.

Definition. We shall say that the vectors defined in the rheonomous space $r - V_m(t)$ at the points of the trajectory (6) by means of functions of the first class

$$v^a = v^a(T), \quad T \in J \tag{9}$$

are *pseudoparallel* whenever the relation

$$\mathrm{D}_T v^a = 0, \tag{10}$$

holds for all $T \in J$.

Definition. The trajectory (6) is said to be a *pseudogedesic* whenever the condition

$$\mathrm{D}_T \frac{\mathrm{d}x^a}{\mathrm{d}T} = 0, \quad g_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}T} \frac{\mathrm{d}x^j}{\mathrm{d}T} \neq 0, \tag{11}$$

holds for all $T \in J$.

The system of equations (10) may be interpreted as a system of m differential equations of the first order for m unknown functions $v^a(T)$. From writing out this system in Cauchy's canonical form

$$\frac{\mathrm{d}v^a}{\mathrm{d}T} = - \begin{Bmatrix} a \\ b\ c \end{Bmatrix} \frac{\mathrm{d}x^b}{\mathrm{d}T} v^c$$

there follows the unique existence of the solution of the system for initial conditions $v_0^a = v^a(T_0)$, where $T_0 \in I$. We also say that the vector v_0^a undergoes a *pseudoparallel displacement* along the trajectory (6) uniquely. Similarly, by means of (11) we may verify the unique local existence of a geodesic which goes through a given point of the rheonomous Riemannian space and which possesses a given nonvanishing tangent vector at this point.

Theorem. *The scalar product of two vectors which undergo a pseudoparallel displacement along the trajectory* (6) *is generally not constant.*

Proof. Let us denote $G_{ab} = D_t g_{ab}$. Evidently, at all points of the trajectory (6) the following equation holds

$$\text{(12)} \qquad \mathrm{D}_T g_{ab} = \mathrm{D}_c g_{ab} \frac{\mathrm{d}x^c}{\mathrm{d}T} + \mathrm{D}_t g_{ab} = G_{ab} \,.$$

Let two fields of pseudoparallel vectors be defined along the trajectory (6) by means of functions $v^a(T)$, $w^b(T)$, $T \in J$. Then

$$\text{(13)} \qquad \mathrm{D}_T v^a = 0 \,, \quad \mathrm{D}_T w^b = 0 \,.$$

Let us investigate if the function $f(T) = G_{ab} v^a w^b$ is a constant on J. From (12) and (13) we have

$$\frac{\mathrm{d}f}{\mathrm{d}T} = G_{ab} v^a w^b \,.$$

Now it is easy to see that in a general case $f \neq$ const. So the theorem is proved.

Remark. In a stationary rheonomous Riemannian space is $G_{ab} = 0$ and $f =$ = const., in accordance with the well-known case of parallel displacement of vectors in a Riemannian space.

In a rheonomous space $r - V_m(t)$, let us consider all regular trajectories that go through two different points $[\underset{1}{x}^a, T_1]$ and $[\underset{2}{x}^a, T_2]$ and let us find among them a trajectory of extreme length, i.e. a trajectory along which the functional (7) attains its extreme value. If there exists such regular trajectory then the corresponding functions $x^a(T)$ satisfy the system of Euler's differential equations

$$\text{(14)} \qquad \frac{\partial F}{\partial x^c} - \frac{\mathrm{d}}{\mathrm{d}T} \frac{\partial F}{\partial \dot{x}^c} = 0 \,,$$

where

$$F = \sqrt{(g_{ab} \dot{x}^a \dot{x}^b)} \neq 0 \,, \quad \dot{x}^a = \frac{\mathrm{d}x^a}{\mathrm{d}T} \,.$$

We calculate easily that

$$\frac{\partial F}{\partial x^c} = \frac{\partial_c g_{ab} \dot{x}^a \dot{x}^b}{2F} \,,$$

$$\frac{\mathrm{d}}{\mathrm{d}T} \frac{\partial F}{\partial \dot{x}^c} = \frac{\mathrm{d}}{\mathrm{d}T} \frac{g_{ac} \dot{x}^a}{F} = \left(\frac{\mathrm{d}}{\mathrm{d}T} \frac{1}{F} \right) g_{ac} \dot{x}^a + \frac{1}{F} \left(\partial_b g_{ac} \dot{x}^a \dot{x}^b + \partial_t g_{ac} \dot{x}^a + g_{ac} \ddot{x}^a \right) .$$

