Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 92 (1967), No. 4, 373--391

Persistent URL: `http://dml.cz/dmlcz/117601`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

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SVAZEK 92 * PRAHA 9. 10. 1967 * ČÍSLO 4

STRONG MAXIMUM PRINCIPLE FOR WEAK SOLUTIONS OF NONLINEAR PARABOLIC DIFFERENTIAL INEQUALITIES

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(Received March 10, 1966)

1. INTRODUCTION

Let $\mathcal{O}$ be a region in E_{n+1}. Let u be a weak solution of the parabolic nonlinear differential inequality (2.1), where the conditions of Section 2 are satisfied. Let $(x_0, t_0) \in \mathcal{O}$ and let us denote $S(x_0, t_0)$ the set of all (x, t) such that $t < t_0$ and there exists a vector function $\varphi(\tau)$, $\varphi \in C^{(1)}(t, t_0)$ where $\varphi(t) = x$, $\varphi(t_0) = x_0$, $(\varphi(\tau), \tau) \in \mathcal{O}$ for $\tau \in \langle t, t_0 \rangle$.

The function u is said to have a maximum (in $S(x_0, t_0)$), near the point $(x_0, t_0) \in \mathcal{O}$, if for any n-dimensional ball K and every $\delta > 0$ such that $(x_0, t_0) \in Q^{(\delta)} = K \times \times \langle t_0 - \delta, t_0 \rangle$, the inequality

$$\operatorname*{supess}_{(x,t)\in Q^{(\delta)}} u(x, t) \geqq \operatorname*{supess}_{(x,t)\in S(x_0,t_0)} u(x, t)$$

holds.

The purpose of this paper is to prove the following two statements:

1. The function u is bounded from above on every compact subset of $\mathcal{O}$.

2. If u has a maximum M near the point $(x_0, t_0) \in \mathcal{O}$ then $u(x, t) = M$ almost everywhere in $S(x_0, t_0)$ (Strong maximum principle of NIRENBERG [3].).

To prove these results, we use the method of DE GIORGI [2]. Using this method, apirori estimates for solutions of nonlinear parabolic equations were obtained by LADYZHENSKAYA and URALTSEVA [6]. For basic results concerning the elliptic equations from this point of view see STAMPACCHIA [4].

This paper is a generalization of Výborný's and author's result for weakly nonlinear parabolic equations [1].

2. DEFINITIONS

2.1. Definition. Let E_n be an Euclidean n-space of variables $x = (x_1, \ldots, x_n)$, and let t be the time-variable. Put $E_{n+1} = E_n \times E_1$, the set of all pairs $P = (x, t)$. Let $\mathcal{O}$ be a domain in E_{n+1}. Let m be the Lebesgue measure in E_{n+1}, μ – the n-dimensional Lebesgue measure and μ_1 – the 1-dimensional Lebesgue measure.

The set of all infinitely differentiable functions in $\mathcal{O}$ with a compact support in $\mathcal{O}$ will be denoted by $\mathcal{D}(\mathcal{O})$.

Let $u \in L_2(\mathcal{O})$. We denote $I(u)$ the interval $(\operatorname{infess}_{P\in\mathcal{O}} u(P), \operatorname{supess}_{P\in\mathcal{O}} u(P))$, $\operatorname{osc}_{\mathcal{O}} u = \operatorname{supess}_{P\in\mathcal{O}} u(P) - \operatorname{infess}_{P\in\mathcal{O}} u(P)$ and put $t = x_{n+1}$. Let $M(u)$ be a set of all $i = (i_1, \ldots \ldots, i_{n+1})$ such that

$$D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \quad (|i| = i_1 + \ldots + i_n)$$

is defined a.e. in $\mathcal{O}$, $\mathbf{q} = (q_j)_{j\in M(u)}$ be a vector. Let us denote $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{e}_\alpha = (\delta_{1\alpha}, \delta_{2\alpha}, \ldots, \delta_{n\alpha}, 0)$ for $\alpha = 1, \ldots, n$. (Here $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $\delta_{\alpha\alpha} = 1$.)

The space of all $u \in L_2(\mathcal{O})$ for which

$$\|u\|_W = \left(\int_{\mathcal{O}} \left(|u|^2 + \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2\right) \mathrm{d}m\right)^{1/2} < +\infty$$

will be denoted by $W_2^{(1,0)}$, briefly W. It is obvious that for $u \in W$ we have $\mathbf{0} \in M(u)$, $\mathbf{e}_\alpha \in M(u)$. Put $\partial u = (D^j u)_{j\in M(u)}$ a.e. in $\mathcal{O}$.

2.2. Definition. Let p be a real variable. In the following we assume all qualities real. Let a_j $(j = 1, \ldots, n)$, c be functions of x, t, p, $\mathbf{q}$ satisfying the following relations for some fixed u $(\mathbf{q} = \partial u)$:

1) $a_j(u - h, \partial u) \in L_2(\mathcal{O})$, $c(u - h, \partial u) \in L_2(\mathcal{O})$ for $h \in I(u)$,

2) $\sum_{j=1}^{n} a_j(p, \mathbf{q})\, q_{\mathbf{e}_j} - c(p, \mathbf{q})\, p \geqq \nu \sum_{j=1}^{n} |q_{\mathbf{e}_j}|^2 - M_1 p^2$ where $\nu > 0$, $0 < p < \operatorname{osc}_{\mathcal{O}} u$, $q_{\mathbf{0}} \in I(u)$,

3) $\sum_{j=1}^{n} |a_j(p, \mathbf{q})| \leqq M_2(\sum_{j=1}^{n} |q_{\mathbf{e}_j}| + p)$ for $0 < p < \operatorname{osc}_{\mathcal{O}} u$, $q_{\mathbf{0}} \in I(u)$,

4) $\sum_{j=1}^{n} (\partial/\partial x_j)(a_j(u - h, \partial u) - a_j(u, \partial u)) + c(u - h, \partial u) - c(u, \partial u) \geqq M_4 \sum_{j=1}^{n} |\partial u/\partial x_j|$ for every $h \in I(u)$ in the sense of distributions.

(Here we do not express the dependence on x and t.)

2.3. Example. Let u be a function with derivatives of the third order continuous in $\mathcal{O}$. Let $c(v, \partial u) \equiv 0$,

$$a_j(v, \partial u) = \left(1 + \frac{u^2 + \dfrac{\partial u}{\partial t} + \left(\dfrac{\partial^3 u}{\partial x_j^3}\right)^3}{1 + u^2 + \left(\dfrac{\partial u}{\partial t}\right)^2 + \left(\dfrac{\partial^3 u}{\partial x_j^3}\right)^4}\right) \frac{\partial u}{\partial x_j} + v \operatorname{arctg} x_j\,.$$

Then conditions 1) to 4) are satisfied for $u > 0$.

2.4. Definition. In the following, let u be a solution of the nonlinear parabolic inequality

$$\frac{\partial u}{\partial t} \leqq \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j(u, \partial u) + c(u, \partial u) + M_3 \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right| \tag{2.1}$$

in the sense of distributions, $u \in W$.

The objective of this paper is to prove the common parabolic strong maximum principle for u. We must slightly modify the formulation of this principle because u is not necessarily a continuous function.

3. PROOF OF THE BASIC INEQUALITY

3.1. Lemma. *Let* $u \in W$; *then* $|u| \in W$ *and* $\| |u| \|_W \leqq \|u\|_W$.

Proof.

$$\left|\frac{\partial |u|}{\partial x_j}\right| \leqq \left|\frac{\partial u}{\partial x_j}\right|$$

a.e. in $\mathcal{O}$.

3.2. Lemma. *Let* $u, v \in W$. *Then* $\max(u, v) \in W$.

Proof. $\max(u, v) = \frac{1}{2}(u + v) + \frac{1}{2}(|u - v|)$.

