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CHARACTERIZATION OF FUNCTIONS WITH ZERO TRACES BY INTEGRALS WITH WEIGHT FUNCTIONS II

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First of all, it should be remarked that results contained in this paper as well as in its part I were included in the preliminary communication [1].

Let Ω be a bounded domain of E_N and let its boundary $\partial\Omega$ fulfil the Lipschitz condition only. Put $L_{p,\alpha}(\Omega)$ the common weighted space with the weight function $\varrho^\alpha(X)$, where $\varrho(X)$ is the distance between the point $X \in \Omega$ and the boundary $\partial\Omega$.

In part I of this paper [2], it was proved that the norms

$$\|u\|_{W^{(k)}_{p,\alpha}(\Omega)} = \Big[\sum_{|j|=0}^{k} \|D^j u\|^p_{L_{p,\alpha}(\Omega)}\Big]^{1/p} \tag{0.2}$$

and

$$\|u\|_V = \Big[\sum_{|j|=0}^{k} \|D^j u\|^p_{L_{p,\alpha-(k-|j|)p}(\Omega)}\Big]^{1/p} \tag{0.4}$$

are equivalent on the space V which is the space of all functions having the norm (0.4) finite. We obtained (see Theorem 1 in the part I), that $V = \mathring{W}$, where $\mathring{W}$ is the closure in the norm (0.2) of the set $\mathscr{D}(\Omega)$ of all infinitely differentiable functions with compact support in Ω.

In Theorem 1 we supposed $\alpha \neq ip - 1$, $i = 1, 2, \ldots, k$, because we cannot use the Hardy's inequality for the "singular" values $\alpha = \alpha_i = ip - 1$.

In this paper, we give certain modification of Hardy's inequality (1.3), which enable us to prove Theorem 3 which completes Theorem 1.

4. MODIFICATION OF HARDY'S INEQUALITY AND AUXILIARY LEMMAS

4.1 Lemma 1. *Let $R > 0$, β and γ be real and put*

$$f(t) = t^\beta \lg^\gamma \frac{R}{t}$$

for $t \in (0, R)$. *Then*

$$f^{(k)}(t) = t^{\beta-k} \lg^{\gamma} \frac{R}{t} \cdot P_k\left(\frac{1}{\lg R/t}\right) \tag{4.1}$$

where $P_k(s)$ *is a polynomial of degree less or equal* k *and* $P_1(s) = \beta - \gamma s$.

Proof. We use mathematical induction:

I. Lemma obviously takes place for $k = 0$ where $P_0(s) \equiv 1$.

II. Derivating (4.1) we obtain

$$f^{(k+1)}(t) = t^{\beta-k-1} \lg^{\gamma} \frac{R}{t} \cdot P_{k+1}\left(\frac{1}{\lg R/t}\right)$$

where

$$P_{k+1}(s) = (\beta - k) P_k(s) - \gamma s P_k(s) + s^2 P'_k(s) .$$

The polynomial $P_{k+1}(s)$ is obviously of a degree less or equal $k + 1$ and $P_1(s) = \beta - \gamma s$.

The following lemma is an easy consequence of lemma 1.

Lemma 2. *If* $t \in (0, R)$, *then*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{1}{\lg R/t}\right) = \frac{1}{t^k} \frac{1}{\lg^2 R/t} P_{k-1}\left(\frac{1}{\lg R/t}\right) \tag{4.2}$$

where $P_k(s)$ *is the polynomial from lemma* 1.

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lg R/t}\right) = \frac{1}{\lg^2 R/t} \cdot \frac{1}{t} = f(t) ,$$

where $f(t)$ is the function from lemma 1 with $\beta = -1$ and $\gamma = -2$. So we obtain (4.2) from (4.1).

4.2 Now, we can prove one modification of the inequality of Hardy.

Lemma 3. *Let* $f(t) \in \mathscr{D}(0, 1)$. *Then*

a) *If* $\beta \neq -1$, γ *real*, $R \geqq e^{2|\gamma|/|\beta+1|}$ *then*

$$\int_0^1 |f(t)|^p t^{\beta} \lg^{\gamma} \frac{R}{t} \mathrm{d}t \leqq \left(\frac{2p}{|\beta + 1|}\right)^p \int_0^1 |f'(t)|^p t^{\beta+p} \lg^{\gamma} \frac{R}{t} \mathrm{d}t . \tag{4.3}$$

b) *If* $\gamma \neq -1$, $R \geqq 1$ *then*

$$\int_0^1 |f(t)|^p t^{-1} \lg^\gamma \frac{R}{t} \, dt \leqq \left(\frac{p}{|\gamma+1|}\right)^p \int_0^1 |f'(t)|^p t^{-1+p} \lg^{\gamma+p} \frac{R}{t} \, dt . \tag{4.4}$$

