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DANDELIN'S FIGURE IN *n*-SPACE

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Some figures known from the projective geometry are generalized in an n-space.

Abstract. A pair of perspective simplexes S, S' in a complex projective space of n dimensions, or *briefly* in an n-space, are always polar reciprocal of each other with respect to a quadric Q (cf. [1], pp. 218, 251; [14]). A particular case of interest arises when n vertices of either simplex lie in their corresponding n primes which are therefore the tangent hyperplanes of Q there, such that the $n(n-1)$ joins of the non-corresponding vertices of the $(n+1)$-th pair of corresponding prime faces of S, S' (being a pair of non-tangent hyperplanes of Q) are $n(n-1)$ generators of Q. We can initiate the whole figure from these $n(n-1)$ generators too. For $n = 3$, it becomes Dandelin's figure [2; 7; 8] of six generators of Q and hence we name it the *Dandelin's figure of $n(n-1)$ lines in an n-space.* It is also indicated here as a consequence how Pascal's theorem for a conic and its dual Brianchon's theorem have an analogue in an n-space [4; 14].

1. PERSPECTIVE $(n-1)$-SIMPLEXES

We are already familiar with the method of symbols ([2], pp. 6–44; [3], pp. 115 to 160; [6]; [9–11]; [13]) for points. We use this method here to prove first the following theorem:

Theorem 1. *Let a, a' be a pair of $(n-1)$-dimensional simplexes, or briefly of $(n-1)$-simplexes,*[1]) *in an n-space, perspective from a point O, A be the point common to the n hyperplanes determined by the n $(n-2)$-spaces of a joined respectively to the n vertices of a' opposite their corresponding n $(n-2)$-spaces, and A' common to the n hyperplanes determined similarly by the n $(n-2)$-spaces of a' joined respectively to the n vertices of a opposite their corresponding n $(n-2)$-spaces. A and A' are then collinear with O* (cf. [18], [19]).

[1]) B. SEGRE calls it an "n-simplex" in his "Lectures on Modern Geometry". Roma 1961.

Proof. Let A_i $(i = 1, \ldots, n)$ be the n vertices of the $(n-1)$-simplex a and A_i' of a', a_i be the $(n-2)$-space of a opposite a vertex A_i and a_i' of a' opposite A_i' such that every pair of corresponding vertices A_i and A_i' of a, a' are collinear with O and therefore a_i corresponds to a_i'; the symbols of the n pairs of points A_i, A_i' and O are then related as

$$\text{(i)} \qquad A_i' - A_i = O ;$$

let U, U' be the points represented by

$$\text{(ii)} \qquad U = A_1 + \ldots + A_n , \quad U' = A_1' + \ldots + A_n' .$$

Then

$$\text{(iii)} \qquad M_i = U - A_i , \quad M_i' = U' - A_i'$$

are points lying respectively in the $(n-2)$-spaces a_i and a_i' such that the two pairs of points U, U' and M_i, M_i' are each collinear with O and are related as

$$\text{(iv)} \qquad U' - U = n \,.\, O , \quad M_i' - M_i = (n-1) \,.\, O .$$

Now A is common to the n hyperplanes $A_i'a_i$ and therefore let the n joins AA_i' meet the n $(n-2)$-spaces a_i respectively in the n points M_i. Or, the n joins $A_i'M_i$ concur at A. Thus, the symbol for A may be taken as

$$\text{(v)} \qquad A = A_i' + M_i = O + U ,$$

and similarly for A' as

$$\text{(vi)} \qquad A' = A_i + M_i' = U' - O .$$

Hence

$$\text{(vii)} \qquad A' - A = U' - U - 2 \,.\, O = (n-2) \,.\, O .$$

Thus A, A' are collinear with O, proving the proposition.

2. PERSPECTIVE SIMPLEXES

$S = Aa$, $S' = A'a'$ (§ 1) now form a pair of simplexes, in the n-space, perspective from O and therefore are polar reciprocal of each other with respect to a quadric Q such that the vertex A of S is the pole of the hyperplane p' of the $(n-1)$-simplex a', the vertex A' of S' is the pole of the hyperplane p of a, the n vertices A_i of S are respectively the poles of the n primes $A'a_i'$ of S' and the n vertices A_i' of S' are respectively the poles of the n primes Aa_i of S with respect to Q.

Again, by definition (Theorem 1), A is common to the n hyperplanes $A_i'a_i$ and therefore every hyperplane Aa_i of the simplex S contains respectively the vertex A_i' of S'.

Now Aa_i is the polar hyperplane of A'_i with respect to the quadric Q and is therefore its tangent hyperplane there.

The n hyperplanes $A'a'_i$ of the simplex S' are similarly related with the n vertices A_i of S: every hyperplane $A'a'_i$ is a tangent prime of Q at A_i as desired.