3.3. Notations. Let us denote R^η the operator of regularization which is positive selfadjoint and for every v with compact support in $\mathcal{O}$ there is a η_0 such that $R^\eta(\partial v/\partial x_j) = (\partial/\partial x_j) R^\eta v$, $R^\eta(\partial v/\partial t) = (\partial/\partial t) R^\eta v$ in $\mathcal{O}$ for $0 < \eta < \eta_0$.

Further, let us denote $\psi(t) = \psi(\delta_1, t_1, t_2, t)$ the continuous function defined for $t_1 < t_2$, $\delta_1 < \frac{1}{2}(t_2 - t_1)$ by

$$\begin{aligned}
\psi(t) &= 0 && \text{for } t \notin (t_1, t_2)\,,\\
\psi(t) &= 1 && \text{for } t \in (t_1 + \delta_1, t_2 - \delta_1),\\
\psi'(t) &= 1/\delta_1 && \text{for } t \in (t_1, t_1 + \delta_1),\\
\psi'(t) &= -1/\delta_1 && \text{for } t \in (t_2 - \delta_1, t_2).
\end{aligned}$$

In a similar manner the function

$$\chi(\delta_2, x_0, r, x) = \psi(\delta_2, -r, r, |x - x_0|)$$

is defined.

If h is a constant we put $v_h = (v)_h = \max(v - h, 0)$.

Let $P_0 = (x_0, t_0)$, $\varrho > 0$, $\sigma > 0$. Then we denote

$$\begin{aligned} K_{\varrho,\sigma}(P_0) &= K_{\varrho,\sigma} = \{P; |x - x_0| < \varrho, t = t_0 - \sigma\}, \\ k_\varrho(x_0) &= k_\varrho = \{x; |x - x_0| < \varrho\}, \\ Q_{\varrho,\sigma}(P_0) &= Q_{\varrho,\sigma} = k_\varrho \times (t_0 - \sigma, t_0), \\ B_{\varrho,\sigma}(P_0) &= \{P; |x - x_0| = \varrho, t_0 - \sigma < t < t_0\}, \\ M^h(u) &= \{P; P \in \mathcal{O}, u(P) > h\} = M^h, \\ \varkappa &= \mu(k_1). \end{aligned}$$

The intersection of M^h and the set $K_{\varrho,\sigma}$, $Q_{\varrho,\sigma}$, $B_{\varrho,\sigma}$ will be denoted by $K^h_{\varrho,\sigma}$, $Q^h_{\varrho,\sigma}$, $B^h_{\varrho,\sigma}$ respectively.

Now we can state the main theorem of this section.

3.4. Theorem. *Let x_0, t_0, ϱ, σ be such that $\bar{Q}_{\varrho,\sigma} \subset \mathcal{O}$. Then there exists a constant $C = C(u)$ such that*

$$\frac{\nu}{2}\int_{Q_{\varrho,\sigma}} |\nabla u_h|^2 \, dm + \tfrac{1}{2}\int_{K_{\varrho,0}} u_h^2 \, d\mu \leqq \tfrac{1}{2}\int_{K_{\varrho,\sigma}} u_h^2 \, d\mu + + C\left(\int_{Q_{\varrho,\sigma}} u_h^2 \, dm + \int_{B_{\varrho,\sigma}} (|\nabla u_h| + u_h)\, u_h \, d\mu\right), \tag{3.1}$$

for a.e. ϱ, t_0, σ and for all $h \in I(u)$, where

$$|\nabla v| = \left(\sum_{j=1}^n \left|\frac{\partial v}{\partial x_j}\right|^2\right)^{1/2}.$$

Proof. Let us denote $\Psi_{\delta_1}(t) = \psi(\delta_1, t_0 - \sigma, t_0, t)$, $X_{\delta_2}(x) = \chi(\delta_2, x_0, \varrho, x)$ and put $\varphi = R^n X_{\delta_2} \Psi_{\delta_1} (R^n u)_h$ in the inequality

$$0 \leqq \int_{\mathcal{O}} \left(u \frac{\partial \varphi}{\partial t} - \sum_{j=1}^n a_j(u, \delta u) \frac{\partial \varphi}{\partial x_j} + c(u, \partial u)\, \varphi + M_3 |\nabla u|\, \varphi\right) dm$$

for $\varphi \geqq 0$, $\varphi \in \mathcal{D}(\mathcal{O})$, which is an other form of (2.1). Thus we obtain (with respect to 2.2 condition 4))

$$\int_{\mathcal{O}} \left(R^n u \frac{\partial}{\partial t} X_{\delta_2} \Psi_{\delta_1} (R^n u)_h - \sum_{j=1}^n R^n a_j(u - h, \partial u) \frac{\partial}{\partial x_j} X_{\delta_2} \Psi_{\delta_1} (R^n u)_h + + [R^n c(u - h, \partial u) + M_5 R^n |\nabla u|]\, X_{\delta_2} \Psi_{\delta_1} (R^n u)_h\right) dm \geqq 0$$

for $\eta < \eta_0$. We denote the integrals in the inequality by I_0, I_j, I_{n+1} respectively. The integration by parts yilds

$$I_0 = - \int_{\mathcal{O}} \frac{\partial}{\partial t} (R^\eta u)\, X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h \, dm = - \int_{\mathcal{O}} \frac{\partial}{\partial t} (R^\eta u)_h \, X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h \, dm =$$

$$= \int_{\mathcal{O}} (R^\eta u)_h^2 \, X_{\delta_2} \frac{\partial \Psi_{\delta_1}}{\partial t} \, dm + \int_{\mathcal{O}} (R^\eta u)_h \, X_{\delta_2} \Psi_{\delta_1} \frac{\partial}{\partial t} (R^\eta u)_h \, dm$$

with respect to properties of R^η. Thus

$$I_0 = \frac{1}{2} \int_{\mathcal{O}} (R^\eta u)_h^2 \, X_{\delta_2} \frac{\partial \Psi_{\delta_1}}{\partial t} \, dm$$

and

$$\lim_{\eta \to 0} I_0 = \frac{1}{2} \int_{\mathcal{O}} u_h^2 X_{\delta_2} \frac{\partial \Psi_{\delta_1}}{\partial t} \, dm \, .$$

Further,

$$\left\| \frac{\partial}{\partial x_j} (X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h) \right\|_{L_2(\mathcal{O})} \leqq C(\delta_1, \delta_2) \left(\left\| \frac{\partial}{\partial x_j} (R^\eta u) \right\|_{L_2(\mathcal{O})} + \| R^\eta u \|_{L_2(\mathcal{O})} + 1 \right) \leqq$$

$$\leqq C(\delta_1, \delta_2, u)$$

and

$$\lim_{\eta \to 0} \int_{\mathcal{O}} \frac{\partial}{\partial x_j} (X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h) \, \varphi \, dm = - \lim_{\eta \to 0} \int_{\mathcal{O}} X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h \frac{\partial \varphi}{\partial x_j} \, dm =$$

$$= - \int_{\mathcal{O}} X_{\delta_2} \Psi_{\delta_1} u_h \frac{\partial \varphi}{\partial x_j} \, dm = \int_{\mathcal{O}} \frac{\partial}{\partial x_j} (X_{\delta_2} \Psi_{\delta_1} u_h) \, \varphi \, dm$$

for $\varphi \in \mathcal{D}(\mathcal{O})$. Thus we have $(\partial/\partial x_j)(X_{\delta_2} \Psi_{\delta_1} (R^\eta u)_h \to (\partial/\partial x_j)(X_{\delta_2} \Psi_{\delta_1} u_h)$ weakly in $L_2(\mathcal{O})$. Further, $R^\eta a_j(u - h, \partial u) \to a_j(u - h, \partial u)$ in $L_2(\mathcal{O})$ and consequently

$$\lim_{\eta \to 0} I_j = \int_{\mathcal{O}} a_j(u - h, \partial u) \frac{\partial}{\partial x_j} (\Psi_{\delta_1} X_{\delta_2} (u)_h) \, dm \, .$$