Proof. a) Let $\beta \neq -1$. We have

$$\frac{d}{dt}\left(t^{\beta+1} \lg^\gamma \frac{R}{t}\right) = t^\beta \lg^\gamma \frac{R}{t} \, . \, P_1\left(\frac{1}{\lg R/t}\right)$$

by lemma 1 with $P_1(s) = (\beta+1) - \gamma s$. We have $t \in (0, 1)$ and $R \geqq e^{2|\gamma|/|\beta+1|}$ and so $R/t \geqq e^{2|\gamma|/|\beta+1|}$. So $0 \leqq s = \lg^{-1} R/t \leqq |\beta+1|/2|\gamma|$ and

$$\left|P_1\left(\frac{1}{\lg R/t}\right)\right| = |P_1(s)| \geqq |\beta+1| - |\gamma| \, s \geqq \frac{|\beta+1|}{2} .$$

So we obtain

$$\int_0^1 |f(t)|^p t^\beta \lg^\gamma \frac{R}{t} \, dt \leqq \frac{2}{|\beta+1|} \left| \int_0^1 |f(t)|^p \frac{d}{dt}\left(t^{\beta+1} \lg^\gamma \frac{R}{t}\right) dt \right| \leqq$$

$$\leqq \frac{2}{|\beta+1|} \left| \left[|f(t)|^p t^{\beta+1} \lg^\gamma \frac{R}{t} \right]_0^1 \right| + \frac{2p}{|\beta+1|} \left| \int_0^1 |f(t)|^{p-1} \frac{d|f|}{dt} \, t^{\beta+1} \lg^\gamma \frac{R}{t} \, dt \right| \leqq$$

$$\leqq \frac{2p}{|\beta+1|} \int_0^1 |f(t)|^{p-1} t^{\beta(p-1)/p} \lg^{\gamma(p-1)/p} \frac{R}{t} \, . \left|\frac{df}{dt}\right| t^{\beta/p+1} \lg^{\gamma/p} \frac{R}{t} \, dt \leqq$$

$$\leqq \frac{2p}{|\beta+1|} \left(\int_0^1 |f(t)|^p t^\beta \lg^\gamma \frac{R}{t} \, dt \right)^{(p-1)/p} . \left(\int_0^1 \left|\frac{df}{dt}\right|^p t^{\beta+p} \lg^\gamma \frac{R}{t} \, dt \right)^{1/p}$$

where we used $f \in \mathscr{D}(0, 1)$. The latter inequality implies (4.3).

b) Let $\gamma \neq -1$. We have $t^{-1} \lg^\gamma R/t = -1/(\gamma+1) \, (d/dt) \, (\lg^{\gamma+1} R/t)$ and in a similar manner as in a) we obtain for $f \in \mathscr{D}(0, 1)$

$$\int_0^1 |f(t)|^p t^{-1} \lg^\gamma \frac{R}{t} \, dt \leqq \frac{1}{|\gamma+1|} \left| \int_0^1 |f(t)|^p \frac{d}{dt}\left(\lg^{\gamma+1} \frac{R}{t}\right) dt \right| \leqq$$

$$\leqq \frac{p}{|\gamma+1|} \int_0^1 |f(t)|^{p-1} t^{-(p-1)/p} \lg^{\gamma(p-1)/p} \frac{R}{t} \, . \left|\frac{df}{dt}\right| t^{(p-1)/p} \lg^{\gamma/p+1} \frac{R}{t} \, dt \leqq$$

$$\leqq \frac{p}{|\gamma+1|} \left(\int_0^1 |f(t)|^p t^{-1} \lg^\gamma \frac{R}{t} \, dt \right)^{(p-1)/p} . \left(\int_0^1 \left|\frac{df}{dt}\right|^p t^{p-1} \lg^{\gamma+p} \frac{R}{t} \, dt \right)^{1/p}$$

wich implies (4.4).

4.3 Lemma 3. *Let $\Omega \in E_N$, $g(X) \in C^\infty(\Omega)$, $g(X) \geqq 0$, $f(t) \in C^\infty(0, \infty)$. Put $F(X) = f(g(X))$. Then*

$$D^i F(X) = \sum C^{(i)}_{n,\beta_j} f^{(n)}(g(X)) \prod_{|j|=1}^{|i|} (D^j g(X))^{\beta_j} \tag{4.5}$$

for $|i| \geqq 1$, where we sum over

$$n + \sum_{|j|=1}^{|i|} \beta_j(|j| - 1) \leqq |i| \tag{4.6}$$

and where

$$1 \leqq \sum_{|j|=1}^{|i|} \beta_j \leqq n\,, \quad \beta_j \text{ non-negative integers.} \tag{4.7}$$

Proof. We use mathematical induction:

I. Let $|r| = 1$; then $D^r F(X) = f^{(1)}(g(X))\, D^r g(X)$ namely (4.5) takes place and $\beta_r = 1$.

II. Let $m = i + r$, $|r| = 1$, i.e. $|m| = |i| + 1$, and let the assertion holds for i. So $D^m F = D^r(D^i F)$ and it follows from (4.5) that $D^r(D^i F)$ is a sum of members of the form

$$A = f^{(n+1)}(g(X))\, D^r g \prod_{|j|=1}^{|i|} (D^j g)^{\beta_j} \tag{1)}$$

and

$$B = f^{(n)}(g(X)) \sum_{|j_0|=1}^{|i|} \Big[\prod_{j \neq j_0} (D^j g)^{\beta_j}\Big] \beta_{j_0}(D^{j_0} g)^{\beta_{j_0}-1} D^{j_0+r} g\,. \tag{2)}$$

Ad 1): Obviously

$$A = (D^r g)^{\beta_r+1} \prod_{j \neq r} (D^j g)^{\beta_j} f^{(n+1)}(g(X)) = f^{(n+1)}(g(X)) \prod_{|s|=1}^{|i|} (D^s g)^{\gamma_s}$$

where $\gamma_s = \beta_s$ for $s \neq r$ and $\gamma_r = \beta_r + 1$. So we have

$$\sum_s \gamma_s = \sum_j \beta_j + 1 \leqq n + 1$$

i.e. the condition (4.7) for γ_s holds. Further

$$(n + 1) + \sum_s \gamma_s(|s| - 1) = (n + 1) + \sum_{j \neq r} \beta_j(|j| - 1) + (\beta_r + 1)(|r| - 1) =$$
$$= (n + 1) + \sum_j \beta_j(|j| - 1) = n + \sum_j \beta_j(|j| - 1) + 1 \leqq |i| + 1 = |m|$$

(because $|r| = 1$ i.e. $|r| - 1 = 0$). So we have obtained (4.6) for $m = i + r$.

Ad 2): We must show, that for the expressions

$$C = \prod_{j \neq j_0}^{|i|} (D^j g)^{\beta_j} (D^{j_0} g)^{\beta_{j_0}-1} D^{j_0+r} g$$

conditions of the type (4.6) and (4.7) for $|i| + 1$ take place. There are two possibilities:

a) If $|j_0| = |i|$ then we can write

$$C = \prod_{s=1}^{|i|+1} (D^s g)^{\gamma_s}$$

where $\gamma_s = \beta_s$ for $|s| \leqq |i|$, $s \neq j_0$; $\gamma_{j_0} = \beta_{j_0} - 1$ and $\gamma_s = 1$ for $s = j_0 + r$.
So we obtain $1 \leqq \sum_s \gamma_s = \sum_j \beta_j \leqq n$ and then (4.7) holds (we multiply C by $f^{(n)}!$).
With respect to (4.6) we have

$$n + \sum_\gamma \gamma_s(|s| - 1) = n + \sum_{\substack{|j|=1 \\ j \neq j_0}}^{|i|} \beta_j(|j| - 1) + (\beta_{j_0} - 1)(|j_0| - 1) +$$

$$+ \gamma_{j_0+r}(|j_0| + 1 - 1) = n + \sum_j \beta_j(|j| - 1) - |j_0| + 1 + |j_0| \leqq |i| + 1$$

namely (4.7) holds for $|m| = |i| + 1$.

b) If $|j_0| < |i|$, we put $j_1 = j_0 + r$; we have also $|j_1| \leqq |i|$ and we can write

$$C = \prod_{s=1}^{|i|} (D^s g)^{\gamma_s}$$

where $\gamma_s = \beta_s$ for $s \neq j_0, j_1$; $\gamma_{j_0} = \beta_{j_0} - 1$ and $\gamma_{j_1} = \beta_{j_1} + 1$. Obviously $\sum_s \gamma_s = \sum_j \beta_j \leqq n$ and further

$$n + \sum_s \gamma_s(|s| - 1) = n + \sum_{s \neq j_0, j_1} \beta_s(|s| - 1) + (\beta_{j_0} - 1)(|j_0| - 1) + (\beta_{j_1} + 1)(|j_1| - 1) =$$

$$= n + \sum_j \beta_j(|j| - 1) - |j_0| + 1 + |j_1| - 1 \leqq |i| + 1,$$

because $-|j_0| + |j_1| = 1$ and (4.7) holds. The expression $D^r(D^i F)$ is a sum of the type (4.5) for which we have conditions corresponding to (4.6) and (4.7) and so the proof is complete.

4.4 Let now $f(s)$ be an infinitely differentiable function in $(0, \infty)$ and let $f(s) \equiv 0$ for $s \leqq \frac{1}{4}$ and $f(s) \equiv 1$ for $s \geqq \frac{3}{4}$. Let $0 \leqq f(s) \leqq 1$.

