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Časopis pro pěstování matematiky, roč. 93 (1968), Praha

ON PERIODIC AND RECURRENT COMPACT GRUPOIDS

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In the paper [1], four problems concerning compact periodic semigroups have been presented. They deal with the formulation of theorems analogous to the theorems on the pointwise periodic mapping of a compact space into itself which have been presented in the same paper. We shall show that the analogy is not incidental and that it is possible to study the periodic semigroups and the pointwise periodic mapping simultaneously as special cases of the compact periodic groupoids. In the present paper, two problems out of four are solved, all consideration and results being formulated by means of the theory of topological groupoids. Cf. e.g. [2].

1. GROUPOIDS

1.1.1 Let G be an arbitrary *groupoid*. If $A \subset G$ and $B \subset G$, then by AB denote the set of all products ab, $a \in A$, $b \in B$. Further denote $A^2 = AA$. By a *subgroupoid*, as it is well known, we mean any non-empty subset $A \subset G$ for which $A^2 \subset A$ holds. If $A^2 = A$ then the subgroupoid A is called *decomposable*. By a *minimal subgroupoid* we understand any subgroupoid not containing any other subgroupoid.

1.1.2 *The product of a non-empty system of subgroupoids of the groupoid G is either the empty set or again a subgroupoid.*

1.1.3 *If A is a subgroupoid of the groupoid G, then A^2 is also a subgroupoid.*

Proof. If $A^2 \subset A$, then $A^2A^2 \subset A^2$.

1.1.4 *Every minimal subgroupoid of the groupoid G is decomposable.*

1.1.5 In this paper $\mathcal{N}$ always means the set of all natural numbers. Define the set $\mathcal{L} = \bigcup_{n=1}^{\infty} L_n$, where L_1 contains only the element 1 from $\mathcal{N}$ and the set L_{n+1} all ordered pairs $\langle\alpha, \beta\rangle$ of elements from $\bigcup_{i=1}^{n} L_i$, one element at least being from L_n. Evidently $L_i \cap L_k = \emptyset$ for $i \neq k$, $i, k \in \mathcal{N}$. The sets L_n are non-empty and finite, hence the set $\mathcal{L}$ is denumerable. The pair $\langle 1, 1\rangle$ let us denote by 2 (cf. [3]).

1.1.6 In the set $\mathcal{L}$ let us define the *addition*. If $\alpha, \beta \in \mathcal{L}$ then put $\alpha + \beta = \langle\alpha, \beta\rangle$. Evidently $\alpha + \beta \in L_n$, $n = \max(i, k) + 1$ if $\alpha \in L_i$ and $\beta \in L_k$. The set $\mathcal{L}$ generates a groupoid with respect to the addition; it is neither associative nor commutative.

1.1.7 Let us further define the *multiplication* in $\mathcal{L}$. Put $\alpha 1 = \alpha$ for $\alpha \in \mathcal{L}$. If $\beta = \langle\beta_1, \beta_2\rangle$, $\beta \in L_{n+1}$ and $\beta_1, \beta_2 \in \bigcup_{i=1}^{n} L_i$, then $\alpha\beta = \alpha\beta_1 + \alpha\beta_2$. Again the set $\mathcal{L}$ with respect to the multiplication generates a groupoid which is neither associative nor commutative.

1.1.8 *The equation* $1\alpha = \alpha = \alpha 1$ *holds for any element* $\alpha \in \mathcal{L}$.

Proof. From 1.1.7 it follows that $\alpha 1 = \alpha$ for all $\alpha \in \mathcal{L}$. By mathematical induction with respect to n we prove that $1\alpha = \alpha$ for every $\alpha \in L_n$. For $n = 1$ the assertion is obvious. Suppose that the assertion holds for all m, $1 \leqq m < n$ and let us prove it for n. If $\alpha \in L_n$ then $\alpha = \langle\alpha_1, \alpha_2\rangle$ where $\alpha_1, \alpha_2 \in \bigcup_{i=1}^{n-1} L_i$. According to the assumption $1\alpha_1 = \alpha_1$ and $1\alpha_2 = \alpha_2$. Hence according to 1.1.7 and 1.1.6 there is $1\alpha = 1\alpha_1 + 1\alpha_2 = \alpha_1 + \alpha_2 = \langle\alpha_1, \alpha_2\rangle = \alpha$.

