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A REMARK ON $\prod$-AUTOMORPHISMS

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In this paper one problem of S. M. Ulam is solved.

S. M. ULAM in his book [1], page 18 of the Russian translation, defines the $\prod$-isomorphism and the $\prod$-automorphism in the following manner. Let E be a set, $m \geqq 2$ a positive integer. The $\prod$-isomorphism is defined as a one-to-one mapping which maps each element of the Cartesian power E^m with coordinates $[x_1, x_2, \ldots, x_m]$ onto the element of E^m with coordinates $[f(x_1), f(x_2), \ldots, f(x_m)]$, where f is some mapping of the set E onto E again. When a set $A \subset E^m$ is given, then a $\prod$-automorphism of the set A is defined as a $\prod$-isomorphism which maps A again onto A. Now in [1] the question is posed whether, for every positive integer n, there exists a set A in E^m which has exactly n $\prod$-automorphisms different on A (one assumes that E is infinite).

The answer to this question is affirmative, even in the case that E has a finite number of at least n elements, and in the case that E is infinite and instead of a finite n the cardinal number $\aleph_0$ is given.

Let a positive integer n be given, and let N be the set of residue classes modulo n (we shall denote its elements by the representatives of these classes). In the set E choose distinct elements $\hat{a}_i$ for all $i \in N$. To each $i \in N$ assign that element a_i of E^m whose first coordinate is $\hat{a}_i$ and all other coordinates are equal to $\hat{a}_{i+1}$. Denote the set of all elements $\hat{a}_i$ (or a_i) for $i \in N$ by $\hat{A}$ (or A respectively). To each $k \in N$ assign the mapping $\hat{f}_k$ of E onto E defined by the equations

$$\hat{f}_k(\hat{a}_i) = \hat{a}_{i+k} \quad \text{for all} \quad i \in N\,,$$
$$\hat{f}_k(x) = x \quad \text{for} \quad x \in E - \hat{A}\,.$$

It is easily verified that each $\hat{f}_k$ induces a $\prod$-automorphism of the set A, and that for different k the $\prod$-automorphisms f_k induced by the mappings $\hat{f}_k$ are different in A. The number of these $\prod$-automorphisms is exactly n. Therefore it remains to prove that there are no further $\prod$-automorphisms of the set A (when we do not consider the values on $E^m - A$).

Let g be some $\prod$-automorphism of A induced by the mapping $\hat{g}$ of E onto E. Let $g(a_0) = a_l$, where $l \in N$. This means that $\hat{g}(\hat{a}_0) = \hat{a}_l$, $\hat{g}(\hat{a}_1) = \hat{a}_{l+1}$ as $\hat{a}_0$, $\hat{a}_1$

are coordinates of the element a_0 and $\hat{a}_l$, $\hat{a}_{l+1}$ are the corresponding coordinates of the element a_l. As a_1 has the first coordinate $\hat{a}_1$ and g is a $\prod$-automorphism of A, the image of a_1 in g must be such an element of A, whose first coordinate is $\hat{a}_{l+1}$. But the only such element is a_{l+1} and so $g(a_1) = a_{l+1}$. Then obviously $\hat{g}(\hat{a}_2) = \hat{a}_{l+2}$ must hold and therefore $g(a_2) = a_{l+2}$. We may continue thus and after a finite number of steps we prove that

$$\hat{g}(\hat{a}_i) = \hat{a}_{i+l} \quad \text{for all} \quad i \in N$$

and therefore $\hat{g} = \hat{f}_l$ on $\hat{A}$; hence $g = f_l$ on A. This is the proof for positive integral n. In the case that instead of n the cardinal number $\aleph_0$ is given, we can construct the set A analogously, only instead of N we must take the set of all integers. The proof of the theorem is also analogous (in proving $g = f_l$ we must progress from zero both to the positive and the negative numbers).

References

[1] *Ulam, S. M.:* A Collection of Mathematical Problems. The Russian translation: Нерешенные математические задачи, Москва 1964.

Výtah

POZNÁMKA O $\prod$-AUTOMORFISMECH

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V tomto článku je dána kladná odpověď na otázku z [1], zda ke každému přirozenému n při dané množině E a přirozeném čislu $m \geqq 2$ existuje množina A v E^m, která má právě n $\prod$-automorfismů různých na A (přičemž E obsahuje alespoň n prvků).

Резюме

ЗАМЕТКА О $\prod$-АВТОМОРФИЗМАХ

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В этой статье дан положительный ответ на вопрос из [1], существует ли для всякого натурального n при заданном множестве E и натуральном числе $m \geqq 2$ множество A в E^m, которое обладает точно n $\prod$-автоморфизмами, различными на A (причем E имеет по меньшей мере n элементов).