Using the last two equations and (8) for the modification of the system (14) we obtain the following form of Euler's equations:

$$(15) \qquad \mathrm{D}_T \frac{\mathrm{d}x^a}{\mathrm{d}T} + G^a_b \frac{\mathrm{d}x^b}{\mathrm{d}T} - \frac{\mathrm{d}x^a}{\mathrm{d}T} \frac{\mathrm{d}}{\mathrm{d}T} \ln \sqrt{\left(g_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}T} \frac{\mathrm{d}x^c}{\mathrm{d}T}\right)} = 0 \,.$$

If we call the (evidently regular) trajectory which is the solution of the system (15) an *E-geodesic* then we may assert that every regular trajectory which is a trajectory of extreme length in a rheonomous space $r - V_m(t)$ is also an E-geodesic.

Euler's equations (15) form a system of differential equations of the second order in which the second derivatives are not explicitly expressed. Let us find their explicit expressions. If we write, for brevity sake,

$$\mathrm{D}_T \frac{\mathrm{d}x^a}{\mathrm{d}T} = z^a \,, \quad \frac{\mathrm{d}x^a}{\mathrm{d}T} = v^a$$

and denote by the symbol Q^a the sum of terms which are on the right-hand side of the a-th equation and do not contain the unknown z^a then we may write (15) in the form

$$(16) \qquad z^a - \frac{1}{F^2} v^a v_b z^b = Q^a \,.$$

By a rather longer calculation we may find out that the determinant of the system (16) is zero. Therefore, in the system od Euler's equations (15) we cannot express explicit second derivatives uniquely and transform the system into the equivalent canonical form. But that means that the usual initial conditions secure neither the uniqueness nor even the existence of an E-geodesic. Further, in a general case, the pseudogeodesic is not a trajectory of extreme lenght.

Let us suposse that the vector field (9) consits wholly of nonvanishing vectors. The set of all directions which are defined by these vectors will be called shortly the *direction field* (9). If there exists a function $f(T)$, $T \in J$, $f(T) \neq 0$ every where in J, and such that the vector field defined by the functions

$$(17) \qquad w^a = f(T)\, v^a(T)$$

is composed of pseudoparallel vectors then we say that the direction field (9) is *pseudoparallel.*

Theorem. *The direction field (9) is pseudoparallel exactly in that case when there exists such a function $k(T)$, $T \in J$ that for all $T \in J$ the equation*

$$(18) \qquad \mathrm{D}_T v^a = k(T)\, v^a$$

holds.

Proof. Let us suppose that the direction field (9) is pseudoparallel. Then there exists such a function $f(T)$, non-vanishing everywhere in J, that for all $T \in J$

$$\mathrm{D}_T(f(T) \cdot v^a) = 0 \tag{19}$$

holds.

Using the notation

$$\frac{\mathrm{D}_T f(T)}{f(T)} = k(T)\,,$$

we arrange (19) easily into the form (18). Conversely, it is easy to show that (19) follows from (18) and thus complete the proof.

We shall use the preceding considerations to introduce another generalization of the parallel displacement.

Definition. Vector field (9), defined along the trajectory (6), is said to be a *δ-parallel* field whenever it satisfies the following conditions:

1. The direction field (9) is pseudoparallel.
2. The magnitude of all vectors of the given field is a non-zero constant.

Vector field (9) which consists of non-zero vectors only is δ-parallel exactly in that case when there exists such a function $k(T)$ that

$$\mathrm{D}_T v^a = k(T)\, v^a \tag{20}$$

and

$$\mathrm{D}_T(g_{ab} v^a v^b) = 0\,. \tag{21}$$

If we put (12) and (20) into the equation (21) then we calculate easily the function $k(T)$ and find out that equation (20) may be written in the form

$$\mathrm{D}_T v^a + \frac{1}{2} \frac{G_{bc} v^b v^c}{g_{bc} v^b v^c}\, v^a = 0 \tag{22}$$

or, conveniently, in short form

$$\delta_T v^a = 0\,. \tag{23}$$

Conversely, it is obvious that a vector field which satisfies the condition (22) or (23) is a δ-parallel field. By means of (22) we may easily verify that under usual initial conditions the δ-parallel displacement of a vector along a trajectory may be realized uniquely. It may be shown that a δ-parallel displacement does not generally preserves the scalar product of two vectors which undergo the displacement. The system of differential equations

$$\delta_T \frac{\mathrm{d}x^a}{\mathrm{d}T} = 0 \tag{24}$$

describes a trajectory which we shall call a *δ-geodesic*. From the writing out of the system (24) into Cauchy's canonical form follows that there exists exactly one δ-geodesic which goes through a given point in the rheonomous Riemannian tangent vector at that point. We shall show another generalization of the parallel displacement.

Definition. Let $Q^a_b(T)$, $T \in J$ be a quadratic tensor field defined along the trajectory (6). We shall say that the vector field (9) is *generally-parallel* with respect to the tensor field $Q^a_b(T)$ if the equation

$$\mathrm{D}_T v^a = Q^a_b v^b \,, \tag{25}$$

holds for all $T \in J$.

Let us find the condition for the tensor field Q^b_a that the generally-parallel displacement defined by the equations (25) preserves the scalar product of any two vectors which undergo that displacement. Such a displacement will be called a *H-parallel* displacement.

Let $v^a(T)$, $w^b(T)$ be two vector fields which are in the above stated sense H-parallel along the trajectory (6). Also,

$$\mathrm{D}_T v^a = Q^a_c v^c \,, \quad \mathrm{D}_T w^b = Q^b_c w^c \,, \quad \mathrm{D}_T(g_{ab} v^a w^b) = 0 \,.$$

Using the first two equations for the modification of the third equation we obtain the relation

$$v^a w^b (G_{ab} + Q_{ab} + Q_{ba}) = 0 \,.$$

Hence, the tensor Q_{ab} may be written in the form

$$Q_{ab} = -\tfrac{1}{2} G_{ab} + E_{ab} \,, \tag{26}$$

where E_{ab} is any antisymmetric tensor. Conversely, it is easy to verify, supposing (26), that the parallel displacement (25) is an H-parallel displacement.

If E_{ab} is a zero tensor at all points of the trajectory (6) then the H-parallel displacement is called a *special H-parallel* displacement. In this case the equations (25) are of the form

$$\mathrm{D}_T v^a + \tfrac{1}{2} G^a_b v^b = 0 \,. \tag{27}$$

From these equations we conclude that under usual initial conditions a given vector may undergo an H-displacement along the trajectory (6) uniquely. Further, we may introduce the notion of a *special H-geodesic* and prove that there exists exactly one special H-geodesic which goes through a given point of the rheonomous space $r - V_m(t)$ and possesses a given tangent vector at that point.

If the rheonomous space $r - V(t)$ is stationary then the relation $G_{ab} = 0$ holds true everywhere. But then the equationes (10), (22) and (27) are mutually identical. So the following theorem holds true:

Theorem. *Let $r - V_m(t)$ be a stationary rheonomous Riemannian space. Then the pseudoparallel, δ-parallel and special H-parallel displacements along a given trajectory nutually merge.*

We shall give a simple physical interpretation of the introduced concepts from the standpoint of classical mechanics. First, to the parameter t we shall assign the physical meaning of *time*. After all, it is consistent with the equation (2) which describes the admissible transformation of this parameter. We shall consider the rheonomous space $r - V_m(t)$ as an m-dimensional Riemannian space the metric of which at every point is a function of time. We shall interpret the parametric equations (6) as equations of motion of a point that is moving in $r - V_m(t)$ where $[x^a(T), T]$ is the so-called *position* of the moving point in time T. We shall call the vector

$$\frac{\mathrm{d}x^a(T)}{\mathrm{d}T} \quad \text{or} \quad \delta_T \frac{\mathrm{d}x^a(T)}{\mathrm{d}T}$$

the *velocity* vector or the *acceleration* vector respectively of the point moving in time T.