Let $R \geqq 1$ and put

$$g_h(t) = \begin{cases} f\left(\dfrac{1}{h \lg R/t}\right) & \text{for} \quad t < R, \\ 1 & \text{for} \quad t \geqq R, \end{cases}$$

where $0 < h < 1$. The function g_h has the following properties

1) $g_h \in C^\infty(0, \infty)$; $0 \leqq g_h \leqq 1$;

2) $g_h(t) \equiv 0$ for $t < Re^{-4/h}$,
$g_h(t) \equiv 1$ for $t \geqq Re^{-4/3h}$.

3) For $t \in I_h = \langle Re^{-4/h}, Re^{-4/3h} \rangle$, the inequality

(4.8) $$\left| \frac{d^k}{dt^k} g_h(t) \right| \leqq \frac{c(k)}{t^k \lg R/t} \quad (k \geqq 1)$$

holds.

Properties 1) and 2) are obvious and so we prove the property 3) only:

By a similar way as in Lemma 4 we can show

$$\frac{d^k}{dt^k} g_h(t) = \sum_{n,\beta_j} c^{(k)}_{n,\beta_j} f^{(n)} \left(\frac{1}{h \lg R/t} \right) \cdot \prod_{j=1}^{k} \left[\frac{d^j}{dt^j} \left(\frac{1}{h \lg R/t} \right) \right]^{\beta_j}$$

where $n + \sum_{j=1}^{k} \beta_j (j-1) \leqq k$ and $1 \leqq \sum_{j=1}^{k} \beta_j \leqq n$.

The function f possesses all derivatives bounded and so

(4.9) $$\left| \frac{d^k}{dt^k} g_h(t) \right| \leqq c_1 \sum_{n,\beta_j} \prod_j \frac{1}{h^{\beta_j}} \left| \frac{d^j}{dt^j} \left(\frac{1}{\lg R/t} \right) \right|^{\beta_j}.$$

By Lemma 2 we have

$$\frac{d^j}{dt^j} \left(\frac{1}{\lg R/t} \right) = \frac{1}{t^j} \frac{1}{\lg^2 R/t} P_{j-1} \left(\frac{1}{\lg R/t} \right)$$

where the expression P_{j-1} is bounded for $t \in I_h$. Further $\frac{4}{3} \leqq h \lg R/t \leqq 4$ and so $1/h$ is equivalent to $\lg R/t$ and we obtain

$$I_{n,\beta_j} = \prod_j \frac{1}{h^{\beta_j}} \left| \frac{d^j}{dt^j} \left(\frac{1}{\lg R/t} \right) \right|^{\beta_j} \leqq c_2 \prod_j \lg^{\beta_j} \frac{R}{t} \cdot \frac{1}{t^{j\beta_j}} \cdot \frac{1}{\lg^{2\beta_j} R/t} =$$

$$= c_2 \prod_j t^{-j\beta_j} \lg^{-\beta_j} \frac{R}{t} = c_2 t^{-\sum_j j\beta_j} . \lg^{-\sum_j \beta_j} \frac{R}{t}.$$

Now $1 \leqq \sum_j \beta_j \leqq n$ and so $\sum_j j\beta_j = \sum_j \beta_j (j-1) + \sum_j \beta_j \leqq k - n + n = k$ and finally

$$I_{n,\beta_j} \leqq c_3 t^{-k} \lg^{-1} \frac{R}{t}.$$

From here and from (4.9) it follows (4.8).

4.5 In Nečas's paper [3] is proved, that for $\Omega \in \mathfrak{N}^{(0),1}$ (for the definition of $\Omega \in \mathfrak{N}^{(0),1}$ see [2]) there exists a function σ continuous in $\overline{\Omega}$ and infinitely differentiable in Ω for which

(4.10) $$c_4 \varrho(X) \leqq \sigma(X) \leqq c_5 \varrho(X)$$

(i.e. $\varrho(X) = \operatorname{dist}(X, \partial\Omega)$ is equivalent to $\sigma(X)$), and further

$$|D^i\sigma| \leqq \frac{c(i)}{\sigma^{i-1}(X)} \quad \text{for} \quad |i| > 0 . \tag{4.11}$$

Put

$$G_h(X) = g_h(\sigma(X)) \tag{4.12}$$

where g_h is the function from section 4.4. The function $G_h(X)$ possesses the following properties:

A. $G_h(X) \in \mathscr{D}(\Omega)$; $0 \leqq G_h \leqq 1$.

B. $G_h(X) \equiv 1$ for $\sigma(X) \geqq Re^{-4/3h}$.

C. For $|i| > 0$, the estimation

$$|D^i G_h(X)| \leqq \frac{c(i)}{\sigma^{|i|}(X)} \cdot \frac{1}{\lg R/\sigma} \tag{4.13}$$

takes place.

