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PERIODIC SOLUTIONS OF KIRCHHOFF'S NETWORKS

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In this paper some conditions for the existence of periodic solutions of Kirchhoff's networks introduced in [1], are presented.

The concepts and symbols used in this paper will have the same meaning as those introduced in [1].

Let $\bar{\mathbf{D}}$ be the set of all (complex) one-dimensional Schwartz distributions. Let $f \in \bar{\mathbf{D}}$ and let us define on $\mathbf{K}$ (the set of all infinitely differentiable functions $\varphi(t)$ with compact support) the functional $f^{(-1)}$ by the relation

(1)
$$(f^{(-1)}, \varphi) = \left(f, -\int_{-\infty}^{t} \varphi(\tau)\,d\tau + \left(\int_{-\infty}^{\infty} \varphi(\tau)\,d\tau\right)\int_{-\infty}^{t} \varphi_0(\tau)\,d\tau\right) + \bar{C}\int_{-\infty}^{\infty} \varphi(\tau)\,d\tau\,,$$

where $\varphi_0(t)$ is a fixed function belonging to $\mathbf{K}$, which satisfies the relation $\int_{-\infty}^{\infty} \varphi_0(\tau) \,.\, d\tau = 1$, and $\bar{C}$ is a constant.

It can be easily verified that the following statements are true: a) $f^{(-1)} \in \bar{\mathbf{D}}$, b) $(f^{(-1)})' = f$, c) *two distributions defined by* (1) *for the same* f *and any* $\varphi_0(t)$ *and* $\bar{C}$ *differ by a constant*, d) $(f')^{(-1)} = f + K$, *K being a constant*, e) *if* $f \in \bar{\mathbf{D}}$ *is regular then* $f^{(-1)}$ *is also regular, the corresponding function being* $\int_0^t f(\tau)\,d\tau + K$.

In view of statements b), d), e) $f^{(-1)}$ will be called the primitive distribution to f.

If $P(\xi) = a_n\xi^n + a_{n-1}\xi^{n-1} + \ldots + a_0$ (a_i being numbers), let us define the operator $P(D)$ on $\bar{\mathbf{D}}$ by the equation $P(D)\,x = a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_0 x$. Defining the sum and the product of two operators defined on $\bar{\mathbf{D}}$ in the usual manner, it can be easily verified that *the product of any two operators $P_1(D)$, $P_2(D)$ is commutative.*

If $f \in \bar{\mathbf{D}}$, $T > 0$, let the functional f_T be defined on $\mathbf{K}$ by

(2)
$$(f_T, \varphi(t)) = (f, \varphi(t + T))\,.$$

Obviously, $f_T \in \bar{\mathbf{D}}$ and $Df_T = (Df)_T$, $(\exp \alpha t)_T = \exp \alpha(t - T)$.

The distribution $f \in \bar{\mathbf{D}}$ will be called T-periodic, if $f = f_T$. Let $\bar{\mathbf{D}}_T$ be the set consisting of all T-periodic distributions. It is clear that if $f, g \in \bar{\mathbf{D}}_T$, then $f + g$, $\alpha f, f' \in \bar{\mathbf{D}}_T$ (α being a number).

Lemma 1. *Let α be a number, $\omega > 0$, $f \in \bar{\mathbf{D}}_T$ with $T = 2\pi/\omega$; if $\alpha \neq in\omega$ for $n = 0, \pm 1, \pm 2, \ldots$, then there is an $x \in \bar{\mathbf{D}}_T$ satisfying the equation*

$$(D - \alpha)\, x = f\,. \tag{3}$$

Moreover, if f is regular, then x is also regular and the corresponding function $x(t)$ has a local integrable derivative (in the usual sense) almost everywhere.

Proof. First note that equation (3) has solutions, since the distribution

$$x = e^{\alpha t}(e^{-\alpha t} f)^{(-1)} \tag{4}$$

satisfies (3). Moreover, it can be easily shown that every solution of $(D - \alpha)\, z = 0$ has the form $z = C \exp \alpha t$, C being a constant.

Let x satisfy (3); then we have $(D - \alpha)\, x_T = f_T$, and, consequently, $(D - \alpha) \cdot (x - x_T) = 0$. Thus, $x - x_T = C \exp \alpha t$. Let us put $\tilde{x} = x + K \exp \alpha t$ with $K = - C(1 - \exp(-\alpha T))^{-1}$; evidently, $\tilde{x}$ is a solution of (3) and we have

$$\tilde{x} - \tilde{x}_T = C \exp \alpha t + K(1 - \exp(-\alpha T)) \exp \alpha t = 0\,,$$

i.e. $\tilde{x}$ is T-periodic.

The proof of the second statement is obvious.

From Lemma 1 the subsequent statement follows immediately by induction.

Lemma 2. *Let $\omega > 0$, $P(\xi) \not\equiv 0$ be a polynomial of the n-th degree each root of which is different from the numbers $i\nu\omega$, $\nu = 0, \pm 1, \pm 2, \ldots$, and let $f \in \bar{\mathbf{D}}_T$ with $T = 2\pi/\omega$; then there is a unique distribution $x \in \bar{\mathbf{D}}_T$ satisfying the equation*

$$P(D)\, x = f\,. \tag{5}$$

Moreover, if f is regular, then x is also regular and the corresponding function $x(t)$ has the (usual) locally integrable derivative of the n-th order almost everywhere.