We may observe further that the join of a vertex A_i of the $(n-1)$-simplex a to a non-corresponding vertex A'_j of a' lies, obviously, in the two hyperplanes $A'a'_i$, Aa_j respectively tangent to Q at A_i and A'_j $(j = 1, \ldots, n;\ i \neq j)$. Thus the line $A_iA'_j$ touches Q at its two distinct points A_i and A'_j. Such is the case only when it is a generator of Q. Thus, the $n(n-1)$ joins $A_iA'_j$ or A'_iA_j are $n(n-1)$ generators of Q. Hence, we have the following theorem:

Theorem 2. *Let a, a' be a pair of perspective $(n-1)$-simplexes in an n-space and A, A' be the pair of points as constructed in Theorem 1. Then the pair of perspective simplexes $S = Aa$, $S' = A'a'$ are polar reciprocal of each other with respect to a quadric Q, with the $n(n-1)$ joins of the non-corresponding vertices of a, a' as its $n(n-1)$ generators; furthermore, n vertices of S, namely those of a, lie respectively in their corresponding n primes of S' concurrent at A', and n vertices of S', namely those of a', also lie respectively in their corresponding n primes of S concurrent at A.*

3. THE QUADRIC Q

3.1. A quadric Q in an n-space is determined by $n(n+3)/2$ linearly independent conditions or can be made to pass through an equal number of independent points [14; 15].

Now every pair of lines $A_iA'_i$, A'_jA_j meet in the point O (§ 1), so that every four points A_i, A_j, A'_i, A'_j are coplanar. Therefore, every pair of joins $A_iA'_j$, A'_iA_j meet in a point M^{ij}, and A_iA_j, $A'_iA'_j$ in L^{ij}, say. Obviously, there are in all $n(n-1)/2$ points like M^{ij} and the same number of points like L^{ij}.

Thus, we can construct a unique quadric Q to pass through the $2n$ vertices A_i, A'_i of the pair of given $(n-1)$-simplexes a, a' perspective from O and the $n(n-1)/2$ points M^{ij}. The three points A_i, A'_j, M^{ij} of every line $A_iA'_j$ lie on Q and therefore this line is a generator of Q. Hence, the polar hyperplane of every vertex A_i of a with respect to Q is its tangent prime there, determined by its $n-1$ generators $A_iA'_j$ through A_i; and that of every vertex A'_i of a' is its tangent prime there determined by its $n-1$ generators A'_iA_j through A'_i.

3.2. Let A''_i be the n points, conjugate to O for Q, on the n lines $A_iA'_i$, concurrent at O, such that $(OA_iA''_iA'_i) = -1$, or following the notations of H. S. M. Coxeter [5], $H(OA''_i, A_iA'_i)$. It is then apparent from the harmonic property of the quadrangle $A_iA'_iA'_jA_j$ that every pair of points A_i, A_j lie on the join $L^{ij}M^{ij}$ such that $H(A''_iA''_j, L^{ij}M^{ij})$.

Thus, the n points A_i'' form an $(n-1)$-simplex a'' in the polar hyperplane p'' of O for Q such that the $n(n-1)/2$ pairs of points L^{ij}, M^{ij} on the $n(n-1)/2$ edges of a'' form the $n(n-1)/2$ pairs of opposite vertices of an $(n-1)$-dimensional *S-configuration* [12] with a'' as its *diagonal simplex*; the $n(n-1)/2$ points L^{ij} lie in one of the 2^{n-1} space $(n-2)$-spaces, say p_1 which is obviously the $(n-2)$-space of perspectivity of the pair of the given $(n-1)$-simplexes a, a' perspective from O.

Thus the three hyperplanes p, p', p'' have p_1 as the common $(n-2)$-space of perspectivity of the three $(n-1)$-simplexes a, a', a'' lying therein respectively, and perspective from the same center O.

Hence, if A, A' be the respective poles of p', p with respect to Q, they are collinear with O as the pole of p'' for Q, and this completes the construction of the pair of simplexes $S = Aa$, $S' = A'a'$ perspective from O and polar reciprocal of each other with respect to Q as desired. Hence we have the following theorem:

Theorem 3. *If a, a' be a pair of $(n-1)$-simplexes, in an n-space, perspective from a point O, the $n(n-1)$ joins of the non-corresponding vertices of a, a' generate a quadric Q. If A', A be the respective poles of the pair of the hyperplanes p, p' of a, a' with respect to Q, the pair of simplexes $S = Aa$, $S' = A'a'$ are perspective to each other from O and polar reciprocal of each other with respect to Q as in the Theorem 2 above.*

Definition. For $n = 3$, this gives Dandelin's figure ([2], p. 45) of six generators. Hence we define its analogue or extension here as *Dandelin's figure of $n(n-1)$ generators* in an n-space.

