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Časopis pro pěstování matematiky, roč. 93 (1968), Praha

NOTE ON STABILITY OF A CONTROL SYSTEM

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It is well known (see [2]) that the systems of homogeneous linear ordinary differential equations have following two properties. The uniform asymptotic stability of origin is equivalent to the exponential asymptotic stability and the stability of origin is equivalent to the boundedness of every solution. In this paper we will show that under some assumptions the same can be extended on a control system

$$\dot{x} = f(x, u), \quad u \in \mathscr{U}. \tag{1}$$

Here we mean by $\mathscr{U}$ the set of all measurable functions $u : \langle 0, \infty) \to U$, where U is a subset of an Euclidean space R^m. We use the usual notation denoting a norm, resp. an inner product, in Euclidean space R^m by $\|.\|$, resp. $(., .)$. We say that (1) has a solution $x(t)$ if there is a function $u \in \mathscr{U}$ such that $x(t)$ solve the equation $\dot{x}(t) = f(x(t), u(t))$ in the sense of Carathéodory (see [1]). We will denote such solution by $x(t, u)$.

Let us start with some definitions. We say that zero-solution of (1) is stable if for each $\varepsilon > 0$ there is such $\delta > 0$ that every solution $x(t, u)$, $u \in \mathscr{U}$, of (1) fulfils the implication $\|x(0, u)\| \leqq \delta \Rightarrow \|x(t, u)\| \leqq \varepsilon$ for every $t > 0$. We say that zero-solution of (1) is exponentially asymptotic stable if it exists such $T > 0$ that for every $\delta > 0$, $u \in \mathscr{U}$ the solution $x(t, u)$ of (1) fulfils the implication

$$\|x(t_0, u)\| \leqq \delta \Rightarrow \|x(t, u)\| \leqq \tfrac{1}{2}\delta \quad \text{for every} \quad t \geqq t_0 + T.$$

We repeate one result from [3]: Let $U \subset R^m$ be compact and $f(x, u) : R^n \times U \to R^n$ fulfil the following assumptions:

1) f is continuous in (x, u) on $R^n \times U$.

2) $f(x, U) = \{f(x, u); u \in U\}$ is convex for every $x \in R^n$.

3) $f(x, U)$ is upper-semicontinuous on R^n.

4) It exists such constant C that for every $x \in R^n$ and every $y \in f(x, U)$ it holds $(x, y) \leqq C(1 + \|x\|^2)$.

Then for every compact $A \subset R^n$ the set of all solutions of (1) starting in A and defined on a finite interval is compact in the topology of the uniform convergence.

Lemma 1. *Let $U \subset R^m$ be compact and $f(x, u) : R^n \times U \to R^n$ fulfil the following assumptions:*

1) *f is continuous in x on R^n uniformly on U and continuous in u on U for every fixed $x \in R^n$.*

2) *$f(x, U)$ is convex for every $x \in R^n$.*

3) *f is homogeneous of degree one in x for every $u \in U$.*

4) *the solution $x(t, u)$ of (1) converges to zero as $t \to \infty$ for every $u \in \mathcal{U}$ and every initial condition.*

Then zero-solution of (1) is stable.

Proof. If zero-solution of (1) is not stable then it exists $\varepsilon > 0$ and there are solutions x_k of (1) and $t_k > 0$ such that $\|x_k(0)\| < 1/k$, $\|x_k(t_k)\| > \varepsilon$, $k = 1, 2, \ldots$ For every $k \geqq k_0$, $k_0 = [1 + 1/\varepsilon]$, we can choose $t_{k1}, t_{k2} \in (0, t_k\rangle$ such that $\|x_k(t_{k1})\| = 1/k$, $\|x_k(t_{k2})\| = \varepsilon$, $1/k < \|x_k(t)\| < \varepsilon$, $t \in (t_{k1}, t_{k2})$. Let us put $T_k = t_{k2} - t_{k1}$, $y_k(t) = k \cdot x_k(t + t_{k1})$, $k \geqq k_0$. Then the functions y_k, $k \geqq k_0$, solve (1) and satisfy $\|y_k(0)\| = 1$, $\|y_k(T_k)\| = k\varepsilon$, $1 < \|y_k(t)\| < k\varepsilon$, $t \in (0, T_k)$, $k \geqq k_0$. It holds $\lim_{k \to \infty} T_k = +\infty$. In fact, if we put $M = \sup \{\|f(x, u)\|;\ \|x\| \leqq 1,\ u \in U\}$, then for every solution x of (1) we have $\mathrm{d}\|x\|/\mathrm{d}t \leqq \|\mathrm{d}x/\mathrm{d}t\| = \|f(x, u)\| \leqq M\|x\|$. Hence $\|x(t)\| \leqq \|x(0)\| \exp(Mt)$ and at last