Lemma 3. *Let $M(p)$ be a square matrix whose elements are polynomials in p, and f a vector over $\bar{\mathbf{D}}$; let $d(p) = \det M(p) \not\equiv 0$ and $N(p)$ be the matrix adjoint to $M(p)$, (i.e., $M(p)\, N(p) = N(p)\, M(p) = I \det M(p)$, I being the unit matrix). Furthermore, let $q(p)$ be a common factor of $d(p)$ and all elements of $N(p)$, and let $d(p) = q(p) \cdot \tilde{d}(p)$, $N(p) = q(p)\, \tilde{N}(p)$; then:*

1. *If the vector ξ over $\bar{\mathbf{D}}$ is a solution of the equation $\tilde{d}(D)\, \xi = f$, then the vector $x = \tilde{N}(D)\, \xi$ is a solution of*

$$M(D)\, x = f\,. \tag{6}$$

2. *If the vector x_1 over $\bar{\mathbf{D}}$ is a solution of (6), then there is a solution ξ_1 of the equation $\tilde{d}(D)\, \xi_1 = f$ such that $x_1 = \tilde{N}(D)\, \xi_1$.*

Proof. From the equation $M(p)\, N(p) = N(p)\, M(p) = I\, d(p)$ it follows that $M(p)\, \tilde{N}(p) = \tilde{N}(p)\, M(p) = I\, \tilde{d}(p)$. 1) Let ξ be a solution of $\tilde{d}(D)\, \xi = f$; then for the vector $x = \tilde{N}(D)\, \xi$ we have: $M(D)\, x = M(D)\, (\tilde{N}(D)\, \xi) = (M(D)\, \tilde{N}(D))\, \xi = \tilde{d}(D)\xi =$

$= f$. 2) Conversely, let the vector x_1 be a solution of (6); choosing a solution ξ_0 of $\tilde{d}(D)\,\xi_0 = f$ and putting $x_0 = \tilde{N}(D)\,\xi_0$, we have $M(D)\,x_0 = f$. Consequently,

$$M(D)\,y = 0 \quad \text{with} \quad y = x_1 - x_0\,. \tag{7}$$

Multiplying (7) by $\tilde{N}(D)$ one gets

$$\tilde{d}(D)\,y = 0\,. \tag{8}$$

Let now u be a solution of $\tilde{d}(D)\,u = y$ and put $\eta = M(D)\,u$. Then we have

$$\tilde{N}(D)\,\eta = \tilde{N}(D)\,(M(D)\,u) = (\tilde{N}(D)\,M(D))\,u = \tilde{d}(D)\,u = y\,. \tag{9}$$

Moreover, by (7),

$$\begin{aligned}\tilde{d}(D)\,\eta = \tilde{d}(D)\,(M(D)\,u) = (\tilde{d}(D)\,M(D)\,u = (M(D)\,\tilde{d}(D))\,u = \\ = M(D)\,(\tilde{d}(D)\,u) = M(D)\,y = 0\,.\end{aligned} \tag{10}$$

Thus, according to (9) we have $x_1 = x_0 + y = \tilde{N}(D)\,\xi_0 + \tilde{N}(D)\,\eta = \tilde{N}(D)\,(\xi_0 + \eta)$, where $\tilde{d}(D)\,\xi_0 = f$, $\tilde{d}(D)\,\eta = 0$ by (10); hence $\tilde{d}(D)\,(\xi_0 + \eta) = f$ which completes the proof.

Let us now consider Kirchhoff's networks. (See [1].)

Let $\mathfrak{N} = (G, R, L, S)$ be a K-network; the vector q over $\bar{\mathbf{D}}$ will be called the solution of $\mathfrak{N}$ on the entire time-axis corresponding to the vector e over $\bar{\mathbf{D}}$, if

A 1. $c^{\backprime}(Lq'' + Rq' + Sq) = c^{\backprime}e$ for every cycle $c^{\backprime}h$,

A 2. $a^{\backprime}q = 0$.

Note. The vector e has the physical meaning of the vector of impressed electromotive forces, q of the vector of electrical charges passed through individual branches.

In the same manner as in [1] it can be shown that A 1, A 2 are equivalent to the equation

$$X^{\backprime}(LD^2 + RD + S)\,Xw = X^{\backprime}e \tag{11}$$

with $q = Xw$, X being a constant matrix the columns of which form a complete set of linearly independent solutions of $a^{\backprime}\xi = 0$.

Theorem 1. *Let $\mathfrak{N}$ be a K-network, and e a vector over $\bar{\mathbf{D}}$ such that $l^{\backprime}e \in \bar{\mathbf{D}}_T$ for every loop $l^{\backprime}h$; further, let $\det X^{\backprime}(Lp^2 + Rp + S)\,X \neq 0$ for $p = in\omega$, $n = 0, \pm 1, \pm 2, \ldots$ with $\omega = 2\pi/T$. Then there is a unique solution q over $\bar{\mathbf{D}}_T$ corresponding to e.*

Moreover, if in addition $\mathfrak{N}$ is a passive K-network and $l^{\backprime}e$ is a regular distribution for every loop $l^{\backprime}h$, then the solution q over $\bar{\mathbf{D}}_T$ is a vector having regular distributions as its components.

Proof. Put $M(p) = X^{\backprime}(Lp^2 + Rp + S)\,X$ and let $d(p) = \det M(p)$; then obviously $d(p) \not\equiv 0$. Further, it is clear that $X^{\backprime}e$ is a vector over $\bar{\mathbf{D}}_T$. If the vector ξ over $\bar{\mathbf{D}}_T$ is the solution of $d(D)\,\xi = \tilde{e} = X^{\backprime}e$ (which exists due to Lemma 2), then according to

Lemma 3 $w = N(D)\,\xi$ is a solution of (11), which is obviously a vector over $\bar{\mathbf{D}}_T$. From this it follows that $q = Xw$ is also a vector over $\bar{\mathbf{D}}_T$.

Suppose that $\tilde{q}$ is another solution of A 1, A 2 over $\bar{\mathbf{D}}_T$; then clearly the vector $\tilde{w}$ fulfilling the equality $\tilde{q} = X\tilde{w}$ is also over $\bar{\mathbf{D}}_T$. Thus, from (11) we have $M(D) . (w - \tilde{w}) = 0$, and, consequently, $d(D)(w - \tilde{w}) = 0$; but due to the assumption on roots of $d(p)$ no solution of the eq. $d(D)\,z = 0$ belongs to $\bar{\mathbf{D}}_T$, unless $z = 0$, so that $w - \tilde{w} = 0$. The first statement of Th. 1 is proved.