3.3. Every pair of corresponding vertices A_i, A_i' of a, a' are obviously coplanar with the corresponding pair of vertices A, A' of S, S'. Therefore, the n pairs of corresponding edges AA_i, $A'A_i'$ of S, S' meet respectively in n points L^i, say, as the n poles of the n hyperplanes Oa_i or Oa_i' with respect to Q. For AA_i, $A'A_i'$ are seen to be the polar lines, for Q, of the pair of the corresponding $(n-2)$-spaces a_i', a_i respectively; these, being perspective to each other from O, lie in the same hyperplane through O. Hence, L^i all lie in the polar hyperplane p'' of O for Q (3.2).

Thus p'' coincides with the prime of perspectivity of S, S' such that the $n(n+1)/2$ points L^i and L^{ij} of intersection of the $n(n+1)/2$ pairs of the corresponding edges of S, S' lie therein by threes on $(n+1)\,n(n-1)/6$ lines [16]. For example, every three points L^{ij}, L^{jk}, L^{ki} *are collinear*, and same is the case with every three points L^i, L^j, L^{ij}.

In fact, O, A, A', A_i, A_i', L^i, L^{ij} form a figure of $(n+2)(n+3)/2$ points lying by threes on $(n+1)(n+2)(n+3)/6$ lines and by $\binom{n+1}{2}$s in $(n+2)(n+3)/2$ hyperplanes such that $n+1$ lines and $n(n+1)/2$ hyperplanes pass through each point. The whole figure is self-reciprocal for the quadric Q, and the $n+1$ vertices of either simplex, S or S', make a *self-conjugate* $(n+2)$ *ad* of points with O for Q

such that the line joining any two contains the pole of the hyperplane determined by the remaining n points for Q (cf. [2], pp. 34–41, Exs. 5–6; [3], pp. 148–149, Exs. 21–22; 9).

3.4. Following H. F. BAKER ([2], p. 46), it can be proved that the equation of the quadric Q, referred to the simplex S replacing A by A_0, is of the form

$$\text{(viii)} \qquad (2-n)(x_0^2 - x_1^2 - \dots - x_n^2) = (x_0 - x_1 - \dots - x_n)^2 .$$

This is the particular case of the equation

$$\text{(ix)} \qquad (1 + b_0 + \dots + b_n)(b_0x_0^2 + \dots + b_nx_n^2) = (b_0x_0 + \dots + b_nx_n)^2$$

for $b_0 = 1,\ b_1 = \dots = b_n = -1$.

The equation (ix) represents the quadric for which S reciprocates into a simplex whose vertex, referred to S, corresponding to the vertex A_i of S has $n+1$ coordinates as

$$x_i = 1 + 1/b_i , \quad x_j = 1 \quad (i, j = 0, 1, \dots, n;\ 1 \neq j) .$$

4. PASCAL'S THEOREM

Let us take a section of Dandelin's figure in an n-space by a hyperplane h such that it meets the n concurrent lines OA_i in n points B_i which determine an $(n-1)$-simplex b, and the $n(n-1)$ generators $A_iA'_j$ of the quadric Q in $n(n-1)$ points C_i^j which then lie on the $(n-2)$-quadric section q of Q by h.

Obviously every pair of points C_i^j, C_j^i lie on the edge B_iB_j of b. For, B_iB_j is the section of the plane OA_iA_j which contains the pair of generators $A_iA'_j$, A'_iA_j of Q.

The section of the hyperplane A_ia_i is an $(n-2)$-space c_i (say) determined by the $n-1$ points C_i^j on the $n-1$ concurrent edges B_iB_j of b through B_i, and that of Oa'_i is the $(n-2)$-space b_i of b opposite its vertex B_i. Therefore the $(n-3)$-space section d_i of a'_i is common to the two $(n-2)$-spaces b_i, c_i. All the n $(n-3)$-spaces d_i then lie in an $(n-2)$-space section d of the hyperplane p' of the $(n-1)$-simplex a' (§ 2) which contains all the n $(n-2)$-spaces a'_i.

Hence, the n $(n-2)$-spaces c_i form an $(n-1)$-simplex c perspective to b with d as the $(n-2)$-space of perspectivity of the two $(n-1)$-simplexes b and c such that every point C_i^j occurs once only, and all the $n(n-1)$ such points are accounted for. Thus we have the following theorem:

Theorem 4. *If b, c are a pair of perspective simplexes in an $(n-1)$-space, let the $n-1$ points of intersection of the $n-1$ concurrent edges of either simplex, say b, through every vertex B_i of b with the $(n-2)$-space c_i of the other simplex, namely c, corresponding to the $(n-2)$-space b_i of b opposite B_i be marked. The $n(n-1)$ such marked points, lying in pairs on the $n(n-1)/2$ edges of b, lie on a quadric q.*