Property A follows from the fact $g_h \in C^\infty$ and $g_h(t) \equiv 0$ for small t; B follows from the fact that $g_h(t) \equiv 1$ for $t \geqq Re^{-4/3h}$. Now, we must prove (4.13):

We use Lemma 4 and obtain

$$I = |D^i G_h| \leqq c_6 \sum_{n,\beta_j} g_h^{(n)}(\sigma(X)) \prod_{|j|=1}^{|i|} |D^j\sigma|^{\beta_j} .$$

With respect to estimations (4.8) (for $t = \sigma(X)$) and (4.11) we have

$$I \leqq c_7 \sum_{n,\beta_j} \frac{1}{\sigma^n \lg R/\sigma} \cdot \prod_j \frac{1}{\sigma^{(|j|-1)\beta_j}} = c_7 \sum_{n,\beta_j} \frac{1}{\lg R/\sigma} \cdot \frac{1}{\sigma^{n+\sum_j(|j|-1)\beta_j}} .$$

Further $n + \sum_j(|j| - 1)\,\beta_j \leqq |i|$ and so (4.13) holds.

5. THE CASE $\alpha = sp - 1$

In the Theorem 1 we supposed that the parameter α is not equal to some "singular" values. For these exceptional values α, we shall, in this Section, characterise functions of $\mathring{W}$ with the help of a certain equivalent norm.

Let s be one of the values $1, 2, \ldots, k$ and let $\alpha = sp - 1$. Let us consider the space $W_{p,\alpha}^{(k)}(\Omega)$, where we (with respect to the equivalence of $\varrho(X)$ – the distance between $X \in \Omega$ and $\partial\Omega$ – and the function $\sigma(X)$; see (4.10)) in the definition put the weight function $\sigma^\alpha(X)$ instead of $\varrho^\alpha(X)$.

Let $[x_r', x_{rN}]$ be the local coordinate systems from the Section 1.1 (see [2]). For $X \in B_r$ is $\varrho(X)$ and so $\sigma(X)$ equivalent to $a_r(x_r') - x_{rN}$, i.e. there are positive constants c

and $\tilde{c}$ independent on r such that

$$c\,\sigma(X) \leqq a_r(x_r') - x_{rN} \leqq \tilde{c}\,\sigma(X) \tag{5.1}$$

for $X = [x_r', x_{rN}] \in B_r$.

Put $M = \max_{\bar{\Omega}} \sigma(X)$ and let R fulfil

$$R > \max(M, e^2/c)$$

(c is the constant from (5.1)).

If $\alpha = sp - 1$ we denote by V_s the space of all functions defined almost everywhere in Ω possessing generalized derivatives $D^i u$ $(0 \leqq |i| \leqq k)$ in Ω such that

$$\begin{aligned}
&D^k u &&\in L_{p,\alpha}(\Omega)\,,\\
&D^{k-1} u &&\in L_{p,\alpha-p}(\Omega)\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&D^{k-s+1} u &&\in L_{p,\alpha-(s-1)p}(\Omega)\,,\\
&\frac{D^{k-s} u}{\lg R/\sigma} &&\in L_{p,\alpha-sp}(\Omega)\,,\\
&\frac{D^{k-s-1} u}{\lg R/\sigma} &&\in L_{p,\alpha-(s+1)p}(\Omega)\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{u}{\lg R/\sigma} &&\in L_{p,\alpha-kp}(\Omega)
\end{aligned} \tag{5.2}$$

(here by $D^l u$ we mean arbitrary derivatives of order l; the weight function is always the corresponding power of the function $\sigma(X)$).

Now, $\alpha = sp - 1$ and so $\alpha - sp = -1$, $\alpha - (s-1)\,p = -1 + p$ etc. and we can write V_s as the space of functions u, for which

$$\begin{aligned}
&D^k u &&\in L_{p,-1+sp}(\Omega)\,,\\
&D^{k-1} u &&\in L_{p,-1+(s-1)p}(\Omega)\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&D^{k-s+1} u &&\in L_{p,-1+p}(\Omega)\,,\\
&\frac{D^{k-s} u}{\lg R/\sigma} &&\in L_{p,-1}(\Omega)\,,\\
&\frac{D^{k-s-1} u}{\lg R/\sigma} &&\in L_{p,-1-p}(\Omega)\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{u}{\lg R/\sigma} &&\in L_{p,-1-(k-s)p}(\Omega)\,.
\end{aligned} \tag{5.3}$$

The norm in V_s we take analogically as in (0.4):

$$(5.4)\quad \|u\|_{V_s} = \left[\sum_{|j|=0}^{k-s} \left\| \frac{D^j u}{\lg R/\sigma} \right\|^p_{L_{p,\alpha-(k-|j|)p}(\Omega)} + \sum_{|j|=k-s+1}^{k} \|D^j u\|^p_{L_{p,\alpha-(k-|j|)p}(\Omega)}\right]^{1/p}.$$

Theorem 3. *Let $\Omega \in \mathfrak{N}^{(0),1}$ and $\sigma(X)$ be the function from* (4.10), $p > 1$. *Let $\mathring{W}$ be the closure of the set $\mathscr{D}(\Omega)$ in the norm of the space $W^{(k)}_{p,-1+sp}(\Omega)$. Then*

$$\mathring{W} = V_s .$$

Proof is very analoguous to the proof of Theorem 1 and so we give it abbreviated.