In order to prove the second statement, let us first recall the fact that due to the assumption of passivity of $\mathfrak{N}$ (see [1]) the elements of the matrix

$$\tilde{A}(p) = (X'(Lp + R + Sp^{-1})\,X)^{-1}\,, \tag{12}$$

which belongs to $\mathfrak{P}_n$, have a pole of at most first order at infinity. But $M^{-1}(p) = d^{-1}(p)\,N(p) = p^{-1}\,\tilde{A}(p)$, so that each element of $M^{-1}(p)$ is regular at infinity; hence, if n is the degree of the polynomial $d(p)$, then the degree of each element of $N(p)$ does not exceed n. If now $l'e$ is a regular distribution for every loop $l'h$, then obviously the elements of $X'e = \tilde{e}$ are regular distributions; consequently, by Lemma 2, the elements of ξ are regular distributions with the corresponding functions having the n-th (usual) derivative almost everywhere. Therefore, $w = \tilde{N}(D)\,\xi$ has regular distributions as its elements, and the same is true for the vector q, q.e.d.

It might seem that the assumptions of Th. 1 could be relaxed if one replaced the condition "det $X'(Lp^2 + Rp + S)\,X \neq 0$ for $p = in\omega$; $n = 0, \pm1, \pm2, \ldots$" by the condition "$\tilde{d}(in\omega) \neq 0$ for $n = 0, \pm1, \pm2, \ldots$", where $\tilde{d}(p)$ is the polynomial obtained from $d(p)$ by removing the greatest common factor of $d(p)$ and all elements of $N(p)$. But this is not true. In order to show it let us first prove the following assertion:

Lemma 4. *Let $M(p)$ be an $r \times r$ matrix ($r \geqq 2$), having polynomials as its elements, $N(p)$ the adjoint matrix, $d(p) = \det M(p) \not\equiv 0$; if α is a root of $d(p)$ with multiplicity $k \geqq 1$, then there is an integer m fulfilling the inequality $0 \leqq m \leqq k - 1$ such that $N(p)$ is divisible by $(p - \alpha)^m$ (i.e. each element of $N(p)$ is divisible by $(p - \alpha)^m$) and such that at least one element of $N(p)$ is not divisible by $(p - \alpha)^{m+1}$.*

Proof. The identity $N(p)\,M(p) = I\,d(p)$ yields $\det N(p) . \det M(p) = [d(p)]^r$, i.e. $\det N(p) = [d(p)]^{r-1}$. Let $N(p)$ be divisible by $(p - \alpha)^{m^*}$, $m^* \geqq 0$; then obviously $\det N(p)$ is divisible by $(p - \alpha)^{\tilde{m}}$ with $\tilde{m} \geqq rm^*$. On the other hand, from the previous equality it follows that $\tilde{m} = (r - 1)\,k$; consequently, $rm^* \leqq (r - 1)\,k$, i.e., $m \leqq k - 1$. q.e.d.

Now, from Lemma 4 it follows that the polynomials $d(p)$ and $\tilde{d}(p)$ have the same roots, i.e. the conditions $d(in\omega) \neq 0$ and $\tilde{d}(in\omega) \neq 0$ are equivalent.

Recalling Th. 4.5 in [1], we can state the following assertion:

Theorem 2. *Let $\mathfrak{N}$ be a dissipative K-network, $T > 0$; further, let e be a vector such that there is a vector g over $\bar{\mathbf{D}}_T$ with $g' = e$. Then $\mathfrak{N}$ possesses a T-periodic solution q. Moreover, two T-periodic solutions of $\mathfrak{N}$ differ by a constant vector.*

Proof. Let $M(p)$, $d(p)$, $N(p)$ have the meaning introduced in the proof of Th. 1. By Th. 4.5 in [1], the matrix $\tilde{A}(p) = (X'(Lp + R + Sp^{-1}) X)^{-1}$ exists and every element of it has no poles in the half-plane $\operatorname{Re} p \geqq 0$ nor at infinity. Hence, $d(p) \not\equiv 0$. Denoting $\tilde{q}(p)$ the greatest common factor of $d(p)$ and all elements of $N(p)$, and putting $\tilde{d}(p) = \tilde{q}^{-1}(p)\, d(p)$, $\tilde{N}(p) = \tilde{q}^{-1}(p)\, N(p)$, then from the identity $M^{-1}(p) = \tilde{d}^{-1}(p)\, \tilde{N}(p) = p^{-1}\, \tilde{A}(p)$ it follows easily that $\tilde{d}(p)$ has no roots on the imaginary axis except the root $p = 0$, which, if it exists, is simple.

Now, using Lemma 2 one obtains that the equation $\tilde{d}(D)\, \xi = \tilde{e} = X'e$ possesses a T-periodic solution. Actually, if $\tilde{d}(p)$ does not have the root $p = 0$, the existence of ξ is a direct consequence of Lemma 2. If $\tilde{d}(0) = 0$, put $\tilde{d}(p) = p\, d^*(p)$. Then, of course, there is a T-periodic ξ fulfilling the equation $d^*(D)\, \xi = X'g$, and, consequently, the equation $D\, d^*(D)\, \xi = \tilde{d}(D)\, \xi = X'g' = X'e$.

Putting finally $w = \tilde{N}(D)\, \xi$, then w is over $\bar{\mathbf{D}}_T$ and is a solution of (11); thus $q = Xw$ is over $\bar{\mathbf{D}}_T$ and is a solution of $\mathfrak{N}$.

Let q_1 be another T-periodic solution of $\mathfrak{N}$, and let w_1 be defined by $q_1 = Xw_1$; it is evident that w_1 is over $\bar{\mathbf{D}}_T$ and that $M(D)\,(w_1 - w) = 0$; consequently $\tilde{d}(D)\,(w_1 - w) = 0$. The constant vector, however, is the unique T-periodic solution of the latter equation, which completes the proof.