$$k\varepsilon = \|y_k(T_k)\| \leqq \exp(MT_k), \quad k \geqq k_0. \tag{2}$$

We see that y_k, $k \geqq k_0$, together with their first derivatives are uniformly bounded on every finite subinterval of $\langle 0, \infty)$. Therefore there is a subsequence $z_1, z_2, \ldots,$ which is locally uniformly convergent on $\langle 0, \infty)$ to a continuous function z. Evidently $\|z(t)\| \geqq 1$ for every $t \geqq 0$. As f fulfils all assumptions of the mentioned theorem from [3] the function z is a solution of (1). We have got a contradiction and the proof is complete.

Theorem A. *Let the assumptions of Lemma 1 are fulfilled. Then zero-solution of the equation (1) is exponentially asymptotic stable.*

Proof. We have to find such $T > 0$ that for every $\delta > 0$ and every solution x of (1) it holds

$$\|x(t_0)\| \leqq \delta \Rightarrow \|x(t)\| \leqq \tfrac{1}{2}\delta, \quad \text{for every} \quad t \geqq t_0 + T. \tag{4}$$

Let δ be fixed. According to lemma 1 zero-solution of (1) is stable and therefore there is such $\eta > 0$ that for every solution x of (1) it holds

$$\|x(t)\| \leqq \eta \Rightarrow \|x(\tau)\| \leqq \tfrac{1}{2}\delta, \quad \tau \geqq t. \tag{5}$$

Denote by $T(z, u)$, where $\|z\| \leqq \delta, u \in \mathscr{U}$, the minimum of such $\tau \geqq 0$ that $x(0, u) = z$, $\|x(\tau, u)\| = \eta$, where $x(t, u)$ is a solution of (1) corresponding to u. Since $\lim_{t \to \infty} x(t, u) = 0$, the set of those τ is nonempty and $T(z, u) < +\infty$ for every z, u.

To prove (4) it suffices to show that $\sup_{z,u} T(z, u) < +\infty$. Let it be not true. Then there are sequences $T_k \to \infty$, $x_k(t)$, $k = 1, 2, \ldots$, of solutions of (1) such that

$$\|x_k(0)\| \leqq \delta, \ \|x_k(t)\| > \eta \quad \text{for} \quad t \in \langle 0, T_k \rangle\,, \quad k = 1, 2, \ldots$$

Similarly as in the proof of Lemma 1 it can be shown that there is a subsequence z_k, of x_k, $k = 1, 2, \ldots$, which is uniformly convergent on every finite subinterval of $\langle 0, \infty)$ to a solution z of (1). Furthermore, $\|z(t)\| \geqq \eta$ fo every $t \geqq 0$. We have got a contradiction.

Lemma 2. *Let us have functions $y_k : \langle 0, \infty) \to R^n$, $k = 0,1, \ldots, s$. Then it holds*

$$\liminf_{(\varepsilon_1,\ldots,\varepsilon_s)\to 0} \limsup_{t\to+\infty} \|y_0(t) + \sum_{k=1}^{s} \varepsilon_k y_k(t)\| \geqq \limsup_{t\to+\infty} \|y_0(t)\|\,. \tag{6}$$

Proof. Let there be such numbers a_k, $k = 1, 2, \ldots, s$, $a_s = 1$, that

$$\limsup_{t\to+\infty} \|\sum_{k=1}^{s} a_k y_k(t)\| < +\infty\,, \tag{7}$$

then

$$\liminf_{(\varepsilon_1,\ldots,\varepsilon_s)\to 0} \limsup_{t\to+\infty} \|y_0(t) + \sum_{k=1}^{s} \varepsilon_k y_k(t)\| \geqq$$

$$\geqq \liminf_{(\varepsilon_1,\ldots,\varepsilon_s)\to 0} \limsup_{t\to+\infty} \|y_0 + \sum_{k=1}^{s-1} (\varepsilon_k - a_k\varepsilon_s) y_k\| - \limsup_{\varepsilon_s\to 0} \limsup_{t\to+\infty} \|\varepsilon_s \sum_{k=1}^{s} a_k y_k\| =$$

$$= \liminf_{(\varepsilon_1,\ldots,\varepsilon_{s-1})\to 0} \limsup_{t\to+\infty} \|y_0 + \sum_{k=1}^{s-1} \varepsilon_k y_k\|\,.$$