In a similar way can find $\lim_{\eta \to 0} I_{n+1}$. We have

$$\frac{1}{2} \int_{\mathcal{O}} (u_h^2 X_{\delta_2}) \frac{\partial \Psi_{\delta_1}}{\partial t} \, dm - \int_{\mathcal{O}} \sum_{j=1}^{n} a_j(u - h, \partial u) \frac{\partial}{\partial x_j} (\Psi_{\delta_1} X_{\delta_2} u_h) \, dm +$$

$$+ \int_{\mathcal{O}} [c(u - h, \partial u) + M_3 |\nabla u|] \, X_{\delta_2} \Psi_{\delta_1} u_h \, dm \geqq 0 \, .$$

Further,

$$\int_{\mathcal{O}} a_j(u - h, \partial u) \frac{\partial \Psi_{\delta_1}}{\partial x_j} X_{\delta_2} u_h \, dm =$$

$$= - \int_{\varrho - \delta_2 < |x - x_0| < \varrho} a_j(u - h, \partial u) \Psi_{\delta_1} \frac{1}{\delta_2} \frac{x_j - x_0}{|x - x_0|} u_h \, dm +$$

$$+ \int_{\mathcal{O}} a_j(u - h, \partial u) \Psi_{\delta_1} X_{\delta_2} \frac{\partial u_h}{\partial x_j} \, dm .$$

Letting $\delta_1 \to 0$, $\delta_2 \to 0$ we obtain

$$\frac{1}{2} \int_{K_{\varrho,0}} u_h^2 \, d\mu - \frac{1}{2} \int_{K_{\varrho,\sigma}} u_h^2 \, d\mu - \sum_{j=1}^{n} \int_{Q_{\varrho,\sigma}} a_j(u - h, \partial u) \frac{\partial u_h}{\partial x_j} \, dm +$$

$$+ \int_{B_{\varrho,\sigma}} v_j a_j(u - h, \partial u) u_h \, d\mu + \int_{Q_{\varrho,\sigma}} [c(u - h, \partial u) + M_3 |\nabla u|] u_h \, dm \geqq 0$$

for a.e. ϱ and a.e. t_0 and σ. We put $v_j = (x_j - x_0)/|x - x_0|$. Now we can use properties 2) and 3) of 2.2 and obtain

$$\text{(3.2)} \quad \nu \int_{Q_{\varrho,\sigma}} |\nabla u_h|^2 \, dm - M_1 \int_{Q_{\varrho,\sigma}} u_h^2 \, dm \leqq \sum_{j=1}^{n} \int_{Q_{\varrho,\sigma}} a_j(u - h, \partial u) \frac{\partial u_h}{\partial x_j} \, dm -$$

$$- \int_{Q_{\varrho,\sigma}} c(u - h, \partial u) u_h \, dm \leqq \frac{1}{2} \int_{K_{\varrho,0}} u_h^2 \, d\mu - \frac{1}{2} \int_{K_{\varrho,\sigma}} u_h^2 \, d\mu +$$

$$+ C \int_{B_{\varrho,\sigma}} (|\nabla u_h| + u_h) u_h \, d\mu + M_3 \int_{Q_{\varrho,\sigma}} |\nabla u_h| u_h \, dm .$$

Next,

$$\text{(3.3)} \qquad M_3 \int_{Q_{\varrho,\sigma}} |\nabla u_h| u_h \, dm \leqq \frac{\nu}{4} \int_{Q_{\varrho,\sigma}} |\nabla u_h|^2 + \frac{M_3^2}{\nu} \int_{Q_{\varrho,\sigma}} u_h^2 \, dm$$

and by (3.2) with respect to (3.3) we obtain the assertion of our theorem.

3.5. Lemma (Caccioppoli). Let $\alpha > 0$ and

$$\alpha \int_0^{\varrho} \omega^2(r) \, dr \leqq \beta + \omega(\varrho) \varphi(\varrho)$$

for $\varrho_1 < \varrho < \varrho_2$. Then

$$\alpha \int_0^{\varrho_1} \omega(r) \, dr \leqq \beta + \alpha^{-1} (\varrho_2 - \varrho_1)^{-2} \int_{\varrho_1}^{\varrho_2} \varphi^2(r) \, dr .$$

Proof. See [2].

3.6. Lemma. *Let $v \in W_q^{(1)}(k_\varrho)$ where $q \geqq 1$. Put $N = \{X; v(x) = 0\}$, $M = k_\varrho - N$, $e \subset k_\varrho$. Then there exists a constant C such that*

$$(3.4) \qquad \left(\int_e |v|^q \, d\mu\right)^{1/q} \leqq \frac{c\varrho^n}{\mu(N)} (\mu(e))^{1/pn} (\mu(M))^{1/qn} \left(\int_M |\nabla v|^q \, d\mu\right)^{1/q},$$

where $1/p = 1 - 1/q$.

Proof. See [5].

4. BASIC THEOREM

4.1. Theorem. *Let $\delta\varrho > 0$, $\delta\sigma > 0$, $\overline{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} \subset \mathcal{O}$. Then*

$$(4.1) \quad \int_{Q_{\varrho,\sigma}} |\nabla u_h|^2 \, dm + \int_{K_{\varrho,0}} u_h^2 \, d\mu \leqq C\left[\frac{1}{(\delta\varrho)^2} + \frac{1}{\delta\sigma} + 1\right] \int_{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} u_h^2 \, dm .$$

Proof. Put $\alpha = \nu/2$, $\omega(r) = (\int_{B_{r,\sigma}} (|\nabla u_h|^2 + u_h^2) \, d\mu)^{1/2}$; then $\int_0^\varrho \omega^2(r) \, dr = \int_{Q_{\varrho,\sigma}} (|\nabla u_h|^2 + u_h^2) \, dm$. Further, put $\varphi(r) = C(\int_{B_{r,\sigma}} u_h^2 \, d\mu)^{1/2}$

$$\beta = \frac{1}{2} \int_{K_{\varrho+\delta\varrho,\sigma}} u_h^2 \, d\mu - \frac{1}{2} \int_{K_{\varrho,0}} u_h^2 \, d\mu + C \int_{Q_{\varrho+\delta\varrho,\sigma}} u_h^2 \, dm .$$

We have by Theorem 3.4,

$$\frac{\nu}{2} \int_0^R \omega^2(r) \, dr \leqq \beta + \omega(R) \, \varphi(R) \, (\varrho < R < \varrho + \delta\varrho)$$

and consequently,

$$(4.2) \qquad \int_0^\varrho \omega^2(r) \, dr \leqq C\left[\int_{Q_{\varrho+\delta\varrho,\sigma}} u_h^2 \, dm + \frac{1}{(\delta\varrho)^2} \int_{Q_{\varrho+\delta\varrho,\sigma}} u_h^2 \, dm\right] + \frac{1}{2} \int_{K_{\varrho+\delta\varrho,\sigma}} u_h^2 \, d\mu - \frac{1}{2} \int_{K_{\varrho,0}} u_h^2 \, d\mu .$$

Integrating by σ over $(\sigma, \sigma + \delta\sigma)$ we obtain (4.1).

4.2. Theorem. *Let $\overline{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} \subset \mathcal{O}$, $0 < \varepsilon < 1$. Put*

$$A^h = \frac{m(Q^h_{\varrho+\delta\varrho,\sigma+\delta\sigma})}{\mu(k_1) \, \varrho^n \sigma} .$$

(Here $\mu(k_1) \, \varrho^n \sigma = m(Q_{\varrho,\sigma})$.) Let

$$(4.3) \qquad A^h < \varepsilon , \quad A^h < \left(\frac{\delta\sigma}{4\sigma}\right)^{1+n/2} .$$

Then

$$(4.4)\quad \int_{Q_{\varrho,\sigma}} u_h^2 \,\mathrm{d}m \leqq C(A^h)^{2/(n+2)}(\varrho^2+\sigma)\left(\frac{1}{(\delta\varrho)^2}+\frac{1}{\delta\sigma}+1\right)\int_{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} u_h^2 \,\mathrm{d}m\,.$$

Proof. Let us denote $\alpha = \mu(k_1)\,\varrho^n(A^h)^{n/(n+2)}$, $\beta = \sigma(A^h)^{2/(n+2)}$. It is obvious that $\alpha\beta = m(Q^h_{\varrho+\delta\varrho,\sigma+\delta\sigma})$ and, by (4.3), $\beta < \delta\sigma/4$, $\alpha < \mu(k_1)\,\varrho^n\varepsilon^{n/(n+2)} < \mu(k_1)\,\varrho^n$. Further put $M = \{t;\, t_0 - \sigma - \delta\sigma < t < t_0,\, \mu(K^h_{\varrho,t}) > \alpha\}$.