For further investigations, the following well-known Lemma will be useful:

Lemma 5. 1. *Each $f \in \bar{\mathbf{D}}_T$ has a finite order.*

2. *If $f \in \bar{\mathbf{D}}_T$ then there are uniquely determined numbers c_n, $n = 0, \pm 1, \pm 2, \ldots$ such that*

$$f = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \quad \omega = 2\pi/T; \tag{13}$$

moreover, there is a positive number M and an integer k such that

$$|c_n| \leqq M |n|^k, \quad n = \pm 1, \pm 2, \ldots \tag{14}$$

3. *If a distribution $f \in \bar{\mathbf{D}}$ admits the representation* (13) *with coefficients fulfilling the inequality* (14), *then $f \in \bar{\mathbf{D}}_T$.*

The Lemma just given permits us to state the following simple assertion:

Theorem 3. *Let the assumptions of Th.* 1 *be satisfied and let $e = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega t)$ be a vector over $\bar{\mathbf{D}}_T$; further, let $A(p) = X(X'(Lp^2 + Rp + S) X)^{-1} X'$. Then the unique T-periodic solution q of $\mathfrak{N}$ corresponding to e is given by*

$$q = \sum_{n=-\infty}^{\infty} A(in\omega)\, c_n \exp(in\omega t)\,. \tag{15}$$

Proof. Let

$$w = \sum_{n=-\infty}^{\infty} \{X'(Li^2n^2\omega^2 + Rin\omega + S) X\}^{-1} X' c_n \exp(in\omega t)\,; \tag{16}$$

since the elements of the matrix $\{\ldots\}^{-1}$ in (16) are rational functions of n, then, using statements 2 and 3 of Lemma 5, it is obvious that series (16) converges and that w is a vector over $\bar{\mathbf{D}}_T$. At the same time, we have $q = Xw$ which is also over $\bar{\mathbf{D}}_T$. But

$$u = X'(LD^2 + RD + S)\,Xw =$$
$$= \sum_{n=-\infty}^{\infty} X'(LD^2 + RD + S)\,XM^{-1}(in\omega)\,X'c_n \exp(in\omega t)$$

with $M(p) = X'(Lp^2 + Rp + S)\,X$. Carrying out the derivatives in the latter equation one obtaines immediately $u = X'e$; the uniqueness of w guaranteed by Th. 1 completes the proof.

In Theorems 1, 3 the "regular" case, i.e. $d(in\omega) \neq 0$ for $n = 0, \pm 1, \pm 2, \ldots$ was considered. Let us now consider the singular case, i.e. if $d(in\omega)$ vanishes for some n, ω being related to the given period T by $\omega = 2\pi/T$. Since the system (11) is linear and the decomposition (13) is true for every T-periodic distribution, we will restrict ourselves for the sake of simplicity to the case that $e = c \exp(i\omega_0 t)$, c being a constant vector. Referring to Lemma 3 it is obvious that in this case every solution q of A 1, A 2 is a vector whose components are regular distributions. Then the following statements are true.

Theorem 4a. *Let $\mathfrak{N}$ be a passive K-network, $M(p) = X'(Lp^2 + Rp + S)\,X$, $d(p) = \det M(p) \not\equiv 0$ and let $N(p)$ be the adjoint matrix to $M(p)$. If $i\omega_0 \neq 0$ is a root of $d(p)$ with multiplicity $k \geqq 1$, then all elements of $N(p)$ have the common factor $q(p) = (p - i\omega_0)^{k-1}$.*

Moreover, let $N(p) = q(p)\,\tilde{N}(p)$ and let $c \neq 0$ be a constant vector; if

A. *$\tilde{N}(i\omega_0)\,X'c = 0$, then there is a nontrivial solution $q = h \exp(i\omega_0 t)$ (h being a constant vector) of $\mathfrak{N}$ corresponding to $e = c \exp(i\omega_0 t)$;*

B. *$\tilde{N}(i\omega_0)\,X'c \neq 0$, then every solution q of $\mathfrak{N}$ corresponding to $e = c \exp(i\omega_0 t)$ is a vector, whose elements are not bounded on $(-\infty, \infty)$.*

Theorem 4b. *Let $\mathfrak{N}$ be a passive K-network, and let $M(p)$, $d(p)$, $N(p)$ have the same meaning as in Th. 4a; if $p = 0$ is the root of $d(p)$ with multiplicity $k \geqq 1$, then either 1. p^{k-1} or 2. p^{k-2} (provided $k \geqq 2$) is the highest power which is a common factor of all elements of $N(p)$. Moreover, if $c \neq 0$ is a constant vector, then the following statements are true:*

1. *If we put $N_1(p) = N(p)/p^{k-1}$ in case 1, and if the equality*

$$N_1(0)\,X'c = 0 \tag{17}$$

is satisfied, then there is a constant non-zero vector q, which is a solution of $\mathfrak{N}$ corresponding to $e = c$. If (17) is not satisfied, then every solution of $\mathfrak{N}$ corresponding to $e = c$ is a vector whose elements are not bounded on $(-\infty, \infty)$.

2. *If we put* $N_2(p) = N(p)/p^{k-2}$ *in case 2, and if there is a constant vector* $\tilde{k}$ *such that the equalities*

$$N_2(0)\, X^{\backprime}c = 0\,, \quad N_2'(0)\, X^{\backprime}c + N_2(0)\, \tilde{k} = 0 \tag{18}$$

are satisfied (the prime in N_2' *denotes the derivative), then a constant non-zero vector* q *exists, which is a solution of* $\mathfrak{N}$ *corresponding to* $e = c$. *If* (18) *are not satisfied, then the elements of any solution of* $\mathfrak{N}$ *corresponding to* $e = c$ *are not bounded on* $(-\infty, \infty)$.

For the proof the following Lemma will be useful.

Lemma 6. *Let* $P(p)$ *be a polynomial,* α *a number; then*

$$P(D)\,(te^{\alpha t}) = (P'(\alpha) + t\, P(\alpha))\, e^{\alpha t}\,. \tag{19}$$

$$P(D)\,(t^2 e^{\alpha t}) = (P''(\alpha) + 2P'(\alpha)\, t + P(\alpha)\, t^2)\, e^{\alpha t}\,. \tag{20}$$

(The proof is obvious.)