Thus it suffices to prove (6) under the assumption that for any nontrivial linear combination $\sum_{k=1}^{s} a_k y_k$ the inequality (7) is not valid. If (6) is not true then it exists a constant $C > 0$ and a sequence $(\varepsilon_{i1}, \ldots, \varepsilon_{is}) \to 0$, as $i \to \infty$, such that

$$\limsup_{t\to+\infty} \|y_0 + \sum_{1}^{s} \varepsilon_{ik} y_k\| < C\,. \tag{8}$$

We can assume that the matrix

$$A = \begin{pmatrix} \varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1s} \\ \varepsilon_{21}, \varepsilon_{22}, \ldots\ldots \varepsilon_{2s} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{pmatrix},$$

has at least one regular $(s \times s)$ submatrix. Otherwise, if the rank r of the matrix A is less then s, then there is a linearly dependent column of A. Let for example the s-column be a linear combination of remaining columns with coefficients $a_1, \ldots, a_{s-1}$. If we substitute the function y_k by $y_k + a_k y_s$, $k = 1, 2, \ldots, s - 1$, and repeate this procedure $(s - r)$-times we get the former problem with the $((s - r) \times \infty)$ matrix (ε_{ik}) of rank $s - r$.

Assuming that rank $A = s$, we can find a regular $(s \times s)$ submatrix

$$\begin{pmatrix} \varepsilon_{i_1 1}, & \ldots, & \varepsilon_{i_1 s} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \varepsilon_{i_s 1}, & \ldots, & \varepsilon_{i_s s} \end{pmatrix}$$

of A. As the rows of A converge to zero with increasing row-index there is a i_{s+1}-row in A such that matrix

$$B = \begin{pmatrix} 1, & \varepsilon_{i_1 1}, & \varepsilon_{i_1 2}, & \ldots, & \varepsilon_{i_1 s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1, & \varepsilon_{i_s 1}, & \varepsilon_{i_s 2}, & \ldots, & \varepsilon_{i_s s} \\ 1, & \varepsilon_{i_{s+1} 1}, & \varepsilon_{i_{s+1} 2}, & \ldots, & \varepsilon_{i_{s+1} s} \end{pmatrix}$$

is regular. According to (8) it exists $T > 0$ such that the functions $y_0 + \sum_1^s \varepsilon_{i_j k} y_k$, $j = 1, 2, \ldots, s + 1$, are bounded on $\langle T, +\infty)$ and, as B is regular, the functions y_k, $k = 1, 2, \ldots, s$, are bounded on $\langle T, +\infty)$, too. We have got a contradiction and Lemma 2 is proved.

Lemma 2 enables us to weaken assumption 4 of Lemma 1 (and consequently also in Theorem A) in the case when the variable x enters linearly the right hand side of equation (1). We show this exactly in

Lemma 3. *Let $U \subset R^m$ be compact, $V \subset U$, and $F(u) : U \to R^{n^2}$ fulfil the following assumptions*:

1) *F is continuous on U.*

2) *$F(U) = \{F(u); u \in U\}$ is convex.*

3) *V is dense in U (in Euclidean topology).*

4) *For every measurable function $u : \langle 0, \infty) \to V$ and every initial condition the solution $x(t, u)$ of the equation*

$$\dot{x} = F(u)\, x \tag{9}$$

converges to zero as $t \to +\infty$.

Then every solution $x(t, u)$, $u \in \mathscr{U}$, of the equation (9) converges to zero as $t \to +\infty$.

Proof. Let such $u \in \mathscr{U}$ and $x_0 \in R^n$ exist that for the corresponding solution $x(t, u)$, $x(0, u) = x_0$, of (9) an inequality $\limsup_{t\to\infty} \|x(t, u)\| > \sigma > 0$, holds. Then there is $t_1 > 1$ such that $\|x(t_1, u)\| > \sigma$.