I) We prove $\mathring{W} \subset V$.

We can consider only functions $u \in \mathscr{D}(\Omega)$ and put $u_i = u\varphi_i \in \mathscr{D}(C_i)$ where $\varphi_1, \varphi_2, \ldots, \ldots, \varphi_m$ is the corresponding decomposition of unit. Then we can write $[a_i(x_i') - x_{iN}]^\alpha$ instead of $\sigma^\alpha(x_i', x_{iN})$ and omit the index i.

Let $v = D^j u$ where $|j| \leqq k$.

a) If $|j| \geqq k - s + 1$, we can prove in a similar manner as in the proof of Theorem 1

$$(5.5)\qquad \|D^j u\|^p_{L_{p,\alpha-(k-|j|)p}(\Omega)} \leqq c_1 \|u\|^p_{W^{(k)}{}_{p,\alpha}(\Omega)} \quad (\alpha = sp - 1)$$

using the inequality of Hardy.

b) Let $|j| = k - s$ and put

$$I = \left\| \frac{v}{\lg R/\sigma} \right\|^p_{L_{p,-1}(\Omega)} = \int_B |v(X)|^p \lg^{-p}(R/\sigma(X))\, \sigma^{-1}(X)\, \mathrm{d}X .$$

We must estimate I by $\|u\|^p_{W^{(k)}{}_{p,-1+sp}(\Omega)}$. From (5.1), it follows

$$\lg^{-p} \frac{R}{\sigma(X)} \leqq \lg^{-p} \frac{cR}{r(X)}$$

where $r(X) = a(x') - x_N$, and so

$$I \leqq c_2 \int_\Delta \mathrm{d}x' \int_{a(x')-\beta}^{a(x')} |v(x', x_N)|^p \lg^{-p}(cR/r(x', x_N)) \cdot \frac{1}{r(x', x_N)}\, \mathrm{d}x_N .$$

The inner integral we can – substituting $a(x') - x_N = t$ – write in the form ($\beta < 1$!)

$$J(x') = \int_0^1 |v(x', a(x') - t)|^p \lg^{-p}(cR/t) \cdot t^{-1}\, \mathrm{d}t$$

and, using the inequality (4.4) for $\gamma = -p$, we obtain

$$J(x') \leqq \left(\frac{p}{p-1}\right)^p \int_0^1 \left|\frac{\partial v}{\partial x_N}(x', a(x') - t)\right|^p t^{-1+p} \, dt =$$

$$= \left(\frac{p}{p-1}\right)^p \int_{a(x')-\beta}^{a(x')} \left|\frac{\partial v}{\partial x_N}(x', x_N)\right|^p [a(x') - x_N]^{-1+p} \, dx_N .$$

Integrating by x' over Δ we have, with respect to (5.1) and $\partial v/(\partial x_N) = D^{k-s+1}u$, the inequality

$$\left\|\frac{D^j u}{\lg R/\sigma}\right\|^p_{L_{p,-1}(\Omega)} \leqq c_3 \|D^{k-s+1}u\|^p_{L_{p,-1+p}(\Omega)} \quad (|j| = k - s) .$$

The right hand side we estimate with the help of (5.5) and obtain

$$\left\|\frac{D^j u}{\lg R/\sigma}\right\|^p_{L_{p,-1}(\Omega)} \leqq c_4 \|u\|^p_{W^{(k)}_{p,\alpha}(\Omega)} ; \quad |j| = k - s . \tag{5.6}$$

c) Let $|j| = k - s - 1$; we want to estimate the norm of $v/\lg (R/\sigma)$ in $L_{p,-1-p}(\Omega)$. By the same way as in case b) we have

$$\left\|\frac{v}{\lg R/\sigma}\right\|^p_{L_{p,-1-p}(\Omega)} \leqq c_2 \int_\Delta J(x') \, dx' ,$$

where

$$J(x') = \int_0^1 |v(x', a(x') - t)|^p \lg^{-p} (cR/t) \,.\, t^{-1-p} \, dt .$$

We estimate this integral by inequality (4.3), where we put $\beta = -1 - p, \gamma = -p$; the condition $cR \geqq e^{|2\gamma|/|\beta+1|}$ follows from our assumption $R \geqq e^2/c$. So we have

$$J(x') \leqq 2^p \int_0^1 \left|\frac{\partial v}{\partial x_N}(x', a(x') - t)\right|^p t^{-1} \lg^{-p} (cR/t) \, dt .$$