1.1.9 If x is an arbitrary element of the groupoid G, put $x^1 = x$. For $\alpha \in L_{n+1}$, put $x^\alpha = x^{\alpha_1} x^{\alpha_2}$, where $\alpha = \langle\alpha_1, \alpha_2\rangle$ and $\alpha_1, \alpha_2 \in \bigcup_{i=1}^{n} L_i$.

1.1.10 *For* $\alpha, \beta \in \mathcal{L}$ *and* x *from the groupoid* G *there holds*:

1. $x^\alpha x^\beta = x^{\alpha+\beta}$;
2. $(x^\alpha)^\beta = x^{\alpha\beta}$.

Proof. The first assertion follows from 1.1.6. The proof of the second relation we perform by mathematical induction with respect to n, $\beta \in L_n$. For $n = 1$ the assertion is obvious. Now let it hold for all m, $1 \leqq m < n$; we shall prove it for n. If $\beta \in L_k$ then $\beta = \langle\beta_1, \beta_2\rangle$ where $\beta_1, \beta_2 \in \bigcup_{i=1}^{n-1} L_i$. According to 1.1.9, 1.1.7 and 1.1.6 $(x^\alpha)^\beta = (x^\alpha)^{\beta_1}(x^\alpha)^{\beta_2} = x^{\alpha\beta_1}x^{\alpha\beta_2} = x^{\alpha\beta_1+\alpha\beta_2} = x^{\alpha\beta}$.

1.1.11 Let Γ be the set of all mappings f of the groupoid G into itself such that for all $x \in G$ there is $f(x) = x^\alpha$ for some $\alpha \in \mathcal{L}$.

Γ is a semigroup with a unit element.

Proof. If $f, g \in \Gamma$ then $f(x) = x^\alpha$, $g(x) = x^\beta$ for $\alpha, \beta \in \mathscr{L}$ and $x \in G$. Hence according to 1.1.10 there is $gf(x) = g(x^\alpha) = (x^\alpha)^\beta = x^{\alpha\beta}$, and therefore $gf \in \Gamma$. Further $e \in \Gamma$, $e(x) = x^1 = x$, $x \in G$.

1.2.1 The product of all subgroups of the groupoid G which contain the element x, is called the *cyclic subgroupoid* Γ_x determined by the element x.

The cyclic subgroupoid Γ_x is the set of all elements x^α of the groupoid G for $\alpha \in \mathscr{L}$.

Proof. Let A be the set of all elements x^α of the groupoid G for $\alpha \in \mathscr{L}$. According to 1.1.10 A is a subgroupoid containing x and hence $\Gamma_x \subset A$. Let $x^\alpha \in A$, $\alpha \in L_n$. We prove by mathematical induction with respect to n that $x^\alpha \in \Gamma_x$. If $n = 1$ then $x^1 = x \in \Gamma_x$. Let the assertion hold for all m, $1 \leqq m < n$; we shall prove that it holds for n. If $\alpha \in L_n$ then $\alpha = \langle \alpha_1, \alpha_2 \rangle$ where $\alpha_1, \alpha_2 \in \bigcup_{i=1}^{n-1} L_i$. From 1.1.7 it follows $x^\alpha = x^{\alpha_1} x^{\alpha_2} \in \Gamma_x^2 \subset \Gamma_x$. Hence $A = \Gamma_x$.

1.2.2 An element x of the groupoid G is called *strongly periodic* if the cyclic subgroupoid Γ_x is decomposable.

An element x of the groupoid G is strongly periodic if and only if there exists $\alpha \in \mathscr{L}$ ($\alpha \neq 1$) such that $x = x^\alpha$.