Remark. *Conversely, if the* $n(n-1)$ *points of intersection of a quadric q with the* $n(n-1)/2$ *edges of a simplex b in an* $(n-1)$*-space be distributed by* $(n-1)$*-s in n* $(n-2)$*-spaces* c_i *such that the* $n-1$ *points of every* c_i *lie on the* $n-1$ *concurrent edges of b through a vertex* B_i *of b, one on each edge* (such a distribution of the $n(n-1)$ points is obviously always possible in $2^{n(n-1)/2}$ ways, for there are two choices for either of the two points of intersection on every edge of b independent of one another), *the behaviour of the n space* $(n-2)$*-spaces* c_i *is not unique.*

They may form a simplex c which need not necessarily [14] be perspective to b in the sense of Theorem 4 unless $n = 3$ in which case it becomes the well known Pascal's theorem for a conic. They may not form a simplex at all, but may be concurrent or even coaxial.

Hence we may say that Theorem 4 is a *partial analogue in an* $(n-1)$*-space of Pascal's theorem for a conic.*

For $n = 4$, N. A. Court [4] has discussed in detail the different cases in regard to the behaviour of the twelve points of intersection of a quadric with the six edges of a tetrahedron (cf. [2], pp. 53–54, Ex. 15).

For higher values of n, the discussion of the several cases arising therefrom forms the subject matter of another paper [17].

It is now not difficult to formulate the *partial analogue in an n-space of Pascal's theorem for a conic* and thus we have the following corollary:

Corollary. *If the* $n+1$ *primes* b_i *of a simplex b be respectively parallel to the* $n+1$ *primes* c_i *of another simplex c in an n-space, the* $n(n+1)$ *points of intersection of the edges of either simplex with the primes of the other lie on a quadric.*

For, every b_i is parallel to the $n(n-1)/2$ edges of c lying in c_i and meets only the other n edges of c concurrent at its vertex C_i opposite c_i, and similarly every c_i meets only the n concurrent edges of b through its vertex B_i opposite b.

5. BRIANCHON'S THEOREM

We are now in a position to state the *partial analogue in an* $(n-1)$*-space of Brianchon's theorem for a conic* as the dual of the Theorem 4. That in an n-space will be the following theorem:

Theorem 5. *If b, c are a pair of perspective simplexes in an n-space, let the n hyperplanes joining the coprimal n* $(n-2)$*-spaces of either simplex, say b, lying in every prime* b_i *of b to the vertex* C_i *of the other simplex, namely c, corresponding to the vertex* B_i *of b opposite* b_i *be constructed. These* $n(n+1)$ *hyperplanes, passing in pairs through the* $n(n+1)/2$ $(n-2)$*-spaces of b, envelop a quadric.*

Remark. *Conversely, if the* $n(n+1)$ *tangent hyperplanes of quadric through the* $n(n+1)/2$ $(n-2)$*-spaces of a simplex b in an n-space be distributed into* $n+1$

groups of n each such that the n hyperplanes of each group pass through the n coprimal $(n-2)$*-spaces of b lying in a prime* b_i *of b, one through each* $(n-2)$*-space, and concur at a point* C_i (such a distribution of the $n(n+1)$ hyperplanes is obviously always possible in $2^{n(n-1)/2}$ ways, for there are two choices for either of the two hyperplanes through every $(n-2)$-space of b independent of one another), *the behaviour of the* $n+1$ *points* C_i *is not unique. They may form a simplex c which need not necessarily be in* [14] *perspective to b in the sense of the Theorem* 5 *unless* $n = 2$ *in which case it becomes the well known Brianchon's theorem for a conic. They may not form a simplex at all, but may be coprimal or lie even in a lower space.*

For $n = 3$, N. A. COURT [4] has discussed in detail the different cases in regard to the behaviour of the twelve tangent planes of a quadric through the six edges of a tetrahedron (cf. [2], p. 54).

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Výtah

DANDELINŮV ÚTVAR V n-ROZMĚRNÉM PROSTORU

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V článku se zobecňují některé útvary známé z trojrozměrné a dvojrozměrné projektivní geometrie na n-rozměrný případ. Jedná se jednak o zobecnění tzv. Dandelinovy skupiny šesti vytvořujících přímek kvadriky, jednak o částečné zobecnění Pascalovy a Brianchonovy věty.

Резюме

ФИГУРА ДАНДЕЛИНА В n-МЕРНОМ ПРОСТРАНСТВЕ

САГИБ РАМ МАНДАН (Sahib Ram Mandan), Харагпур (Индия)

В статье обобщаются некоторые фигуры, известные из проективной геометрии трехмерного пространства и плоскости, на n-мерный случай. Во-первых, обобщается т. наз. группа Данделина шести производящих прямых поверхности второго порядка, во-вторых, частично обобщаются теоремы Паскаля и Брианшона.