Let $\xi_1, \xi_2, \ldots, \xi_n \in R^n$ be linearly independent. We denote by y_q the solution of (9), corresponding to u, for which $y_q(t_1) = \xi_q$, $q = 1, 2, \ldots, n$. If we put $y_0(t) = x(t, u)$, $t \in \langle 0, \infty)$ then according to Lemma 2 there is a neighborhood $G \subset R^n$ of $x(t_1, u)$ such that $\limsup_{t\to\infty} \|z(t)\| > \sigma$ for all solutions $z(t)$ of (9), corresponding to u and fulfilling the initial condition $z(t_1) \in G$.

We can choose a sequence of measurable functions $u_k : \langle 0, t_1\rangle \to V$, $k = 1, 2, \ldots$, $u_k \to u$ uniformly on $\langle 0, t_1\rangle$. Then the solutions $x(t, u_k)$, $x(0, u_k) = x(0, u)$, of (9) converge uniformly on $\langle 0, t_1\rangle$ to $x(t, u)$.

In fact, if we denote $M = \sup \{\|F(u)\|;\ u \in U\}$, then for $t \in \langle 0, t_1\rangle$ we can estimate

$$\|x(t, u_k) - x(t, u)\| = \left\|\int_0^t (F(u_k(t))\, x(t, u_k) - F(u(t))\, x(t, u))\, \mathrm{d}t\right\| \leqq$$

$$\leqq \left\|\int_0^t F(u_k(t))\,(x(t, u_k) - x(t, u))\, \mathrm{d}t\right\| + \left\|\int_0^t (F(u_k(t)) - F(u(t))\, x(t, u)\, \mathrm{d}t\right\| \leqq$$

$$\leqq M \int_0^t \|x(t, u_k) - x(t, u))\|\, \mathrm{d}t + \left\|\int_0^t (F(u_k(t)) - F(u(t)))\, x(t, u)\, \mathrm{d}t\right\|,$$

$$\|x(t, u_k) - x(t, u)\| \leqq (\exp M t_1) \sup_{t\in\langle 0, t_1\rangle} \left\|\int_0^t (F(u_k(t)) - F(u(t)))\, x(t, u)\, \mathrm{d}t\right\|.$$

From the continuity of F immediately follows that

$$\sup_{t\in\langle 0, t_1\rangle} \left\|\int_0^t (F(u_k(t)) - F(u(t)))\, x(t, u)\, \mathrm{d}t\right\| \to 0 \quad \text{as} \quad k \to \infty\,.$$

Hence, there is an integer k_0 such that $x(t_1, u_{k_0}) \in G$ and $\|x(t_1, u_{k_0})\| > \sigma$. Now we define

$$v_1(t) = \begin{Bmatrix} u_{k_0}(t), & t \in \langle 0, t_1\rangle \\ u(t), & t \in (t_1, \infty) \end{Bmatrix}.$$

Then $\limsup_{t\to\infty} \|x(t, v_1)\| > \sigma$, where $x(t, v_1)$ fulfils the initial condition $x(0, v_1) = x(0, u)$.

Further, we find a number $t_2 > \max(2, t_1)$ such that $\|x(t_2, v_1)\| > \sigma$ and repeate the previous procedure. Thus we construct sequences t_k and $v_k : \langle 0, \infty) \to U$,

$k = 1, 2, \ldots, t_k > \max(k, t_{k-1})$, v_k mapping $\langle 0, t_k \rangle$ into V, coinciding with v_{k-1} on $\langle 0, t_{k-1} \rangle$ and with u on (t_k, ∞) so that an inequality $\|x(t_k, v_k)\| > \sigma$ holds for solutions $x(t, v_k)$ of (9), $x(0, v_k) = x(0, u)$, $k = 1, 2, \ldots$

Let us define $v(t) = v_k(t)$, $t \in \langle t_{k-1}, t_k)$, $t_0 = 0$, $k = 1, 2, \ldots$ Then $v : \langle 0, \infty) \to V$ is measurable and inequalitities $\|x(t_k, v)\| > \sigma$, $k = 1, 2, \ldots$, hold for the solution $x(t, v)$ of (9). which fulfils the condition $x(0, v) = x(0, u)$. That contradicts assumption 4.