Proof of Th. 4a. Let $i\omega_0 \neq 0$ be a root of $d(p)$ with multiplicity $k \geqq 1$. Then due to the assumption on passivity of $\mathfrak{N}$ (see [1]) it follows that $Z(p) = p^{-1}\, M(p) \in \mathfrak{P}_n$; consequently, $M^{-1}(p) = d^{-1}(p)\, N(p) = p^{-1}\, Z^{-1}(p)$ with $Z^{-1}(p) \in \mathfrak{P}_n$. Since each pole $i\omega$ (ω real) of $Z^{-1}(p)$ is simple, it follows that all elements of $N(p)$ necessarily have the common factor $q(p) = (p - i\omega_0)^{k-1}$.

A: Let $d(p) = q(p)\, \tilde{d}(p)$. (Evidently $\tilde{d}(p)$ has a simple root $i\omega_0$.) Choosing arbitrarily a constant vector η, let

$$\xi = \frac{1}{\tilde{d}'(i\omega_0)}\, \tilde{c}te^{i\omega_0 t} + \eta e^{i\omega_0 t} \quad \text{with } \tilde{c} = X^{\backprime}c\,. \tag{21}$$

Using Lemma 6 one obtains

$$\tilde{d}(D)\, \xi = \frac{\tilde{c}}{\tilde{d}'(i\omega_0)}\, (\tilde{d}'(i\omega_0) + t\, \tilde{d}(i\omega_0))\, e^{i\omega_0 t} + \eta\, \tilde{d}(i\omega_0)\, e^{i\omega_0 t} = \tilde{c}e^{i\omega_0 t}\,.$$

According to Lemma 3 the vector $x = \tilde{N}(D)\, \xi$ is a solution of the equation

$$M(D)\, x = \tilde{c} \exp\,(i\omega_0 t)\,, \tag{22}$$

i.e. of (11). Using (21), for x one obtains:

$$\begin{aligned} x &= \tilde{N}(D) \left(\frac{1}{\tilde{d}'(i\omega_0)}\, \tilde{c}te^{i\omega_0 t} + \eta e^{i\omega_0 t} \right) = \\ &= \frac{1}{\tilde{d}'(i\omega_0)}\, (\tilde{N}'(i\omega_0) + t\, \tilde{N}(i\omega_0))\, \tilde{c}e^{i\omega_0 t} + \tilde{N}(i\omega_0)\, \eta e^{i\omega_0 t} = \\ &= \left\{ \frac{1}{\tilde{d}'(i\omega_0)}\, \tilde{N}'(i\omega_0)\, \tilde{c} + \tilde{N}(i\omega_0)\, \eta \right\} e^{i\omega_0 t}\,. \end{aligned} \tag{23}$$

Since $c \neq 0$ implies $\tilde{c} \neq 0$ it follows from (22) that x cannot be a zero vector; hence statement A is proved. Observe also that according to Lemma 3 every solution of (22) with the form $x = h \exp\,(i\omega_0 t)$ can be represented by equation (23).

B: Let $q^*(p)$ be the greatest common factor of $d(p)$ and all elements of $N(p)$, and let $d(p) = q^*(p)\, d^*(p)$, $N(p) = q^*(p)\, N^*(p)$. Then from Lemma 4 it is obvious that $\tilde{N}(i\omega_0)\, X`c \neq 0$ if and only if $N^*(i\omega_0)\, X`c \neq 0$. From the considerations made above (properties of matrices belonging to $\mathfrak{P}_n$) it follows further that $d^*(p)$ has no zeros in the open right half-plane, the zeroes $i\omega$, $\omega \neq 0$ on the imaginary axis are simple and the zero $p = 0$ (if it exists) is of multiplicity at most two. Thus, each solution of the equation $d^*(D)\, \xi = \tilde{c} \exp(i\omega_0 t)$ has the form

$$\xi = \frac{1}{\tilde{d}'(i\omega_0)} \tilde{c} t e^{i\omega_0 t} + \eta e^{i\omega_0 t} + \sum_k r_k e^{i\omega_k t} + \sum_n P_n(t)\, e^{\alpha_n t} + bt\,, \tag{24}$$

where η, r_k, b are constant vectors, $\omega_k \neq \omega_0$ and $P_n(t)$ are vector-polynomials, $\operatorname{Re} \alpha_n < 0$. According to Lemma 3 every solution of (22) has the form $x = N^*(D)\, \xi$. Hence, one has

$$x = \frac{1}{\tilde{d}'(i\omega_0)} N^*(i\omega_0)\, \tilde{c} t e^{i\omega_0 t} + gt + z\,, \tag{25}$$

where g is a constant vector and

$$z = \left\{ \frac{1}{\tilde{d}'(i\omega_0)} \tilde{N}'(i\omega_0)\, \tilde{c} + \tilde{N}(i\omega_0)\, \eta \right\} e^{i\omega_0 t} + \sum_k \tilde{N}(i\omega_k)\, r_k e^{i\omega_k t} + \sum_n Q_n(t)\, e^{\alpha_n t} + l\,, \tag{26}$$

$Q_n(t)$ being vector-polynomials, l a constant vector. For any choice of η, r_k, $P_n(t)$, b, however, the elements of z are bounded as $t \to \infty$, so that by (25) the elements of x are not bounded and the same is true for $q = Xx$. Thus Th. 4a is proved.

Proof of Th. 4b. Let $p = 0$ be the root of $d(p)$ with multiplicity $k \geqq 1$. From the identity $M^{-1}(p) = d^{-1}(p)\, N(p) = p^{-1}\, Z^{-1}(p)$ and from the properties of the matrix $Z^{-1}(p)$ it follows that one of the subsequent three cases takes place: a) $M^{-1}(p)$ has no pole at $p = 0$, b) the pole $p = 0$ is simple, c) the pole $p = 0$ is of order two. Case a), however, cannot occur due to Lemma 4. Hence, the first assertion of the theorem follows.