If $\mu_1(M) > \beta$ then

$$\alpha\beta = m(Q^h_{\varrho+\delta\varrho,\sigma+\delta\sigma}) = \int_0^{\sigma+\delta\sigma} \mu(K^h_{\varrho+\delta\varrho,t})\,\mathrm{d}t \geqq \int_M \mu(K^h_{\varrho,t})\,\mathrm{d}t > \alpha\beta$$

which is not possible. Hence, $\mu_1(M) \leqq \beta$.

By Theorem 3.4 we have

$$\int_{K_{r,0}(x_0,t)} u_h^2\,\mathrm{d}\mu \leqq \int_{K_{r,s}(x_0,t)} u_h^2\,\mathrm{d}\mu + C\int_{Q_{r,s}(x_0,t)} u_h^2\,\mathrm{d}m + C\int_{B_{r,s}(x_0,t)} (|\nabla u_h| + u_h)\,u_h\,\mathrm{d}\mu$$

for a.e. $t_0 - \sigma < t < t_0$, $0 < s < \delta\sigma$, $\varrho < r < \varrho + \delta\varrho$. Let us take a fixed t. If $t - s \notin M$ then $\mu(K^h_{r,s}(x_0, t)) < \alpha$ and, by Lemma 3.6,

$$\int_{K_{r,0}(x_0,t)} u_h^2\,\mathrm{d}\mu \leqq C(\varepsilon)\,[\mu(K^h_{r,s}(x_0, t))]^{2/n}\int_{K_{r,s}(x_0,t)} |\nabla u_h|^2\,\mathrm{d}\mu$$

and $\mu(K^h_{r,s}(x_0, t)) \leqq \alpha$.

We obtain

$$\int_{K_{r,0}(x_0,t)} u_h^2\,\mathrm{d}\mu \leqq C\alpha^{2/n}\int_{K_{r,s}(x_0,t)} |\nabla u_h|^2\,\mathrm{d}\mu +$$
$$+ C\int_{Q_{r,2\beta}(x_0,t)} u_h^2\,\mathrm{d}m + C\int_{B_{r,2\beta}(x_0,t)} (|\nabla u_h| + u_h)\,u_h\,\mathrm{d}\mu$$

for $t - s \notin M$, $0 < s < 2\beta$.

Integrating by s over the set $0 < s < 2\beta$, $t - s \notin M$ (the measure of this set is less or equal 2β and greater then β because $\mu_1(M) \leqq \beta$) we get

$$(4.5)\quad \beta\int_{K_{r,0}(x_0,t)} u_h^2\,\mathrm{d}\mu \leqq C\alpha^{2/n}\int_{Q_{r,2\beta}(x_0,t)} |\nabla u_h|^2\,\mathrm{d}m +$$
$$+ C\beta\int_{Q_{r,2\beta}(x_0,t)} u_h^2\,\mathrm{d}m + C\beta\int_{B_{r,2\beta}(x_0,t)} (|\nabla u_h| + u_h)\,u_h\,\mathrm{d}\mu\,.$$

Further $\alpha^{2/n} = (\mu(k_1))^{2/n} \varrho^2 \sigma^{-1} \beta$ and, by (4.5), integrating by $t \in (t_0 - \sigma, t_0)$ we obtain

$$\int_{Q_{\varrho,\sigma}} u_h^2 \, dm \leqq \int_{Q_{r,\sigma}(x_0,t_0)} u_h^2 \, dm \leqq C \int_{t_0-\sigma}^{t_0} \left(\frac{\varrho^2}{\sigma} \int_{Q_{r,2\beta}(x_0,t)} |\nabla u_h|^2 \, dm + \right.$$
$$\left. + \int_{Q_{r,2\beta}(x_0,t)} u_h^2 \, dm + \int_{B_{r,2\beta}(x_0,t)} (|\nabla u_h| + u_h) \, u_h \, d\mu \right) dt \leqq$$
$$\leqq C\beta \left(\frac{\varrho^2}{\sigma} \int_{Q_{r,\sigma+\frac{1}{2}\delta\sigma}(x_0,t_0)} |\nabla u_h|^2 \, dm + \int_{Q_{r,\sigma+\frac{1}{2}\delta\sigma}(x_0,t_0)} u_h^2 \, dm + \right.$$
$$\left. + \int_{B_{r,\sigma+\frac{1}{2}\delta\sigma}(x_0,t_0)} (|\nabla u_h| + u_h) \, u_h \, d\mu \right)$$

for $\varrho < r < \varrho + \frac{1}{2}\delta\varrho$ because $2\beta < \delta\sigma/2$.

Now, let us integrate the latter inequality by r over $(\varrho, \varrho + \frac{1}{2}\delta\varrho)$. Using the inequality of Hölder and Theorem 4.1 we obtain

$$\int_{Q_{\varrho,\sigma}} u_h^2 \, dm \leqq C\beta \left(\frac{\varrho^2}{\sigma} \left(1 + \frac{1}{(\delta\varrho)^2} + \frac{1}{\delta\sigma} \right) + 1 + \frac{1}{\delta\varrho} + \right.$$
$$\left. + \frac{1}{\delta\varrho} \left(1 + \frac{1}{(\delta\varrho)^2} + \frac{1}{\delta\sigma} \right)^{1/2} \right) \int_{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} u_h^2 \, dm \leqq C(A^h)^{2/(n+2)} (\varrho^2 + \sigma) \, .$$
$$. \left(1 + \frac{1}{(\delta\varrho)^2} + \frac{1}{\delta\sigma} \right) \int_{Q_{\varrho+\delta\varrho,\sigma+\delta\sigma}} u_h^2 \, dm \, .$$

5. BOUNDS FOR A SOLUTION

5.1. Theorem. *Let $\varrho > 0$, $\sigma > 0$, $0 < \nu < 1$. Then there is a $\Gamma = \Gamma(\nu, \sigma/\varrho^2)$ such that*

$$\text{(5.1)} \qquad \operatorname*{supess}_{P \in Q_{\nu\varrho,\nu^2\sigma}} u(P) \leqq h + \Gamma^{-1/2} \varrho^{-n/2} \sigma^{-1/2} \left(\int_{Q^h_{\varrho,\sigma}} |u - h| \right)^2 dm^{1/2}$$

for $\bar{Q}_{\varrho,\sigma} \subset \mathcal{O}$, $h \in I(u)$ such that

$$m(Q^h_{\varrho\sigma}) < \Gamma \varrho^n \sigma \, .$$

Proof. The number Γ will be chosen later in the proof. Put

$$\lambda = \Gamma^{-1/2} \varrho^{-n/2} \sigma^{-1/2} w_0^{1/2} \, ,$$

where $w_0 = \int_{Q^h_{\varrho,\sigma}} |u - h|^2 \, dm$, $\lambda_m = h + \lambda - 2^{-m}\lambda$, $\varrho_m = \nu\varrho + (1 - \nu) \, 2^{-m}\varrho$, $\sigma_m = K\varrho_m^2$, $K = \sigma\varrho^{-2}$.