Theorem B. *Let $U \subset R^m$ be an arbitrary set and $F : U \to R^{n^2}$ be such matrix-function that for every $u \in \mathcal{U}$ and every initial condition $x_0 \in R^n$ the solution $x(t, u)$, $x(0, u) = x_0$, of equation (9) exists and is bounded on the interval $\langle 0, \infty)$. Then zero-solution of equation (9) is stable.*

Proof. If the solution $x \equiv 0$ of (8) is not stable then there are $u_k \in \mathcal{U}$, $t_k > 0$, $k = 1, 2, \ldots$, such that $\|x(0, u_k)\| \leqq 1$, $k = 1, 2, \ldots$, and $\limsup_{k \to \infty} \|x(t_k, u_k)\| = \infty$. We keep the functions u_k, $k = 1, 2, \ldots$, fixed to the end of the proof.

Denote by P the set of all $x \in R^n$ for which there are sequences ϑ_k, τ_k, $k = 1, 2, \ldots$, such that $x(0, u_k(t + \tau_k)) = x$, $k = 1, 2, \ldots$ and $\limsup_{k \to \infty} \|x(\vartheta_k, u_k(t + \tau_k))\| = +\infty$. We prove at first three auxiliary propositions.

Proposition A. $P \neq \emptyset$.

Really, for each $\tau \geqq 0$, $k = 1, 2, \ldots$ denote by $Y_{k,\tau}$ the fundamental matrix-solution of (9), where $u(t) = u_k(t + \tau)$, $t \in \langle 0, \infty)$, for which $Y_{k,\tau}(\tau) = E$ (unit-matrix). Let e_i be the i-column of E, $i = 1, 2, \ldots, n$. Then we can write $x(0, u_k) = \sum_{i=1}^{n} a_{ki} e_i$, $k = 1, 2, \ldots$ As $\|x(0, u_k)\| \leqq 1$, we have $\sum_{i=1}^{n} |a_{ki}| \leqq n$, $k = 1, 2, \ldots$ and

$$\limsup_{k \to \infty} \|x(t_k, u_k)\| = \limsup_{k \to \infty} \| Y_{k,0}(t_k)\, x(0, u_k)\| =$$

$$= \limsup_{k \to \infty} \| \sum_{i=1}^{n} a_{ki} Y_{k,0}(t_k)\, e_i\| \leqq n \sum_{i=1}^{n} \limsup_{k \to \infty} \| Y_{k,0}(t_k)\, e_i\| .$$

That implies $e_i \in P$ at least for one of indices $1, 2, \ldots, n$. Proposition A is proved.

Now, let P_1, resp. P_2, be the set of those e_i, $i = 1, 2, \ldots, n$, which do, resp. do not, belong into P. By renumeration let $e_1, \ldots, e_q \in P_1$, $e_{q+1}, \ldots, e_n \in P_2$ (P_2 may be empty). If there exist such $\alpha_i \in R$, $i = 1, 2, \ldots, q$, that $\sum_{i=1}^{q} \alpha_i e_i \notin P$, then we take the point e_r, where $r = \max\{i = 1, \ldots, q;\ \alpha_i \neq 0\}$, out of P_1 and put the point $\sum_{i=1}^{n} \alpha_i e_i$ into P_2.

We repeate this procedure as many times as all notrivial linear combinations of elements from P_1 belong into P. Let finally, again with a renumeration if necessary, the points which remain in P_1 be $e_1, \ldots, e_s$. We denote the linear hull of P_2 by L and choose an orthonormal basis $e'_{s+1}, \ldots, e'_n$ in L.

Proposition B. $L \cup P = R^n$.

Evidently $e_1, \ldots, e_s, e'_{s+1}, \ldots, e'_n$ form a basis of R^n. Let us take $x \in R^n - L$, then $x = y + z$, where $y = \sum_{i=1}^{s} a_i e_i$, $\sum_{i=1}^{s} |a_i| \neq 0$, $z \in L$. It holds $\sup \{\|Y_{k,\tau}(t)\, z\|;\ t, \tau \geqq 0, k = 1, 2, \ldots\} < +\infty$ and there are sequences k_r, ϑ_r, τ_r, $r = 1, 2, \ldots$, such that $\lim_{r\to\infty} \|Y_{k_r,\tau_r}(\vartheta_r)\, y\| = +\infty$. Hence

$$\lim_{r\to\infty} \|Y_{k_r,\tau_r}(\vartheta_r)\, x\| \geqq \lim_{r\to\infty} \|Y_{k_r,\tau_r}(\vartheta_r)\, y\| - \sup_{r=1,2,\ldots} \|Y_{k_r,\tau_r}(\vartheta_r)\, z\| = +\infty\,.$$

This implies $x \in P$ and Proposition B is proved.