The proof of assertion 1 is the same as the proof of A, B in Th. 4a. Thus, let us prove 2. Denoting $\tilde{d}(p) = d(p)/p^{k-2}$ ($\tilde{d}(p)$ has a double zero at $p = 0$), $\tilde{c} = X`c$, and choosing constant vectors $\tilde{k}$, h put

$$\xi = \frac{1}{\tilde{d}''(0)} \tilde{c} t^2 + \tilde{k} t + h\,. \tag{27}$$

Using Lemma 6 it can be easily verified that ξ fulfils the equation $\tilde{d}(D)\, \xi = \tilde{c}$. By Lemma 3, however, $x = N_2(D)\, \xi$ is a solution of $M(D)\, x = \tilde{c}$. We have

$$x = \frac{1}{\tilde{d}''(0)} N_2(0)\, \tilde{c} t^2 + \left\{ \frac{1}{\tilde{d}''(0)} N_2'(0)\, \tilde{c} + N_2(0)\, \tilde{k} \right\} t + \left\{ \frac{1}{\tilde{d}''(0)} N_2''(0)\, \tilde{c} + N_2'(0)\, \tilde{k} + N_2(0)\, h \right\}. \tag{28}$$

But from (28) it follows that if (18) are satisfied for a certain $\bar{k}$, then x is a constant vector, q.e.d.

The proof of the last assertion is obvious from (28) and from the proof of B in Th. 4a.

Note. The second equation (18) cannot be omitted, since $\det N_2(0) = 0$, whenever case 2 occurs. (This follows easily from the identity $\det N(p) = [d(p)]^{n-1}$.)

In the subsequent considerations the following result will be helpful:

Lemma 7. *Let* $M(p) = \tilde{L}p^2 + \tilde{R}p + \tilde{S}$, $\tilde{L}, \tilde{R}, \tilde{S}$ *be positive semidefinite,* ω *a real number,* v *a complex n-vector; then equation*

$$(29) \qquad M(i\omega)\, u = v$$

has a solution for u *if and only if for every solution* ξ *of equation*

$$(30) \qquad \begin{aligned} M(i\omega)\, \xi &= 0\,, \\ \bar{\xi}' v &= 0\,. \end{aligned}$$

Proof. Let $\xi = \sigma + i\tau$ be a solution of (30); now $M(i\omega) = (\tilde{S} - \omega^2 \tilde{L}) + i\omega\tilde{R} = P + iQ$, where Q is positive semidefinite for $\omega \geqq 0$, negative semidefinite for $\omega < 0$. Now (30) can be written as

$$(31) \qquad P\sigma - Q\tau = 0\,, \quad P\tau + Q\sigma = 0\,.$$

From (31) it follows that

$$(32) \qquad -\tau' P\sigma + \tau' Q\tau = 0\,, \quad \sigma' P\tau + \sigma' Q\sigma = 0\,.$$

Obviously $\sigma' P\tau = \tau' P\sigma$ and, hence, by (32) one has $\tau' Q\tau + \sigma' Q\sigma = 0$ and by the semidefiniteness of Q, $\tau' Q\tau = \sigma' Q\sigma = 0$. By Lemma 5, 3 of [1] one has $Q\sigma = Q\tau = 0$ and by (31) $P\sigma = P\tau = 0$. Hence $M(i\omega)\,\bar{\xi} = 0$. Thus the complex conjugate of a solution of (30) is also a solution of (30). From this and from the well-known fact that (29) has a solution if and only if for every solution ξ of (30), $\xi' v = 0$, the proof follows immediately.

Theorem 5. *Conditions* A *of Theorem* 4a, (17) *and* (18) *of Theorem* 4b *are equivalent to the condition that for every solution* y *of equation*

$$(33) \qquad M(i\omega_0)\, y = 0\,,$$

$$(34) \qquad \bar{y}' X' c = 0\,.$$

Proof. By Theorem 4a, 4b, conditions A, (17), (18) respectively are necessary and sufficient for the existence of a solution of the equation,

$$(35) \qquad M(i\omega_0)\, x = X' c\,.$$

Using Lemma 7 one can easily finish the proof.

Note. From the physical point of view this result is very plausible; in case A of Theorem 4a the solution $q = h \exp(i\omega_0 t)$ of $\mathfrak{N}$ is not determined uniquely, since $(h + Xy) \exp(i\omega_0 t)$, where y is a solution of (33), is also a solution of $\mathfrak{N}$; now the vector $i\omega_0 y \exp(i\omega_0 t)$ corresponds to currents that may exist in the network without electromotive forces; eq. (34) states, therefore, that the total power produced by these currents is zero.

In what follows condition A of Theorem 4a and condition (17) will be examined more closely.

Lemma 8. *Let $\mathfrak{N}$ be a regular passive K-network, ω_0 a real number, let $M(p) = X'(Lp^2 + Rp + S)X$, $d(p) = \det M(p)$ and $N(p)$ be the matrix adjoint to $M(p)$. Let $i\omega_0$ be a root of $d(p)$ with multiplicity $k \geqq 1$ and let $(p - i\omega_0)^{k-1}$ be the greatest common factor of all elements of $N(p)$, i.e. $N(p) = (p - i\omega_0)^{k-1} \tilde{N}(p)$. Then the columns and rows of the matrix $\tilde{N}(i\omega_0)$ are solutions of (33).*

Proof. From relations $M(p)\,N(p) = N(p)\,M(p) = I\,d(p)$, where I is the unit matrix, and from $d(p) = (p - i\omega_0)^{k-1}\,\tilde{d}(p)$, $N(p) = (p - i\omega_0)^{k-1}\,\tilde{N}(p)$ one obtains $M(p)\,\tilde{N}(p) = \tilde{N}(p)\,M(p) = I\,\tilde{d}(p)$. Now substituting $p = i\omega_0$ and using the fact that $\tilde{d}(i\omega_0) = 0$ one can finish the proof.

The following well-known result will be useful: (See [4], pp. 35).