Remark. We may imagine geometrically the motion of a point in $r - V_m(t)$ as a movement of a "very small motorcar", for example on an expanding sphere. The length of the corresponding trajectory that is determined by the relation (7) is the difference of readings on the tachometer of the car at the times T_2, T_1. The length of the trajectory which is a part of the parametric t-curve (the car is "stationary") is zero.

In a rheonomous space $r - V_m(t)$ let be given a vector field, so-called field of *force*, by means of functions

$$p^a = p^a(x^b, t)\,, \quad \{x^b, t\} \in \Omega\,.$$

In our considerations we shall suppose that the motion of every point in $r - V_m(t)$ is described by the system of differential equations

$$M\delta_T \frac{\mathrm{d}x^a}{\mathrm{d}T} = p^a\,, \tag{28}$$

where $M = \text{const.} > 0$ is the so-called *mass* of the moving point. It is easy to transform the system (28) into Cauchy's canonical form. From it there follows immediately:

Theorem. *In a rheonomous Riemannian space a mass point of given initial position and non-zero velocity moves in the field of force uniquely.*

The next theorem follows from (28) and (23).

Theorem. *If no force is acting on a mass point of a rheonomous Reimannian space, i.e. $p^a = 0$, then this point moves along a δ-geodesic with a constant scalar velocity.*

The scalar function

$$E_{\text{kin}} = \tfrac{1}{2} M g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}T} \frac{\mathrm{d}x^b}{\mathrm{d}T} \tag{29}$$

which is defined along the trajectory (6) is called the *kinetic energy*. If we differentiate each side of the equation (29) we obtain the relation:

$$\frac{\mathrm{d}E_{\text{kin}}}{\mathrm{d}T} = \tfrac{1}{2} M G_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}T} \frac{\mathrm{d}x^b}{\mathrm{d}T} + M g_{ab} \left(\mathrm{D}_T \frac{\mathrm{d}x^a}{\mathrm{d}T} \right) \frac{\mathrm{d}x^b}{\mathrm{d}T}. \tag{30}$$

If no force is acting on the mass point the trajectory of which we investigate, then, according to (30), (28) and (22),

$$\frac{\mathrm{d}E_{\text{kin}}}{\mathrm{d}T} = 0 \quad \text{or} \quad E_{\text{kin}} = \text{const.}$$

So the following theorem holds true:

Theorem. *The kinetic energy of a mass point on which no force is acting in $r - V_m(t)$ is constant.*

Similarly, it is possible to generalize further theorems of classical mechanics of the mass point. Let us state without a proof that the equations (28) may be written in the following equivalent form:

$$\frac{\mathrm{d}}{\mathrm{d}T} \frac{\partial E_{\text{kin}}}{\partial \dot{x}^e} - \frac{\partial E_{\text{kin}}}{\partial x^e} = g_{ea} p^a + M \left(G_{ea} - \frac{1}{2} g_{ea} \frac{G_{bc} \dot{x}^b \dot{x}^c}{g_{bc} \dot{x}^b \dot{x}^c} \right) \dot{x}_a .$$

In the case when the rheonomous Riemannian space is stationary the second term on the right-hand side of the preceding equation is zero. So we obtain the well-known Lagrange equation of II. kind.

References

[1] *А. В. Гохман:* Дифференциальная геометрия и классическая динамика систем. 1966 г. Труды геометрического семинара, Том I.

[2] *F. Nožička:* Fundamentální principy mechaniky a jejich ekvivalence. Aplikace matematiky 1959, č. *4*.

[3] *П. К. Рашевский:* Риманова геометрия и тензорный анализ. Москва 1967.

[4] *J. A. Schouten, D. J. Struik:* Einführung in die neueren Methoden der Differential geometrie I., II. Groningen—Batavia 1935, 1939.

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