Proof. Let x be a strongly periodic element of the groupoid G; then Γ_x is a decomposable subgroupoid. Therefore $x \in \Gamma_x = \Gamma_x^2$. From 1.2.1 we get $x = x^{\alpha_1} x^{\alpha_2}$ for $\alpha_1, \alpha_2 \in \mathscr{L}$ and thus $x = x^\alpha$, where $\alpha \in \mathscr{L}$ and $\alpha \neq 1$.

If $x = x^\alpha$, $\alpha \in \mathscr{L}$ and $\alpha \neq 1$ then $x = x^\alpha = x^{\alpha_1} x^{\alpha_2} \in \Gamma_x^2$ for $\alpha = \langle \alpha_1, \alpha_2 \rangle$, $\alpha_1, \alpha_2 \in \mathscr{L}$. According to 1.1.3, Γ_x^2 is a subgroupoid containing the element x and hence $\Gamma_x \subset \Gamma_x^2 \subset \Gamma_x$. From here it follows $\Gamma_x = \Gamma_x^2$. The element x is strongly periodic.

1.2.3 A grupoid whose each element is strongly periodic is called a *strongly periodic groupoid.*

1.2.4 Example – F. Let X be an arbitrary non-empty set and f a mapping of this set into itself. Let us define multiplication in X:

$$xy = f(x) \quad \text{for} \quad x, y \in X .$$

Evidently X is a groupoid. Further $\Gamma_x = \{x, f(x), \ldots, f^m(x), \ldots\}$ and $\Gamma_x^2 = \{f(x), f^2(x), \ldots, f^m(x), \ldots\}$. Hence the element x is strongly periodic in the groupoid X if and only if there is a natural number m such that $x = f^m(x)$. According to [4] it means that the mapping f is *periodic at the point* x. If the groupoid X is strongly periodic then the mapping f is *pointwise periodic.*

1.2.5 Example – S. Let S be an arbitrary semigroup (an associative groupoid).

Then $\Gamma_x = \{x, x^2, \ldots, x^m, \ldots\}$ and $\Gamma_x^2 = \{x^2, x, \ldots, x^m, \ldots\}$. Obviously, the element x is in the semigroup S *strongly periodic* if and only if there is a natural number m such that $x = x^{m+1}$. If the groupoid S is strongly periodic then we say that S is a *strongly periodic semigroup.*

1.2.6 Theorem. *The groupoid G is strongly periodic if and only if its every subgroupoid is decomposable.*

Proof. If each subgroupoid of the groupoid G is decomposable then each element x is strongly periodic. Conversely let the groupoid G be strongly periodic. If A is its subgroupoid then $A^2 \subset A$. Let $x \in A$. Then $\Gamma_x \subset A$ and hence $x \in \Gamma_x = \Gamma_x^2 \subset A^2$. Therefore $A = A^2$ and each subgroupoid is decomposable.

1.2.7 Corollary – F. *A mapping f of a non-empty set X into itself is pointwise periodic if and only if for any subset $A \subset X$ there holds*

$$f(A) \subset A \Rightarrow f(A) = A .$$

See Proposition 1 – F in [1].

1.2.8 Corollary – S. *A semigroup S is strongly periodic if and only if its each subsemigroup is decomposable.*

See Proposition 1 – S in [1].

1.3.1 An element x of a groupoid G is called *strongly regular* if the cyclic subgroupoid Γ_x is minimal. By a *strongly regular groupoid* we understand a groupoid whose each element is strongly regular.

1.3.2 *Every strongly regular element of the groupoid G is strongly periodic. Every strongly regular groupoid is strongly periodic.*

Proof follows from 1.1.4.