Now, we return to coordinate x_N and integrate the inequality by x' over Δ; instead of $a(x') - x_N$ we write $\sigma(x', x_N)$ and obtain

$$\left\|\frac{v}{\lg R/\sigma}\right\|^p_{L_{p,-1-p}(\Omega)} \leqq c_5 \left\|\frac{D^{k-s}u}{\lg R/\sigma}\right\|^p_{L_{p,-1}(\Omega)}$$

because $\partial v/(\partial x_N) = D^{k-s}u$. The norm on the right hand side of the latter inequality we estimate by (5.6) and obtain

$$\left\|\frac{D^j u}{\lg R/\sigma}\right\|^p_{L_{p,-1-p}(\Omega)} \leqq c_6 \|u\|^p_{W^{(k)}_{p,\alpha}(\Omega)} ; \quad |j| = k - s - 1 . \tag{5.7}$$

By a similar manner we estimate (with the help of (4.3)) also the derivatives of smaller order and obtain

$$\left\| \frac{D^j u}{\lg R/\sigma} \right\|^p_{L_{p,-1-np}(\Omega)} \leqq c \|u\|^p_{W^{(k)}_{p,\alpha}(\Omega)}$$

for $|j| = k - s - n$; $n = 1, 2, \ldots, k - s$. These inequalities and (5.6) and (5.5) say that

$$\|u\|_{V_s} \leqq c_8 \|u\|_{W^{(k)}_{p,\alpha}(\Omega)} .$$

From here it follows $\mathring{W} \subset V$.

II) We prove $V \subset \mathring{W}$.

Let $u \in V$ and put $u_h(X) = G_h(X)\, u(X)$ where G_h is the function from (4.12). The function u_h has a compact support in Ω and it follows from the property B (see Section 4.5) that $u - u_h = 0$ outside of the set

$$P_h = \{X \in \Omega \mid \sigma(X) \geqq Re^{-4/3h}\} .$$

For bounded Ω and $\alpha_1 < \alpha_2$ we have $\sigma^{\alpha_1} \lg^{-p}(R/\sigma) \geqq c_9 \sigma^{\alpha_2}$ and so $D^j u \in L_{p,\alpha}(\Omega)$ for $|j| \leqq k - s$. For derivatives $D^j u$ with $|j| > k - s$ we use results of the proof of Theorem 1 and obtain $u \in W^{(k)}_{p,\alpha}(\Omega)$ and so $V \subset W^{(k)}_{p,\alpha}(\Omega)$ ($\alpha = sp - 1$!).

We shall show $\|u - u_h\|_{W^{(k)}_{p,\alpha}(\Omega)} \to 0$ as $h \to 0$. It is sufficient to integrate over P_h; obviously $\operatorname{mes}(P_h) \to 0$ as $h \to 0$.

Let $|i| \leqq k$; we use (2.4) where we write G_h instead of F_h, and obtain

$$\|D^i u(1 - G_h)\|^p_{L_{p,\alpha}(\Omega)} \leqq \int_{P_h} |D^i u|^p \sigma^\alpha(X)\, dX \to 0 \quad \text{as} \quad h \to 0$$

because $u \in W^{(k)}_{p,\alpha}(\Omega)$ namely $D^i u \in L_{p,\alpha}(\Omega)$.

Further, using (4.13)

$$I_{m,n}(h) = \|D^m u D^n G_h\|_{L_{p,\alpha}(\Omega)} = \int_{P_h} |D^m u|^p \,.\, |D^n G_h|^p \sigma^\alpha(X)\, dX \leqq$$

$$\leqq c_{10} \int_{P_h} |D^m u|^p \lg^{-p}(R/\sigma) \,.\, \sigma^{\alpha - |n|p}\, dX \leqq$$

$$\leqq c_{11} \int_{P_h} |D^m u|^p \lg^{-p}(R/\sigma) \,.\, \sigma^{|\alpha| - (k - |m|)p}(X)\, dX$$

for $m + n = i$, $|n| \geqq 1$.

If $|m| \leqq k - s$ then the latter integral converges to zero because $u \in V_s$ and so $D^m u/\lg(R/\sigma) \in L_{p,\alpha-(k-|m|)p}$. If $|m| > k - s$, then this integral is smaller or equal to

$$c_{12} \int_{P_h} |D^m u|^p \sigma^{\alpha - (k - |m|)p}\, dX$$

because R is sufficiently great, and from the fact, that $u \in V$ and so $D^m u \in L_{p,\alpha-(k-|m|)p}$ it follows, that the latter integral also converges to zero.

The rest of the proof is the same as in the proof of Theorem 1: regularizing the function u_h we obtain a function $v \in \mathcal{D}(\Omega)$ which aproximates u in the norm of the space $W^{(k)}_{p,\alpha}(\Omega)$ and so $u \in \mathring{W}$ i.e. $V \subset \mathring{W}$.

Remark 1. In this Section we were using the weight function which was a power of $\sigma(X)$. This is obviously unessential; if we change the constant R in a suitable manner, we can prove similar assertions for spaces V_s where we suppose $D^{k-s-n}u/\lg(R/\varrho) \in \in L_{p,-1-np}$ with the weight function $\varrho^{-1-np}(X)$ (see [1]).