Let us put $\varrho = \varrho_{m+1}$, $\sigma = \sigma_{m+1}$, $\delta\varrho = \varrho_m - \varrho_{m+1}$, $\delta\sigma = \sigma_m \smallsetminus \sigma_{m+1}$ in Theorem 4.2. Obviously $\delta\varrho = (1 - \nu)\,\varrho/2^{m+1}$,

$$\delta\sigma = \frac{\sigma}{\varrho^2}(\varrho_m + \varrho_{m+1})\,\delta\varrho = \frac{(1-\nu)}{2^{m+1}}\left(2\nu + \frac{3(1-\nu)}{2^{m+1}}\right)\sigma \geqq \frac{C(\nu)\,\sigma}{2^m}$$

and, by Theorem 4.2,

$$(5.2) \qquad w_{m+1} = \int_{Q_{\varrho_{m+1},\sigma_{m+1}}} u^2_{\lambda_{m+1}}\,\mathrm{d}m \leqq \int_{Q_{\varrho_{m+1},\sigma_{m+1}}} u^2_{\lambda_m}\,\mathrm{d}m \leqq$$

$$\leqq C(\varrho^n\sigma)^{-2/(n+2)}\,(m(Q^{\lambda_m}_{\varrho_m,\sigma_m}))^{2/(n+2)}\,(\varrho^2 + \sigma)\left(\frac{1}{\varrho^2} + \frac{1}{\sigma} + 1\right)2^{2m}w_m\,.$$

We must suppose that

$$(5.3) \qquad a_m = m(Q^{\lambda_m}_{\varrho_m,\sigma_m}) \leqq \left(\frac{C(\nu)}{4\,.\,2^m}\right)^{1+n/2}\mu(k_1)\,\varrho^n\sigma\,,$$

$$(5.4) \qquad a_m \leqq \varepsilon\mu(k_1)\,\varrho^n\sigma\nu^{n+2}\,, \quad \text{i.e.} \quad \Gamma \leqq \varepsilon\nu^{n+2}\,\mu(k_1)\,.$$

If (5.3), (5.4) hold, then, by (5.2)

$$(5.5) \qquad w_{m+1} \leqq C(K + K^{-1})\,(\varrho^n\sigma)^{-2/(n+2)}\,a_m^{2/n+2}2^{2m}w_m$$

if

$$(5.6) \qquad a_m \leqq C_0 2^{-m(1+n/2)}\varrho^n\sigma\,,$$

where $C_0 = (\frac{1}{4}C(\nu))^{1+n/2}\,\mu(k_1)$.

It is obvious that

$$\int_{Q^{\lambda_m}\varrho_m\sigma_m} \frac{\lambda^2}{2^{2(m+1)}}\,\mathrm{d}m \leqq \int_{Q_{\varrho_m,\sigma_m}} u^2_{\lambda_m}\,\mathrm{d}m$$

for all m, so that

$$(5.7) \qquad a_m\lambda^2 2^{-2(m+1)} \leqq w_m$$

where $\lambda^2 = \varrho^{-n}\sigma^{-1}w_0\Gamma^{-1}$.

Thus, we have

$$(5.8) \qquad w_{m+1} \leqq C_1\left(\frac{\Gamma}{w_0}\right)^{2/(n+2)} 2^{m(2+4/(n+2))}\,w_{m-1}^{1+2/(n+2)}$$

for (5.6).

Let us take Γ so small that (5.6) is necessarily true for $m = 1$ and

$$\Gamma \leqq C_0 2^{-4}\,, \quad \Gamma \leqq (C_1^{-1}2^{-(n+4)(2+2/(n+2))})^{(n+2)/2}\,.$$

Put $\omega = n + 4$. Now we prove by induction that

1) (5.6) holds for m odd,

2) if m is even then

$$w_m \leqq w_0 2^{-m\omega} . \tag{5.9}$$

a) We have (5.6) for $m = 1$ and (5.9) for $m = 0$.

b) Let us have (5.6) for m and (5.9) for $m - 1$.

Then by (5.7)

$$a_{m+2} \leqq a_{m+1} \leqq 2^{-\omega(m-1)} w_0 \lambda^{-2} 2^{2(m+1)} =$$
$$= \Gamma \varrho^n \sigma 2^{\omega+2} 2^{m(2-\omega)} \leqq C_0 2^{(m+2)(1+n/2)} \varrho^n \sigma$$

and so we have (5.6) for $m + 2$. Now, we can use (5.8) and write

$$w_{m+1} \leqq C_1 \left(\frac{\Gamma}{w_0}\right)^{2/(n+2)} 2^{m(2+4/(n+2))} 2^{-\omega m(1+2/(n+2))}$$
$$2^{\omega(1+2/(n+2))} w_0^{1+2/(n+2)} \leqq 2^{-\omega(m+1)} w_0$$

and consequently, (5.9) for $m + 1$.

Now $0 \leqq \int_{Q_{\nu\varrho,\nu^2\sigma}} u^2_{h+\lambda} \leqq w_m \to 0$, i.e. $u_{h+\lambda} = 0$ a.e. in $Q_{\nu\varrho,\nu^2\sigma}$.

5.2. Theorem. *The solution of* (2.1) *is locally bounded from above.*

Proof. This theorem follows immediately from Theorem 5.1.

5.3. Consequence. *If* $m(Q^h_{\varrho,\sigma}) \leqq \frac{1}{4}\Gamma\varrho^n\sigma$ *and* $M = \operatorname{supess}_{P\in Q_{\varrho,\sigma}} u(P) > h$, *then*

$$\operatorname{supess}_{P\in Q_{\nu\varrho,\nu^2\sigma}} u(P) \leqq \frac{h+M}{2} < M . \tag{5.10}$$

Proof. We use the estimate

$$\left(\int_{Q^h{}_{\varrho,\sigma}} |u - h|^2 \, dm\right)^{1/2} \leqq \frac{M-h}{2} \Gamma^{1/2} \varrho^{n/2} \sigma^{1/2}$$

and Theorem 5.1.

5.4. Consequence. *Let* $m(Q^h_{\varrho,\sigma}) \to c$ *as* $h \to M$ *where* $c < \frac{1}{4}\Gamma\varrho^n\sigma$. *Then* $\operatorname{supess}_{P\in Q_{\nu\varrho,\nu^2\sigma}} u(P) <$ $< M$.

5.5. Definition. A function u is said to have its maximum on a set of measure zero on $Q_{\varrho,\sigma}$ if $m(Q^h_{\varrho,\sigma}) \to 0$ as $h \to M$.

5.6. Remark. Let $\bar{Q}_{\varrho,\sigma} \subset \mathcal{O}$. We can change u on a set of measure zero so that $u(P) \leqq \operatorname{supess}_{P \in Q_{\varrho,\sigma}} u(P)$ for every $P \in Q_{\varrho,\sigma}$. Further, this fact will be supposed.

5.7. Lemma. *Let $\bar{Q}_{\varrho,\sigma} \in \mathcal{O}$, $\varrho < 1$. Then to every $\varepsilon_0, \varepsilon_1$, $0 < \varepsilon_0 < \varepsilon_1 < 1$ there exists a ν_0, $0 < \nu_0 < 1$ such that to every ν, $\nu_0 < \nu < 1$ there exists $K_0(\varepsilon_1, \varepsilon_0, \nu) = K_0$ such that if $\sigma \leqq K_0 \varrho^2$,*

$$\int_{K_{\varrho,\sigma}} u_h^2 \, \mathrm{d}\mu \leqq \varepsilon_0 \mu(k_1) \, \varrho^n (M - h)^2 \tag{5.11}$$

and (4.2) is satisfied for t_0, σ and $\delta\varrho \sim (1 - \nu)\,\varrho$, $\varrho\nu \sim \varrho$, then

$$\int_{K_{\nu\varrho,0}} u_h^2 \, \mathrm{d}\mu \leqq \varepsilon_1 \mu(k_1) \, (\nu\varrho)^n \, (M - h)^2 \, . \tag{5.12}$$