1.3.3 *An element x of the groupoid G is strongly regulac if and only if to each α from $\mathscr{L}$ there is β in $\mathscr{L}$ such that $x = x^{\alpha\beta}$.*

Proof. If x is a strongly regular element then the subgroupoid Γ_x is minimal. Let $y = x^\alpha$, $\alpha \in \mathscr{L}$. Obviously $y \in \Gamma_x$ and hence $\Gamma_y \subset \Gamma_x$. From here it follows that $\Gamma_y = \Gamma_x$. Therefore $x \in \Gamma_y$ and there exists β in $\mathscr{L}$ such that $x = y^\beta = (x^\alpha)^\beta = x^{\alpha\beta}$.

Conversely let there exist to each $\alpha \in \mathscr{L}$ an element $\beta \in \mathscr{L}$ such that $x = x^{\alpha\beta}$. If $A \subset \Gamma_x$, $A^2 \subset A \neq \emptyset$ then for some α from $\mathscr{L}$ there is $x^\alpha = y \in A$ and hence $\Gamma_y \subset A$. According to the assumption there is β in $\mathscr{L}$ such that $x = x^{\alpha\beta} = (x^\alpha)^\beta = y^\beta$. Therefore $x \in \Gamma_y$ and hence $\Gamma_x \subset \Gamma_y$ which means that $\Gamma_y = A = \Gamma_x$. The subgroupoid Γ_x is minimal.

1.3.4 Note. An element x of the groupoid X (cf. 1.2.4) is strongly regular if and only if it is strongly periodic. An element x of the semigroup S is strongly regular if and only if it is idempotent. The semigroup S is strongly regular if and only if it is a semigroup of idempotents.

1.4.1 A groupoid G is called *palintropic* if for all $x \in G$ and $\alpha, \beta \in \mathscr{L}$ there is $x^{\alpha\beta} = x^{\beta\alpha}$. Cf. [5].

1.4.2 *A groupoid G is palintropic if and only if the semigroup Γ is commutative.*

Proof is obvious.

1.4.3 In a palintropic groupoid G let us define the relation: $x \sim y$ if and only if there exist α, β in $\mathscr{L}$ such that $x^\alpha = x^\beta$. Evidently $x \sim x$ and $x \sim y$ implies $y \sim x$. If $x \sim y$, $y \sim z$ then for $\alpha, \beta_1, \beta_2, \gamma$ from $\mathscr{L}$ there is $x^\alpha = y^{\beta_1}$ and $y^{\beta_2} = z^\gamma$. Using 1.1.10 and 1.4.1 we get $x^{\alpha\beta_2} = (x^\alpha)^{\beta_2} = (y^{\beta_1})^{\beta_2} = y^{\beta_1\beta_2} = y^{\beta_2\beta_1} = (y^{\beta_2})^{\beta_1} = (z^\gamma)^{\beta_1} = z^{\gamma\beta_1}$ which means that $x \sim z$. Hence the relation $\sim$ is an equivalence on G. By G_x denote the equivalence class that contains the element x. Evidently $\Gamma_x \subset G_x$ since $x \sim x^\alpha$ for all $\alpha \in \mathscr{L}$.

1.4.4 *If G is a strongly regular palintropic groupoid then $\Gamma_x = G_x$.*

Proof. If $y \in G_x$ then there are $\alpha, \beta \in \mathscr{L}$ such that $x^\alpha = y^\beta$. According to 1.3.3 there is $\gamma \in \mathscr{L}$ such that $y = y^{\beta\gamma}$ and hence $y = x^{\alpha\gamma}$. Therefore it is $y \in \Gamma_x$ and consequently $G_x \subset \Gamma_x$. According to 1.4.3 there is $G_x = \Gamma_x$.

1.4.5 Let $\alpha \in \mathscr{L}$ and $\emptyset \neq A \subset G$, G being a groupoid. By $A^{[\alpha]}$ denote the set of all elements x^α, $x \in A$.

If G is a palintropic groupoid then for each $x \in G$ and $\alpha \in \mathscr{L}$ there is $G_x^{[\alpha]} \subset G_x$.