Lemma 9. *Let M be an n by n matrix over the commutative field T and let N be the adjoint matrix of M. Let $1 \leqq \varrho < n$ and let B be a ϱ by ϱ submatrix of N which arose from N by deleting the rows $i_1, \ldots, i_{n-\varrho}$ and the columns $j_1, \ldots, j_{n-\varrho}$; let C be an $n - \varrho$ by $n - \varrho$ submatrix of M which arose from M by deleting the rows $i_{n-\varrho+1}$ $\ldots, i_n$ and the columns $j_{n-\varrho+1}, \ldots, j_n$. Then $\det B = (\det M)^{\varrho-1} \det C$.*

In [5] the following assertion was proved:

Lemma 10. *Let $U(p)$ be an n by n matrix the elements of which are entire analytic functions, and let $u(p) = \det U(p)$; if α is a root of $u(p)$ with multiplicity exactly equal to k, $0 \leqq k \leqq n$, then the rank of $U(\alpha)$ is not smaller than $n - k$.*

Lemma 11. *Under the hypotheses of Lemma 8 the rank of $\tilde{N}(i\omega_0)$ is equal to the multiplicity k of the root $i\omega_0$ of $\det M(p)$.*

Proof. Since by Lemma 10 the rank of $M(i\omega_0)$ is at least $n - k$, there are at most k linearly independent solutions of (33). By Lemma 8 the columns of $\tilde{N}(i\omega_0)$ form a system of solutions of (33), the rank of $\tilde{N}(i\omega_0)$ thus being at most k. Now to prove our Lemma it is sufficient to prove that at least one subdeterminant of order k of matrix $\tilde{N}(i\omega_0)$ does not vanish. Thus let $M^*(p)$ be an $n - k$ by $n - k$ submatrix of $M(p)$ such that $\det M^*(i\omega_0) \neq 0$ (cfr Lemma 10). By Lemma 9 there exists a k by k submatrix $N^*(p)$ of $N(p)$ such that

$$\det N^*(p) = [\det M(p)]^{k-1} \det M^*(p) \tag{36}$$

for every p. As the elements of $N^*(p)$ have a common factor $(p - i\omega_0)^{k-1}$, one can write $N^*(p) = (p - i\omega_0)^{k-1}\,\tilde{N}^*(p)$, where $\tilde{N}^*(p)$ is a k by k submatrix of $\tilde{N}(p)$. Further $\det N^*(p) = (p - i\omega_0)^{k(k-1)} \det \tilde{N}^*(p)$, $\det M(p) = (p - i\omega_0)^k\,\tilde{\tilde{d}}(p)$, $\tilde{\tilde{d}}(i\omega_0)$ being different from zero. Hence by (36) one obtains

$$(p - i\omega_0)^{k(k-1)} \{\det \tilde{N}^*(p) - [\tilde{\tilde{d}}(p)]^{k-1} \det M^*(p)\} = 0$$

for every p. Hence $\det \tilde{N}^*(i\omega_0) = [\tilde{\tilde{d}}(i\omega_0)]^{k-1} \det M^*(i\omega_0) \neq 0$, q.e.d.

Theorem 6. *Under the hypotheses of Lemma* 8 *the rank of* $M(i\omega_0)$ *is* $n - k$. *Moreover, the set of all rows (and also the set of all columns) of* $\tilde{N}(i\omega_0)$ *is a complete set of solutions of* (33).

Proof follows from Lemmas 8, 10 and 11.

Theorem 7. *Let* $\mathfrak{N}$ *be a passive K-network,* $M(p) = X'(Lp^2 + Rp + S)X$. *Let* $i\omega_0$ *be a root of* $\det M(p)$ *and let* $e = ce^{i\omega_0 t}$. *Let every solution* y *of* (33) *fulfil* $y'\tilde{c} = 0$, *where* $\tilde{c} = X'c$. *Let* W *be the linear subspace of the complex Euclidean space* E_n *the elements of which are solutions of* (45), W' *its orthogonal complement in* E_n, *i.e., the direct sum* $W \dotplus W' = E_n$.

If the rank of $\det M(i\omega_0)$ *is* $n - k$, $0 < k < n$, *then* $\dim W = k$ *and there exists a unique solution* x^* *of* (35) *in* W'. *If* x *is a solution of* (35), *then* $x = x^* + y$, *where* $y \in W$, *and conversely, if* $y \in W$, *then* $x = x^* + y$ *is a solution of* (35). *Moreover, both* $\tilde{c}$ *and its complex conjugate* $\bar{\tilde{c}}$ *are elements of* W'.

Proof. Evidently, if x is a solution of (35), then $x = a + b$, where $a \in W$, $b \in W'$. As $-a \in W$, one has $M(i\omega_0)(a + b) - M(i\omega_0)a = \tilde{c}$. Consequently, there is a solution b of (35) which is from W'. Now let $b_1, b_2 \in W'$, $M(i\omega_0)b_i = \tilde{c}$ for $i = 1, 2$. Subtracting the latter equation from the former one obtains $M(i\omega_0)(b_1 - b_2) = 0$. Thus $b_1 - b_2$ is an element of both W and W', which implies $b_1 = b_2 = x^*$.

By hypothesis $\bar{\tilde{c}} \in W'$ and by Lemma 7, $\tilde{c} \in W$. The remaining assertions of the theorem are obvious.

Concluding the previous considerations let us present a statement which has an interesting physical meaning:

Theorem 8. *Let the assumptions of Theorem* 7 *be satisfied. Then the number* $\Phi = \bar{c}'h$ *does not depend on the choice of solution* $q = h\exp(i\omega_0 t)$ *of* $\mathfrak{N}$ *corresponding to* $e = c\exp(i\omega_0 t)$. *Moreover,* $\Phi = \bar{\tilde{c}}'x^*$, *where* $\tilde{c}$ *and* x^* *are defined in Theorem* 7.

Proof. As every solution q of $\mathfrak{N}$ corresponding to $e = c\exp(i\omega_0 t)$ is given by $q = Xx\exp(i\omega_0 t)$, x being a solution of (35), one has $\Phi = \bar{c}'h = \bar{c}'Xx = (\bar{X}'\bar{c})'x = \bar{\tilde{c}}'x$. By Theorem 7, $\bar{\tilde{c}}'x = \bar{\tilde{c}}'(x^* + y)$, where $\bar{\tilde{c}}'y = 0$, which proves the theorem.