Remark 2. It is easily seen from part II of the proof of Theorem 3 that we could use the function G_h instead of F_h in the proof of Theorem 1.

References

[1] *О. В. Бесов, Я. Кадлец, А. Куфнер:* О некоторых свойствах весовых классов, ДАН СССР *171* (1966), 514—516.

[2] *J. Kadlec* and *A. Kufner:* Characterization of functions with zero traces by integrals with weight functions I, Čas. pěst. mat. 91 (1966), 463—471.

[3] *J. Nečas:* Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle, Ann. Scuola Norm. Sup. Pisa, ser. 3, 16, *4* (1962), 305—326.

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Výtah

CHARAKTERIZACE FUNKCÍ S NULOVÝMI STOPAMI POMOCÍ INTEGRÁLŮ S VAHOU II

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Práce navazuje bezprostředně na část I stejnojmenného článku autorů [2]. Vedle několika pomocných vět je v práci dokázána jistá modifikace Hardyho nerovnosti — lemma 3, nerovnosti (4.3) a (4.4) — a v odstavci 4.5 je zavedena a zkoumána důležitá pomocná funkce G_h.

Dále je zaveden prostor V_s funkcí u definovaných skoro všude na Ω, jejichž zobecněné derivace $D^j u$ řádu $|j|$, $0 \leqq |j| \leqq k$ mají následující vlastnosti

$$(D^j u) \lg(R/\sigma) \in L_{p,\alpha-(k-|j|)p}(\Omega) \quad \text{pro} \quad |j| = 0, 1, \ldots, k-s\,;$$

$$D^j u \in L_{p,\alpha-(k-|j|)p}(\Omega) \quad \text{pro} \quad |j| = k-s+1, k-s+2, \ldots, k\,;$$

přitom je všude $\alpha = sp - 1$, kde s je některé z čísel $1, 2, \ldots, k$. Norma v prostoru V_s je definována vzorcem (5.4). Funkce $\sigma(X)$ je ekvivalentní s vzdáleností $\varrho(X)$ bodu $X \in \Omega$ od hranice $\partial\Omega$ oblasti Ω (viz odst. 4.5); je třeba zdůraznit, že v prostorech $L_{p,\gamma}(\Omega)$ vystupuje jako váhová funkce vždy příslušná mocnina $\sigma^\gamma(X)$ funkce $\sigma(X)$. R je vhodně zvolená kladná konstanta.

Ve větě 3, která je doplněním věty 1 z [2], je dokázáno, že pro $\alpha = sp - 1$ jsou prostory V_s a $\mathring{W}$ (kde $\mathring{W}$ je uzávěr nekonečně diferencovatelných funkcí s kompaktním nosičem v Ω v normě (0.2)) totožné a že pro $u \in \mathring{W}$ jsou normy (0.2) a (5.4) ekvivalentní.

Резюме

ХАРАКТЕРИСТИКА ФУНКЦИЙ С НУЛЕВЫМИ СЛЕДАМИ ПРИ ПОМОЩИ ИНТЕГРАЛОВ С ВЕСОМ II

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Работа является непосредственным продолжением статьи авторов [2]. Кроме некоторых вспомогательных утверждений доказывается в работе некоторое видоизменение неравенства Харди – лемма 3, неравенства (4.3) и (4.4) – и в п. 4.5 введена и исследована важная вспомогательная функция G_h.

В дальнейшем введено пространство V_s функций u, определенных почти всюду в Ω, обобщенные производные $D^j u$ порядка $|j|$ $(0 \leqq |j| \leqq k)$, которые имеют следующие свойства:

$$(D^j u) \lg R/\sigma \in L_{p,\alpha-(k-|j|)p}(\Omega) \quad \text{для} \quad |j| = 0, 1, \ldots, k - s\,;$$

$$D^j u \in L_{p,\alpha-(k-|j|)p}(\Omega) \quad \text{для} \quad |j| = k - s + 1,\ k - s + 2, \ldots, k\,;$$

при этом всюду $\alpha = sp - 1$, где s равно одному из чисел $1, 2, \ldots, k$. Норма в пространстве V_s определена формулой (5.4). Функция $\sigma(x)$ эквивалентна расстоянию $\varrho(X)$ точки $X \in \Omega$ от границы $\partial\Omega$ области Ω (см. п. 4.5); нужно отметить, что в пространстве $L_{p,\gamma}(\Omega)$ весовой функцией является всегда степень $\sigma^\gamma(x)$ функции $\sigma(x)$. R – достаточно большая положительная константа.

В теореме 3, являющейся дополнением теоремы 1 из [2], доказано, что для $\alpha = sp - 1$ имеет место тождество $V_s = \mathring{W}$ (где $\mathring{W}$ – замыкание бесконечно дифференцируемых финитных в Ω функций по норме (0.2)) и что для $u \in \mathring{W}$ нормы (0.2) и (5.4) эквивалентны.