Proof. By (4.2)

$$\int_{K_{\nu\varrho,0}} u_h^2 \, \mathrm{d}\mu \leqq \int_{K_{\varrho,\sigma}} u_h^2 \, \mathrm{d}\mu + C_2 \, \mu(k_1) \left(1 + \frac{1}{(1 - \nu)^2 \, \varrho^2}\right) \varrho^n \sigma (M - h)^2 \, ,$$

since $\int_0^\varrho \omega^2(r) \, \mathrm{d}r \geqq 0$. Thus, by (5.11),

$$\int_{K_{\nu\varrho,0}} u_h^2 \, \mathrm{d}\mu \leqq \mu(k_1) \, \varrho^n (M - h)^2 \left[\varepsilon_0 + \sigma C_2 \left(1 + \frac{1}{(1 - \nu)^2 \, \varrho^2}\right)\right] =$$

$$= \mu(k_1) \, (\varrho\nu)^n \, (M - h)^2 \, \varepsilon_1^* \, ,$$

where

$$\varepsilon_1^* = \frac{1}{\nu^n} \left[\varepsilon_0 + \sigma C_0 \left(1 + \frac{1}{(1 - \nu)^2 \, \varrho^2}\right)\right] \leqq \frac{1}{\nu^n} \left[\varepsilon_0 + \frac{2C_0}{(1 - \nu)^2} \frac{\sigma}{\varrho^2}\right].$$

Now, it is sufficient to choose ν_0, K_0 so that

$$\frac{1}{\nu_0^n} \left[\varepsilon_0 + \frac{2C_0}{(1 - \nu)^2} K_0\right] \leqq \varepsilon_1 \, .$$

5.8. Remark. If $\varepsilon_0 = 1/2$, $\varepsilon_1 = 8/9$ we can take $\nu_0 = (3/4)^{1/n}$, $K_0 = (1 - \nu)^2/12C_0$.

5.9. Lemma. *Let*

$$\int_{K_{\nu\varrho,0}} u_h^2 \, \mathrm{d}\mu \leqq \varepsilon_1 \, \mu(k_1) \, (\nu\varrho)^n \, (M - h)^2 \, , \tag{5.13}$$

$$1 > a > \sqrt{\varepsilon_1} \, , \quad h^* - h = a(M - h) \, , \quad \eta = \varepsilon_1 a^{-2} \, .$$

Then

$$\mu(K^{h*}_{v\varrho,0}) \leqq \eta\, \mu(k_1)\,(\varrho v)^n .$$

Proof. If $\mu(K^{h*}_{v\varrho,0}) > \eta\, \mu(k_1)\,(\varrho v)^n$ then

$$\int_{K_{v\varrho,0}} u_h^2 \,\mathrm{d}\mu \geqq \int_{K^{h*}{}_{v\varrho,0}} u_h^2 \,\mathrm{d}\mu > (h^* - h)^2\, \eta\, \mu(k_1)\,(\varrho v)^n = \varepsilon_1\, \mu(k_1)\,(\varrho v)^n\,(M - h)^2 ,$$

which is not possible by (5.13).

5.10. Remark. We can take $a = \sqrt{(14/15)}$, $\eta = 20/21$ for $\varepsilon_1 = 8/9$.

5.11. Definition. Let $0 < v < 1$, $\bar{Q}_{\varrho,\sigma} \subset \mathcal{O}$. Let us denote T the set of all $t \in \langle t_0 - - \sigma, t_0)$ for which

$$\int_{K_{\varrho v,0(x_0,s)}} u_h^2 \,\mathrm{d}\mu \leqq \int_{K_{\varrho,0(x_0,t)}} u_h^2 \,\mathrm{d}\mu + C\left(1 + \frac{1}{(1 - v)^2\, \varrho^2}\right) \int_{Q_{\varrho,s-t(x_0,t)}} u_h^2 \,\mathrm{d}m$$

for a.e. $s > t$, $s \in (t_0 - \sigma, t_0)$. By Theorem 4.1

$$\mu_1((t_0 - \sigma, t_0) - T) = 0 .$$

5.12. Theorem. *Let* $t_0 - \sigma \in T$, $\int_{K_{\varrho,\sigma}} u_\lambda^2 \,\mathrm{d}\mu \leqq \varepsilon_0\, \mu(k_1)\, \varrho^n (M - \lambda)^2$, $0 < \varepsilon_0 < 1$, $\varepsilon_0 < \varepsilon_1 < 1$, $v_0 < v < 1$, $\sigma \leqq K_0 \varrho^2$, *where* v_0, K_0 *are from Lemma 5.7. Then* $m(Q^h_{\varrho v,\sigma v^2}) \to 0$ *as* $h \to M$. *Further, there exists* $b = b(\varepsilon_0, \varepsilon_1, v, K_0)$, $0 < b < 1$ *such that* $m(Q^h_{\varrho v,\sigma v^2}) \leqq \frac{1}{4}\Gamma \varrho^n \sigma v^{2+n}$ *for* $h = \lambda + b(M - \lambda)$, $\sigma = K_0 \varrho^2$.

Proof. We use Lemmas 5.7 and 5.9 for $a = (\varepsilon_1)^{1/4}$, $\eta = (\varepsilon_1)^{1/2}$ and obtain $\mu(K^{h*}_{\varrho v,s}) \leqq \eta\, \mu(k_1)\,(\varrho v)^n$ where $h^* = \lambda + a(M - \lambda)$ for a.e. $s \in (t_0 - \sigma, t_0)$.

Thus, we can use Lemma 3.6 for a.e. s, $q = 1$, $u = u_h - u_k$ where $h^* < h < k$ and obtain

$$|h - k|\, \mu(K^k_{\varrho v,s}) \leqq C(\eta)\, \varrho \int_{K^h{}_{\varrho v,s} - K^k{}_{\varrho v,s}} |\nabla u| \,\mathrm{d}\mu .$$

Integrating by $s \in (t_0 - \sigma v^2, t_0)$ we obtain

$$|h - k|\, m(Q^k_{\varrho v,\sigma v^2}) \leqq C\varrho \int_{Q^h{}_{\varrho v,\sigma v^2} - Q^k{}_{\gamma v,\varrho v^2}} |\nabla u| \,\mathrm{d}m .$$

Put $k = M - 2l$, $h = M - l$, $D(l) = Q^{M-2l}_{\varrho v,\sigma v^2} - Q^{M-l}_{\varrho v,\sigma v^2}$. We have

$$\left(\int_{D(l)} |\nabla u| \,\mathrm{d}m\right)^2 \leqq m(D(l)) \int_{D(l)} |\nabla u|^2 \,\mathrm{d}m$$

and, using Theorem 4.1,

$$l^2[m(Q^{M-l}_{\varrho\nu,\sigma\nu^2})]^2 \leqq C\varrho^2 m(D(l)) \int_{D(l)} |\nabla u|^2 \, dm \leqq$$

$$\leqq C\varrho^2 m(D(l)) \int_{Q^{2M-2l}_{\varrho\nu,\sigma\nu}} |\nabla u|^2 \, dm \leqq C(\eta, \nu)\left[1 + \frac{\varrho^2}{\sigma}\right] m(D(l))\, l^2 \varrho^n \sigma .$$

Hence

$$(5.14) \qquad [m(Q^{M-l}_{\varrho\nu,\sigma\nu^2})]^2 \leqq C(\eta, \nu, K_0)\, m(D(l))\, \varrho^n \sigma .$$

Put $l_m = 2^{-m}(M - h^*)$ $(m = 1, 2, \ldots)$. Obviously $\sum_{m=1}^{\infty} m(D(l_m)) \leqq \mu(k_1)\, \varrho^n \sigma$,

$$k[m(Q^{M-l_k}_{\varrho\nu,\sigma\nu^2})]^2 \leqq \sum_{m=1}^{k} [m(Q^{M-l_m}_{\varrho\nu,\sigma\nu^2})]^2 \leqq C(\eta, \nu, K_0)\, \varrho^{2n}\sigma^2 ,$$

i. e. $m(Q^{M-l_m}_{\varrho\nu,\sigma\nu^2}) \to 0$ as $m \to 0$, and for k sufficiently large $k = k(\varepsilon_0, \varepsilon_1, \nu, K_0)$, $m(Q^{M-l_k}_{\varrho\nu,\sigma\nu^2}) \leqq \frac{1}{4}\Gamma \varrho^n \sigma \nu^{n+2}$.