Note. The number $i\omega_0\Phi$ represents, from the physical point of view, the power supplied to the network by sources of electromotive forces represented by e. Thus Theorem 8 states that if there exists a sinusoidal solution of $\mathfrak{N}$, then the power supplied to $\mathfrak{N}$ is uniquely determined.

Note. As mentioned earlier the solution q of the network represents physically the electrical charges. Consequently, the vector $i = q'$ represents currents in individual branches. Recalling the proofs of Th. 4a and 4b one obtains that a) condition A in Th. 4a is also a necessary and sufficient condition for q' to have the form $\tilde{h}\exp(i\omega_0 t)$. b) If in Th. 4b case 1 occurs, then there is a solution q such that q' is a constant vector; in case 2, however, the first equation of (18) is a necessary and sufficient condition for the existence of q such that q' is a constant vector.

Note. Theorem 6 deals with the case where the fact that $i\omega_0$ is a root of $\det M(p)$ with multiplicity k implies that $(p - i\omega_0)^{k-1}$ is the common factor of all elements of $N(p)$. From Theorem 4b it follows that case 2 of this theorem remains unsolved. Now it will be shown that the zero root of det $M(p)$ cannot be considered as an exceptional case. Really, *the condition* $\det M(0) = 0$ *is equivalent to the following condition: There exists a non zero cycle* $c'h$ *of the graph of* $\mathfrak{N}$ *such that* $\sum_{i=1}^{r} c_i^2 S_{ii} = 0$.

Proof. The latter equality may be written as $v'X'SXv = 0$, where $v \neq 0$. As S is positive semidefinite, the latter equation is equivalent to $X'SXv = 0$, $v \neq 0$, q.e.d.

On the other hand, one is usually more interested in solutions of network with $e = \text{const}$, the elements of e being real, which have constant time derivatives, than in constant solutions. This case will be treated in what follows.

Theorem 9. *Let* $\mathfrak{N}$ *be a dissipative K-network, i.e.* $\det X'RX \neq 0$, *and let* $\det X'SX = 0$. *Then there exists a unique real constant vector* $\tilde{a} \neq 0$ *such that* $q = \tilde{a}t + \tilde{b}$ *is a real solution of* $\mathfrak{N}$ *with* $e = c = \text{const}$, *c real.*

Proof. Consider the equation $M(D)(at + b) = X'c$, where $M(D) = X'(LD^2 + RD + S)X$, or

$$X'RXa + X'SX(at + b) = X'c\,. \tag{37}$$

Now a and b are to be chosen so that (37) is satisfied. Let Y be a real constant matrix the columns of which form a complete set of linearly independent solutions of

$$X'SXw = 0\,. \tag{38}$$

Evidently, a has to fulfil (38), i.e. $a = Yd$ for a certain d. Substituting into (37) one obtains

$$X'RXYd + X'SXb = X'c\,. \tag{39}$$

Now it will be shown that there exists exactly one d such that (39) has a real solution for b. This happens if and only if for every solution Yu of (38) one has $(Yu)'(X'c - X'SXYd) = 0$, or

$$Y'Xc - Y'X'RXYd = 0\,. \tag{40}$$

Since $X'RX$ is the matrix of a positive definite quadratic form and the columns of Y are linearly independent, there is a unique solution d of (40). Thus there exists exactly one $a = Yd$ and at least one b such that (37) is satisfied. Putting $\tilde{a} = Xa$ and $\tilde{b} = Xb$ one can finish the proof.

Note. Of course, $\tilde{b}$ in Theorem 9 is not determined uniquely. Denote by W the set of all real solutions w of (38). Obviously, W is a linear subspace of E_n (real n-dimensional Euclidean space). Let W' be its orthogonal complement in E_n. Then it is easy to show that each solution of (37) can be written as $at + b^* + w$, where b^* is a uniquely determined vector from W' and w is an arbitrary vector from W.

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Výtah

PERIODICKÁ ŘEŠENÍ KIRCHHOFFOVÝCH SÍTÍ

VÁCLAV DOLEŽAL a ZDENĚK VOREL, Praha

Článek navazuje na práci [1] a pojednává o existenci resp. unicitě periodických řešení Kirchhoffových sítí.

Pojem řešení K-sítě na celé časové ose je definován rovnicemi A 1, A 2, kde e je daný vektor, jehož komponenty jsou distribucemi, a q je hledané řešení.

Věta 1 zabývá se „regulárním případem", tj. představuje podmínky, za kterých existuje jediné T-periodické řešení dané K-sítě. Věty 4a, 4b, 7, 9 pozorují speciální „singulární případy", tj. udávají podmínky existence řešení pasivní K-sítě tvaru $q = h \exp i\omega t$, kdy $e = c \exp i\omega t$ (h, c jsou konst. vektory, ω reál. číslo), jakož i dimensi prostoru všech řešení tohoto typu. Věta 8 pak ukazuje, že číslo $\bar{c}'h$, které fysikálně představuje energii dodávanou do sítě, nezávisí na výběru řešení $q = h \exp i\omega t$.

Резюме

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ СЕТЕЙ КИРХГОФФА

ВАЦЛАВ ДОЛЕЖАЛ и ЗДЕНЕК ВОРЕЛ, Прага

Статья примыкает к работе [1] и посвящена вопросам существования и единственности периодических решений сетей Кирхгоффа.

Понятие решения K-сети на всей оси времени определено уравнениями А 1 А 2, где e — данный вектор, компоненты которого являются обобщенными функциями, и q — искомое решение.

Теорема 1 посвяшена „регулярному случаю"; она содержит условия, при которых существует одно единственное T-периодическое решение данной K-сети. В теоремах 4a, 4b, 7, 9 изучаются частные „особые случаи", т. е. приводятся условия существования решения пассинвой K-сети вида $q = h \exp i\omega t$, когда $e = c \exp i\omega t$ (h, c — постоянные векторы, ω — действ. число), равно как и размерность пространства всех решений этого типа. В теореме 8 показано, что число $\bar{c}'h$, которое с физической точки зрения представляет энергию, доставляемую в сеть, не зависит от выбора решения $q = h \exp i\omega t$.