Proposition C. *For every $x \in P$ there are sequences $k_r, T_r, \tau_r, r = 1, 2, \ldots$, such that $x(0, u_{k_r}(t + \tau_r)) = x$, $\lim_{r\to\infty} \|x(T_r, u_{k_r}(t + \tau_r))\| = +\infty$ and $x(T_r, u_{k_r}(t + \tau_r)) \in P$ for $r = 1, 2, \ldots$*

To prove that we fix $x \in P$ and take corresponding sequences $\vartheta_k, \tau_k, k = 1, 2, \ldots$, which occur in the definition of P. If such sequence $k_r, r = 1, 2, \ldots$, exists that $x(\vartheta_{k_r}, u_{k_r}(t + \tau_{k_r})) \in P$, $r = 1, 2, \ldots$, then there is nothing to be proved. Thus we assume the existence of such index k_0 that for every $k \geqq k_0$ we have $x(\vartheta_k, u_k(t + \tau_k)) \in \in L$.

Let us put $t_k = \inf \{\vartheta; \tau \in \langle \vartheta, \vartheta_k \rangle \Rightarrow x(\tau, u(t + \tau_k)) \in L\}$, $k \geqq k_0$. Evidently $x(t_k, u_k(t + \tau_k)) \in L$, $k \geqq k_0$, and we can write $x(t_k, u_k(t + \tau_k)) = \sum_{i=s+1}^{n} a_{ki} e'_i$, $k \geqq k_0$. Denote $M = \sup \{\|Y_{k,\tau}(t)\, e'_i\|;\ t, \tau \geqq 0,\ k \geqq k_0,\ i = s + 1, \ldots, n\}$, then for $k \geqq k_0$ we have

$$\|x(\vartheta_k, u_k(t + \tau_k))\| = \|Y_{k,t_k+\tau_k}(\vartheta_k - t_k)\, x(t_k, u_k(t + \tau_k))\| =$$
$$= \|Y_{k,t_k+\tau_k}(\vartheta_k - t_k) \sum_{i=s+1}^{n} a_{ki} e'_i\| \leqq M \sum_{i=s+1}^{n} |a_{ki}| \leqq nM \|x(t_k, u_k(t + \tau_k))\|\,.$$

Hence $\limsup_{k\to\infty} \|x(t_k, u_k(t + \tau_k))\| = +\infty$.

Now we can choose $T_k < t_k$, $k \geqq k_0$, so that

$$x(T_k, u_k(t + \tau_k)) \in P\,, \quad \|x(T_k, u_k(t + \tau_k))\| > \|x(t_k, u_k(t + \tau_k))\| - \frac{1}{k}\,.$$

Proposition C is proved.

To complete the proof of Theorem B we take $x \in P$. Then according to Proposition C there exist an integer k_1 and positive numbers T_1, τ_1 such that

$$x(0, u_{k_1}(t + \tau_1)) = x\,, \quad x(T_1, u_{k_1}(t + \tau_1)) \in P\,, \quad \|x(T_1, u_{k_1}(t + \tau_1))\| > 1\,.$$

As $x(T_1, u_{k_1}(t + \tau_1)) \in P$ there exist again an integer k_2 and positive numbers T_2, τ_2 such that

$$x(0, u_{k_2}(t + \tau_2)) = x(T_1, u_{k_1}(t + \tau_1)), \quad x(T_2, u_{k_2}(t + \tau_2)) \in P,$$

$\|x(T_2, u_{k_2}(t + \tau_2))\| > 2$. Similarly we construct sequences $k_r, T_r, \tau_r, r = 1, 2, \ldots$

If we now put $u(t) = u_{k_r}(t - \sum_{q=1}^{r-1} T_q + \tau_r)$, for $t \in \langle \sum_{q=1}^{r-1} T_q, \sum_{q=1}^{r} T_q)$, then the solution $x(t, u)$ of equation (9), fulfilling the initial condition $x(0, u) = x$, is not bounded and the proof is complete.

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