Proof follows from the relation $\Gamma_x \subset G_x$ (cf. 1.4.3).

1.4.6 Note. The groupoid X (see 1.2.4) is palintropic. Evidently $x \sim y$ in X if and only if there exist natural numbers n, m such that $f^n(x) = f^m(y)$. Each semigroup is palintropic and $x \sim y$ in the semigroup S if and only if there exist natural numbers n, m such that $x^n = y^m$.

1.5.1 Groupoid G is called *complete* if for all $\alpha \in \mathscr{L}$ there is $G = G^{[\alpha]}$.

1.5.2 *The groupoid G is complete if and only if it holds $f(G) = G$ for each mapping f from the semigroup Γ.*

Proof is obvious.

1.5.3 *Let x be an element of a complete palintropic groupoid G. If G_x is a finite set then the element x is strongly regular.*

Proof. If $\alpha \in \mathscr{L}$ then there is $f \in \Gamma$ such that $f(y) = y^\alpha$ for all $y \in G$. The completeness of the groupoid G implies according to 1.5.2 that $f(G) = G$. Hence there is a sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of G such that $f(x_n) = x_{n-1}$ and $x_0 = x$. Obviously $x_n \in G_x$ and therefore there exist natural numbers r and s $(r < s)$ such that $x_r = x_s$. It holds therefore $x = f^s(x_s) = f^s(x_r) = f^{s-r}(x) = gf(x)$ where $g = f^{s-r-1}$. Hence there is $\beta \in \mathscr{L}$ such that $g(y) = y^\beta$ for all $y \in G$. It holds therefore $x = gf(x) = (x^\alpha)^\beta = x^{\alpha\beta}$ which according to 1.3.3 means that the element x is strongly regular.

1.5.4 *Every palintropic strongly regular groupoid is complete.*

Proof. If $x \in G$ and $\alpha \in \mathscr{L}$ then according to 1.3.3 there is β in $\mathscr{L}$ such that $x = x^{\alpha\beta}$. From 1.4.1 there follows $x = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha = y^\alpha$, $y = x^\beta$.

1.5.5 *Let G be a palintropic groupoid whose all sets G_x are finite The groupoid G is complete if and only if it is strongly regular.*

Proof follows from 1.5.3 and 1.5.4.

1.5.6 *If G is a palintropic complete groupoid then for all $x \in G$ and $\alpha \in \mathscr{L}$ there is $G_x^{[\alpha]} = G_x$.*

Proof follows from 1.4.5 and 1.5.1.

1.5.7 *If G is a complete palintropic groupoid whose all sets G_x are finite then each mapping from the semigroup Γ is one-to-one.*

Proof. If $f(x) = f(y)$ then $x \sim y$ and the rest of the proof follows from 1.5.6.

1.5.8 Note. The groupoid X (see 1.2.4) is complete if and only if it holds $f(X) = X$. The semigroup S is complete if and only if to each its element x and to each natural number n there is y in S such that $y^n = x$. Cf. [6].

2. COMPACT GROUPOIDS

2.1.1 The groupoid G is a *topological groupoid* if it is a Hausdorff topological space and if to any open set U containing the product xy in G there exists an open set V containing x and an open set W containing y so that $VW \subset U$. By $\bar{A}$ let us denote the closure of the subset $A \subset G$. Evidently each subgroupoid of a topological groupoid is a topological groupoid with respect to the relative topology.