However, $M - l_k = \lambda + (1 - (1 - a)/2^k)(M - \lambda)$ and Theorem is proved.

6. MAXIMUM PRINCIPLE

6.1. Theorem. *Let $t_0 - \sigma \in T$ and let u be not a.e. equal to $M = \operatorname{supess}_{Q_{\varrho,\sigma}} u(P)$ on $K_{\varrho,\sigma}$. Then u has its maximum on $Q_{\varrho,\sigma}$ on a set of measure zero.*

Proof. Obviously it is sufficient to prove that $m(Q^h_{\varrho\nu,\sigma\nu^2}) \to 0$ as $h \to M$ for all ν sufficiently close to 1. Thus, let ν be fixed and $h < M$. Then obviously ε_0, $0 < \varepsilon_0 < 1$, exists such that

$$\int_{K_{\varrho,\sigma}} u_h^2 \, d\mu \leqq \varepsilon_0\, \mu(k_1)\, \varrho^n (M - h)^2 .$$

We can suppose $(1 - \nu^n) < \varepsilon_0 < 1$ and $\varepsilon_1 = \varepsilon_0^{1/2}$, $K_0 = K_0(\varepsilon_0, \varepsilon_1, \nu)$. Using Theorem 5.12 we obtain

$$(6.1) \qquad m(Q^h_{\nu\varrho,\nu^2\tau}(x_0, t_0 - \sigma + \tau)) \to 0$$

as $h \to M$ and $\tau \leqq K_0\varrho^2$, $\tau \leqq \sigma$. Let us denote T_1 the set of all τ, $0 < \tau \leqq \sigma$ for which (6.1) holds. We have $(0, K_0\varrho^2\rangle \subset T_1$. Put $\tau_1 = \sup_{(0,\tau)\subset T_1} \tau$. Obviously $\tau_1 \in T_1$. Let $\tau_1 < \sigma$. Then there is $\tau \in (0, \tau_1\rangle$ sufficiently close to τ_1 such that $t_0 - \sigma + \tau \in T$ and

$$\lim_{h\to M} \mu(K^h_{\varrho,t_0-\sigma+\tau}(x_0, t_0)) \leqq \mu(k_1)\, \varrho^n(1 - \nu^n) < \varepsilon_0\, \mu(k_1)\, \varrho^n .$$

There exists $h < M$ such that

$$\int_{K_{\varrho,t_0-\sigma+\tau}} u_h^2 \leqq \mu(K^h_{\varrho,t_0-\sigma+\tau})(M-h)^2 < \varepsilon_0\,\mu(k_1)\,\varrho^n(M-h)^2\,.$$

Let us repeat our reasoning for $\sigma - \tau$ instead for σ and obtain

$$m(Q^h_{\nu\varrho,\nu^2\tau^*}(x_0, t_0 - \sigma + \tau + \tau^*)) \to 0$$

as $h \to M$, $\tau^* \leqq K_0\varrho^2$, $\tau^* \leqq \sigma - \tau$.

Thus we can take τ^* such that $\tau_1 - \tau < \tau^* < K_0\varrho^2$, $\tau^* < \sigma - \tau$, i.e. $\tau_1 < \tau + \tau^* \in T_1$. Hence $\tau_1 = \sigma$.

6.2. Theorem. *Let $t_0 - \sigma \in T$ and let u be not equal to M a.e. on $K_{\varrho,\sigma}$. Then*

$$\operatorname*{supess}_{P\in Q_{\varrho\nu,\sigma\nu^2}} u(P) < M\,.$$

Proof. This Theorem is an immediate consequence of Theorem 6.1 and Consequence 5.4.

6.3. Theorem. *Let $0 < \nu < 1$ and let a $\sigma_1 > 0$ exist such that*

$$\operatorname*{supess}_{P\in Q_{\varrho\nu,\sigma_1}} u(P) = M\,.$$

Then $u(P) = M$ a.e. on $Q_{\varrho,\sigma} - Q_{\varrho,\sigma_1}$.

Proof. If $u(P) = M$ a.e. in K_{ϱ,σ_2} does not hold for some $\sigma_2 > \sigma_1$ such that $t_0 - \sigma \in T$, then we obtain a contradiction with the assertion of Theorem 6.2.

6.4. Theorem. *Let*

$$\operatorname*{supess}_{P\in Q_{\varrho\nu,\sigma_1}} u(P) = M$$

for some ν, $0 < \nu < 1$ and every σ_1, $0 < \sigma_1 < \sigma$. Then $u(P) = M$ a.e. in $Q_{\varrho,\sigma}$.

Proof. This Theorem is a consequence of Theorem 6.3.

7. MINIMUM PRINCIPLE AND HÖLDER CONTINUITY

7.1. Theorem. *Let a_j $(j = 1, \ldots, n)$, c, u be functions satisfying*

1) $a_j(u - h, \partial u) \in L_2(\mathcal{O})$, $c(u - h, \partial u) \in L_2(\mathcal{O})$ *for all* $h \in I(u)$,

2) $\sum_{j=1}^n a_j(p, \mathbf{q})\, q_{\mathbf{e}_j} - c(p, \mathbf{q})\, p \geqq \nu \sum_{j=1}^n |q_{\mathbf{e}_j}|^2 - M_1 p^2$ *where* $\nu > 0$, $-\operatorname*{osc}_{\mathcal{O}} u < p < 0$, $q_{\mathbf{o}} \in I(u)$,

3) $\sum_{j=1}^{n} |a_j(p, \mathbf{q})| \leqq M_2(\sum_{j=1}^{n} |q_{\bullet j}| + |p|)$ *for* $- \operatorname{osc}_{\mathcal{O}} u < p < 0,\ q_\bullet \in I(u)$,

4) $\sum_{j=1}^{n} (\partial/\partial x_j)\, a_j(u - h, \partial u) - a_j(u, \partial u)) + c(u - h, \partial u) - c(u, \partial u) \leqq M_4 |\nabla u|$ *for every* $h \in I(u)$ *in the sense of distributions,*

5) $\partial u/\partial t \geqq \sum_{j=1}^{n} (\partial/\partial x_j)\, a_j(u, \partial u) + c(u, \partial u) + M_3 |\nabla u|$ *in the sense of distributions,* $u \in W$.

Then the function $u_h = -\min(u - h, 0)$ *satisfies* (3.1) *and consequently all our theorems hold for the function* $-u$.

Proof. It is sufficient to modify the proof of Theorem 3.4. The other theorems are based only on Theorem 3.4.

7.2. Theorem. *Let* a_j $(j = 1, \ldots, n)$, c, u *be functions satisfying* 1) *to* 4) *of section* 2.2, (2.1) *and* 1) *to* 5) *from* Theorem 7.1. *Then* u *is locally Hölder continuous in* $\mathcal{O}$.

Proof. In Remarks 5.8 and 5.10 we take $v = v_0$ and suppose $\sigma = K_0 \varrho^2$.

Let us denote $M = \operatorname{supess}_{P \in Q_{\varrho,\sigma}} u(P)$, $m = \operatorname{infess}_{P \in Q_{\varrho,\sigma}} u(P)$, $\lambda = (M + m)/2$; $u_\lambda^+ = u_\lambda$, $u_\lambda^- = \lambda - u + u_\lambda^+$. Then $\mu(K_{\varrho,\sigma}^\lambda) \leqq \frac{1}{2}\mu(k_1)\, \varrho^n$ or $\mu(K_{\varrho,\sigma} - K_{\varrho,\sigma}^\lambda) \leqq \frac{1}{2}\mu(k_1)\, \varrho^n$.