2.1.2 *If G is a topological groupoid then any mapping from the semigroup Γ is continuous.*

Proof. If $f \in \Gamma$ fthen there is α in $\mathcal{L}$ such that $f(x) = x^\alpha$ for all $x \in G$. By mathematical induction with respect to n we prove that f is a continuous mapping of the topological space G into itself for any $\alpha \in L_n$. For $n = 1$ we get the identical mapping which is obviously continuous. Let now the mapping $x \to x^\beta$ (where $\beta \in L_m$) be continuous for all m, $1 \leqq m < n$; we shall show that the mapping f: $x \to x^\alpha$ is continuous, too. Since $\alpha \in L_n$, there is $\alpha = \langle \alpha_1, \alpha_2 \rangle$ where $\alpha_1, \alpha_2 \in \bigcup_{i=1}^{n-1} L_i$. According to the assumption the mappings g: $x \to x^{\alpha_1}$ and h: $x \to x^{\alpha_2}$ are continuous. According to 2.1.1 to any open set U containing the element $x^\alpha = x^{\alpha_1}x^{\alpha_2}$ there exists an open set V containing x^{α_1} and an open set W containing x^{α_2} so that $VW \subset U$. The mapping g as well as h is continuous and hence to the open sets V, W respectively there exist open sets V_1, W_1 respectively containing the element x so that $V_1^{[\alpha_1]} \subset V$, $W_1^{[\alpha_2]} \subset W$ respectively. The set $U_1 = V_1 \cap W_1$ is open and contains x. Further it is obviously $U_1^{[\alpha]} \subset V_1^{[\alpha_1]}W_1^{[\alpha_2]} \subset VW \subset U$. Hence the mapping f is continuous.

2.1.3 *Let G be a topological groupoid. If $A \subset G$ and $B \subset G$ then $\bar{A}\bar{B} \subset \overline{(AB)}$.*

Proof. If $x \in \bar{A}\bar{B}$ then $x = ab$, $a \in \bar{A}$ and $b \in \bar{B}$. Let U be an arbitrary open set containing x. Then there exist open sets V and W such that $a \in V$, $b \in W$ and $VW \subset U$. Obviously $V \cap A \neq \emptyset \neq B \cap W$ and hence $\emptyset \neq VW \cap AB \subset U \cap AB$. Consequently $x \in \overline{(AB)}$.

2.1.4 *The closure of a subgroupoid of a topological groupoid is again a subgroupoid.*

Proof. If $A^2 \subset A$ then according to 2.1.3 there is $\bar{A}^2 \subset \overline{(A^2)} \subset \bar{A}$. Cf. Theorem 2.1 in [2].

2.1.5 *If A, B are two compact subsets of a topological groupoid then AB is also a compact set.*

Proof. If A, B are compact sets then according to 8.3.18 in [7] the set $A \times B$ is also a compact set. Let g be a mapping of the set $A \times B$ on AB, $g(a, b) = ab$ for $a \in A$ and $b \in B$. Evidently g is a continuous mapping and hence according to 8.3.15 in [7] AB is compact.

2.1.6 *If A, B are two connected subsets of a topological groupoid then AB is also a connected set.*

Proof. According to 10.1.21 in [7] the set $A \times B$ is a connected set if only A, B are connected. The mapping g (from 2.1.5) is continuous and hence according to 10.1.12 in [7] AB is a connected set.

2.1.7 Note. Hausdorff topological space X (see 1.2.4) is a topological groupoid if and only if the mapping f is continuous. By a *topological semigroup* we shall understand an associative topological groupoid.

2.2.1 *Let x be an element of a palintropic topological groupoid G. Let all the mappings f from the semigroup Γ be open. If G_x is a compact set then there exists an open set U containing the element x and a mapping f from the semigroup Γ such that f is constant on $U \cap G_x$.*