In these cases (for simplification we take $t_0 - \sigma \in T$) either

$$\int_{K_{\varrho,\sigma}} (u_\lambda^+)^2 \, \mathrm{d}\mu \leqq \tfrac{1}{2}\mu(k_1)\, \varrho^n (M - \lambda)^2 \tag{7.1}$$

or

$$\int_{K_{\varrho,\sigma}} (u_\lambda^-)^2 \, \mathrm{d}\mu \leqq \tfrac{1}{2}\mu(k_1)\, \varrho^n (M - \lambda)^2 \,. \tag{7.2}$$

Consider, e.g. (7.2). Then we use Theorem 5.12 and obtain for $h = \lambda + b(M - \lambda)$, $m(Q^h_{\varrho v_0, \sigma v_0^2}) \leqq \frac{1}{4} \Gamma \varrho^n \sigma v_0^{2+n}$. By (5.10) we have

$$\operatorname*{supess}_{P \in Q_{\varrho v_0^2, \sigma v_0^4}} u(P) \leqq \lambda + \frac{b + 1}{2} (M - \lambda)\,;$$

thus,

$$\operatorname*{osc}_{Q_{\varrho v_0^2, \sigma v_0^4}} u \leqq \left(\frac{1}{2} + \frac{b + 1}{4} \right) 2(M - \lambda) = b^* \operatorname*{osc}_{Q_{\varrho,\sigma}} u$$

because $2(M - \lambda) = \operatorname*{osc}_{Q_{\varrho\sigma}} u$. Here $0 < b^* < 1$.

We have

$$\operatorname*{osc}_{Q_{\varrho v, \sigma v^2}} u < b^* \operatorname*{osc}_{Q_{\varrho,\sigma}} u\,, \quad b^* < 1 \tag{7.3}$$

for $\sigma = K_0 \varrho^2$ and $v = \sqrt[n]{(9/16)} = (9/16)^{1/n}$.

Hölder continuity of u follows from (7.3).

The assertions indicated in the introduction are immediate consequences of theorems in Sections 4. to 7.

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Výtah

SILNÝ PRINCIP MAXIMA PRO SLABÁ ŘEŠENÍ NELINEÁRNÍ PARABOLICKÉ DIFERENCIÁLNÍ NEROVNOSTI

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Nechť $\mathcal{O}$ je oblast v E_{n+1} (eukleidovském prostoru n proměnných a času t). Nechť u je řešení nelineární diferenciální nerovnosti

$$\frac{\partial u}{\partial t} \leqq \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j(u, \partial u) + c(u, \partial u) + M \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|$$

ve smyslu zobecněných funkcí. Funkce a_j, c jsou funkce proměnných x, t a nějaké soustavy derivací funkce u, kterou souhrně označíme ∂u. Nechť tyto funkce mají následující vlastnosti:

1) $a_j(u - h, \partial u), c(u - h, \partial u) \in L_2(\mathcal{O})$.

2) Existuje konstanta $\nu > 0$ a N tak, že

$$\sum_{j=1}^{n} a_j(u - h, \partial u) \frac{\partial u}{\partial x_j} - c(u - h, \partial u)(u - h) \geqq \nu \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 - N(u - h)^2,$$

3) $\sum_{j=1}^{n} |a_j(u - h, \partial u)| \leqq N \left(\sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right| + (u - h) \right)$,

4) $\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_j(u - h, \partial u) - a_j(u, \partial u)) + c(u - h, \partial u) - c(u, \partial u) \geqq - N \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|$,

pro všechna h z oblasti hodnot funkce u a $u(x, t) > h$.

O řešení u se předpokládá, že $u \in L_2(\mathcal{O})$ a $\partial u / \partial x_i \in L_2(\mathcal{O})$ $(i = 1, 2, \ldots, n)$.

Pro $(x_0, t_0) \in \mathcal{O}$ označíme $S(x_0, t_0)$ množinu všech (x, t) takových, že $t < t_0$ a že existuje vektorová funkce $\varphi(\tau), \varphi \in C^{(1)}(t, t_0)$ pro kterou $\varphi(t) = x$, $\varphi(t_0) = x_0$, $(\varphi(\tau), \tau) \in \mathcal{O}$ pro všechna $\tau \in \langle t, t_0 \rangle$.

Dále řekneme, že u nabývá svého maxima (v $S(x_0, t_0)$) v bodě $(x_0, t_0) \in \mathcal{O}$, když pro každou n-rozměrnou kouli K a pro $\delta > 0$ takové, že $(x_0, t_0) \in Q^{(\delta)} = K \times \times \langle t_0 - \delta, t_0 \rangle$, platí

$$\operatorname*{supess}_{(x,t) \in Q^{(\delta)}} u(x, t) \geqq \operatorname*{supess}_{(x,t) \in S(x_0,t_0)} u(x, t) .$$

Z těchto předpokladů vyplývá, že funkce u je v $\mathcal{O}$ lokálně ohraničená shora. Když u nabývá svého maxima (v $S(x_0, t_0)$) v bodě (x_0, t_0), pak u je skoro všude v $S(x_0, t_0)$ konstantní.

Jestliže kromě těchto podmínek vyhovuje funkce u analogickým podmínkám, zaručujícím platnost principu minima, pak u v $\mathcal{O}$ lokálně vyhovuje Hölderově podmínce.

Резюме

СТРОГИЙ ПРИНЦИП МАКСИМУМА ДЛЯ СЛАБЫХ РЕШЕНИЙ НЕЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ НЕРАВЕНСТВ

ЯН КАДЛЕЦ (Jan Kadlec), Прага

Пусть $\mathcal{O}$ — область в E_{n+1} (евклидовом пространстве n пространственных и переменной t). Пусть u — решение нелинейного дифференциального неравенства

$$\frac{\partial u}{\partial t} \leqq \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_j(u, \partial u) + c(u, \partial u) + M \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|$$

в смысле обобщенных функций. Здесь a_j, c — функции от x, t и некоторого

набора производных функции u, обозначнного в совокупности через ∂u. Пусть эти функции обладают следующими свойствами:

1) $a_j(u - h, \partial u)$, $c(u - h, \partial u) \in L_2(\mathcal{O})$.
2) Существует постоянная $\nu > 0$ и N так, что

$$\sum_{j=1}^{n} a_j(u - h, \partial u) \frac{\partial u}{\partial x_j} - c(u - h, \partial u)(u - h) \geqq \nu \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 - N(u - h)^2 .$$

3) $\sum_{j=1}^{n} |a_j(u - h, \partial u)| \leqq N \left(\sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right| + (u - h) \right).$

4) $\sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_j(u - h, \partial u) - a_j(u, \partial u)) + c(u - h, \partial u) - c(u, \partial u) \geqq -N \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|$

для всех постоянных h на области значений функции u и $u(x, t) > h$.

Относительно решения u предпологается, что

$$u \in L_2(\mathcal{O}) \quad \text{и} \quad \frac{\partial u}{\partial x_i} \in L_2(\mathcal{O}) \quad (i = 1, 2, \ldots, n) .$$

Для $(x_0, t_0) \in \mathcal{O}$ обозначим $S(x_0, t_0)$ множество всех (x, t) таких, что $t < t_0$ и существует векторная функция $\varphi(\tau)$, $\varphi \in C^{(1)}(t, t_0)$, для которой $\varphi(t) = x$, $\varphi(t_0) = x_0$, $(\varphi(\tau), \tau) \in \mathcal{O}$ для всех $\tau \in \langle t, t_0 \rangle$.

Далее, скажем, что u достигает своего максимума (в $S(x_0, t_0)$) в точке $(x_0, t_0) \in \in \mathcal{O}$ если для всякого n-мерного шара K и $\delta > 0$ таких, что $(x_0, t_0) \in Q^{(\delta)} = = K \times \langle t_0 - \delta, t_0 \rangle$, имеет место

$$\operatorname*{supess}_{(x,t) \in Q^{(\delta)}} u(x, t) \geqq \operatorname*{supess}_{(x,t) \in S(x_0, t_0)} u(x, t) .$$

Из наших предположений следует, что функция u локально органичена сверху в $\mathcal{O}$. Если u достигает своего максимума (в $S(x_0, t_0)$) в точке (x_0, t_0), то u равна постоянной почти всюду в $S(x_0, t_0)$.

Если кроме описанных условий u удовлетворяет аналогичным условиям, обеспечивающим, что имеет место тоже принцип минимума, то u в $\mathcal{O}$ локально удовлетворяет условию Гельдера.