Proof. For $\alpha, \beta \in \mathscr{L}$ put $A_{\alpha,\beta} = \mathscr{E}[y \in G;\ y^\alpha = x^\beta]$. Obviously $G_x = \bigcup_{\alpha,\beta\in\mathscr{L}} A_{\alpha,\beta}$. According to 7.1.20 [7] and 2.1.2 the sets $A_{\alpha,\beta}$ are closed. According to 1.4.5 it is possible to make all these considerations in the topological space G_x. The set $\mathscr{L}$ is denumerable (see 1.1.5) and hence ([4], pp. 53 and 54) there exist α_0, β_0 in $\mathscr{L}$ and an open subset V of the topological space G_x such that $V \subset A_{\alpha_0,\beta_0}$. According to our assumption the set $V^{[\alpha_0]} = \{x^{\beta_0}\}$ is open. Hence the set W of all elements $y \in G$, $y^{\beta_0} = x^{\beta_0}$ is open according to 2.1.2 and $x \in W$. If f is a mapping from the semigroup Γ such that $f(y) = y^{\beta_0}$ for all $y \in G$ then f is constant on W. In the topological space G there is then an open set U, $U \cap G_x = W$.

2.2.2 *Let x be an element of a complete palintropic topological groupoid G. Let all mappings f from the semigroup Γ be open. Then the set G_x is compact if and only if it is finite.*

Proof. If G_x is a finite set then it is obviously compact. Conversely let G_x be a compact set. Then according to 2.2.1 to any element $y \in G_x$ there exists an open set U_y containing the element y and a mapping f_y from the semigroup Γ such that the mapping f_y is constant on $U_y \cap G_x$. In fact, there is $G_x = G_y$. The compactness of the set G_x implies that there exists a finite number of elements $y_i \in G_x$ $(i = 1, 2, \ldots, n)$ so that $G_x \subset \bigcup_{i=1}^{n} U_i$, $U_i = U_{y_i}$. The mappings f_i $(f_i = f_{y_i})$ are constant on $U_i \cap G_x$. From 1.4.2 there follows that the mapping $f = f_1 f_2 \ldots f_n$ is constant on $U_i \cap G_x$ and hence the set $f(G_x)$ is finite. 1.5.6 implies that also the set $G_x = f(G_x)$ is finite since according to 1.1.11 there is $f \in \Gamma$.

2.2.3 Theorem. *If G is a palintropic compact groupoid then the following two properties are equivalent:*

1. *The groupoid G is complete. All mappings f from the semigroup Γ are open. The sets G_x are closed.*

2. *The groupoid G is strongly regular. All cyclic subgroups of G are finite.*

Proof. $1 \Rightarrow 2$. The sets G_x are according to 8.3.1 in [7] compact. The rest follows from 2.2.2, 1.5.3 and 1.4.4.

$2 \Rightarrow 1$. Realizing that (8.3.24 in [7]) each mapping f from the semigroup Γ is homeomorphic, the proof follows from 1.5.4, 1.4.4 and 1.5.7.

2.2.4 Corollary – **F.** *A continuous mapping f of the compact space X into itself is pointwise periodic if and only if the following is true:*

1) *f is an open mapping,*

2) *$f(X) = X$;*

3) *the sets $\mathscr{E}[y \in X; f^n(y) = f^m(x)$ for some natural n, $m]$ are closed for all $x \in X$.*

See Proposition 4 – F in [1].

2.2.5 Corollary – **S.** *A compact semigroup S is a semigroup of idempotents if and only if there holds*

1) *S is a complete semigroup,*

2) *the mapping f_n ($f_n(x) = x^n$ for all $x \in S$ and for all natural numbers n) are open,*

3) *the sets $\mathscr{E}[y \in S;\ y^n = x^m$ for some natural n, $m]$ are closed for all $x \in S$.*

See Problem 4 – S in [1].

2.3.1 An element x of the topological groupoid G is called *recurrent* if $x \in \overline{(\Gamma_x^2)}$.

An element x of the topological groupoid G is recurrent if and only if to any open set U containing x there is $\alpha \in \mathscr{L}$ ($\alpha \neq 1$) such that $x^\alpha \in U$.

A *recurrent groupoid* is a topological groupoid whose each element is recurrent.

2.3.2 Note. An element x of the topological groupoid X (cf. 2.1.7) is recurrent if and only if to each open set U containing x there is a natural number m such that $f^m(x) \in U$. If the groupoid X is recurrent then we say that the mapping f is *recurrent.* Cf. [4].

An element x of the topological semigroup S is *recurrent* if and only if to every open set U containing x there is a natural number m such that $x^{m+1} \in U$. A topological semigroup whose each element is recurrent is called a *recurrent semigroup.* Cf. [1].

2.3.3 *If K is a closed subgroupoid of a compact groupoid then there is no recurrent element in $K - K^2$.*

Proof. 1.1.3 and 2.1.5 implies that K^2 is a closed subgroupoid of the groupoid G. If $x \in K$ and $x \in \overline{(\Gamma_x^2)}$ then $\Gamma_x \subset K$ and hence $\Gamma_x^2 \subset K^2$. Therefore $x \in \overline{(\Gamma_x^2)} \subset \overline{(K^2)} = K^2$.

2.3.4 Theorem. *A compact groupoid G is recurrent if and only if each of its closed subgroupoids is decomposable.*

Proof. Let the compact groupoid G be recurrent. If A is a closed subgroupoid then according to 2.3.3 there is $A^2 = A$.

Let every closed subgroupoid of the compact groupoid G be decomposable. According to 2.1.4 there is $\overline{\Gamma}_x$ a closed subgroupoid and hence 2.1.3 implies $x \in \overline{\Gamma}_x = (\overline{\Gamma}_x)^2 \subset \overline{(\Gamma_x^2)}$.

2.3.5 Corollary — F. *A continuous mapping f of a compact space into itself is recurrent if and only if there holds for any closed subset $A \subset X$*

$$f(A) \subset A \Rightarrow f(A) = A\,.$$

See Proposition 5 — F in [1].

2.3.6 Corollary — S. *A compact semigroup S is recurrent if and only if its each closed subset is decomposable.*

See Proposition 5 — S in [1].

2.4.1 Theorem. *Let e be a recurrent element of a compact groupoid G. If H, K are two closed connected subsets with the properties*

1) $H \cup K = G$,
2) $H \cap K = \{e\}$,
3) $H^2 \cap H \neq \emptyset \neq K \cap K^2$,
4) $H^2 \cap K^2 = \{e^2\}$

then e is an idempotent.

Proof. Property 1) implies that e^2 lies either in H or in K. Suppose that it lies in K. Evidently $e^2 \in H^2$ and hence $H^2 \cap K \neq \emptyset \neq H^2 \cap H$. According to 2.1.6 H^2 is a connected set and hence $e \in H^2$ since e is a dividing point in G. Property 2) implies evidently $e \in K$. If $e \notin K^2$ then $K^2 \subset K$ since K^2 is a connected set (see 2.1.6) and $K \cap K^2 \neq \emptyset$. Obviously K is a closed subgroupoid and $e \in K - K^2$ which is a contradiction since according to 2.3.3 the element e is not recurrent. Hence $e \in K^2$ which means that $e \in H^2 \cap K^2$. According to Property 4, however, $e = e^2$.

2.4.2 Corollary — F. *Let the continuous mapping f of the compact space X into itself be recurrent at the point e. If H, K are two closed connected subsets with the properties*

1) $H \cup K = X$,
2) $H \cap K = \{e\}$,
3) $f(H) \cap H \neq \emptyset \neq K \cap f(K)$,
4) $f(H) \cap f(K) = \{f(e)\}$

then e is a fixed point of the mapping f.

2.4.3 Note. If f is a one-to-one mapping then Property 4, in the assumptions of the Corollary — F can be omitted.

See Proposition 6 — F in [1].

2.4.4 Corollary — S. *Let e be a recurrent element of a semigroup S. If H, K are two closed connected subsets with the properties*

1) $H \cup K = S$,

2) $H \cap K = \{e\}$,

3) $H^2 \cap H \neq \emptyset \neq K \cap K^2$,

4) $H^2 \cap K^2 = \{e^2\}$

then e is an idempotent.

See Problem 6 – S in [1].

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