Časopis pro pěstování matematiky

Jan Kučera

Multiple Fourier integral

Časopis pro pěstování matematiky, Vol. 88 (1963), No. 3, 295--310

Persistent URL: `http://dml.cz/dmlcz/117463`

Časopis pro pěstování matematiky, roč. 88 (1963), Praha

MULTIPLE FOURIER INTEGRAL

JAN KUČERA, Praha

(Received December 18, 1961)

In the present paper the Fourier integral for a complex function of several real variables is defined and some criteria for its convergence are presented.

INTRODUCTION

All the convergent integrals, which occur in this paper, will be Lebesgue integrals, unless otherwise stated. The r-dimensional Euclidean space (where r is a positive integer) will be denoted by E_r. For any set $M \subset E_r$, the symbol $L(M)$ represents the set of all complex functions f, which are defined on the set M and have a convergent Lebesgue integral $\int_M |f| \, dx$. The boundary of any set $M \subset E_r$ will be denoted $H(M)$. The scalar product of points

$$x = (x_1, x_2, \ldots, x_r) \in E_r, \quad y = (y_1, y_2, \ldots, y_r) \in E_r$$

will be written $(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_r y_r$ and the norm of x will be $\|x\| = \sqrt{(x, x)}$. Further, we shall use the symbols $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, for the positive, negative, part of a real function f, respectively. For the sum, product, of a family of sets we shall use the Greek letter $\sum$, $\prod$, respectively. Finally, the set of all points $x \in M$, where $M \subset E_r$, which have a property $P(x)$, will be denoted $E(x \in M; P(x))$.

Definition 1. A real function f, which is defined on a set $M \subset E_r$, will be called monotone on the set M, if it is monotone (as a function of one variable) on every set

$$M \cap E(x \in E_r;\ x_k = x_k^0,\ 1 \leqq k \leqq r,\ k \neq i),$$

for all points $(x_1^0, x_2^0, \ldots, x_{i-1}^0, x_{i+1}^0, \ldots, x_r^0) \in E_{r-1}$ and for $i = 1, 2, \ldots, r$.

Definition 2. Let f be a complex function, which is defined on E_r and satisfies either of the following conditions:

I. $f \in L(E_r)$.

II. There exist functions f_j ($j = 1, 2, 3, 4$) which are non-negative and monotone on every closed octant [1]) and have the following properties: $f = (f_1 - f_2) + i(f_3 - f_4)$

[1]) Closed octant is each of the intervals $I \subset E_r$, which is the Cartesian product of r intervals either $(-\infty, 0\rangle$ or $\langle 0, +\infty)$.

$\lim_{\|x\| \to +\infty} f_j(x) = 0$ $(j = 1, 2, 3, 4)$. In the following we shall say that such functions f have property BV.

Then the integral

$$(1) \qquad \left(\frac{1}{2\pi}\right)^r \cdot \int_{E_r} \left(\int_{E_r} f(u + x) \, . \, e^{i(u,\xi)} \, du \right) d\xi \,, \quad \text{for} \quad x \in E_r \,,$$

where the internal integral in case II is the improper Lebesgue integral and the external integral is taken in the sense of Cauchy's principal value,[2]) will be called the Fourier integral of the function f.

Lemma 1. (Riemann-Lebesgue). *Let* $-\infty \leqq a < b \leqq +\infty$; $\alpha, \beta \in \langle a, b \rangle$; $\gamma \in E_1$. *Let us distinguish two cases:*

I. *If* $f \in L(\langle a, b \rangle)$, *then*

$$(2) \qquad \lim_{A \to +\infty} \int_\alpha^\beta f(x) \sin A(x - \gamma) \, dx = 0 \,,$$

uniformly with respect to α, β, γ.

II. *Let* $K \subset E_r$ *be a compact set and let the function* $g(x, t)$, *which is defined for* $x \in (a, b)$, $t \in K$, *form a family of equicontinuous functions* $f_t(x) = g(x, t)$ *on* (a, b) *for* $t \in K$. *Finally let a function* $\varphi \in L(\langle a, b \rangle)$ *exist such that* $|g(x, t)| \leqq \varphi(x)$ for $x \in (a, b)$, $t \in K$. *Then*

$$(3) \qquad \lim_{A \to +\infty} \int_\alpha^\beta g(x, t) \sin A(x - \gamma) \, dx = 0$$

uniformly with respect to α, β, γ, t.

Proof. I. See [1]. II. To prove this, let $\varepsilon > 0$ be given. Then there are real numbers a_1, b_1 such that $a < a_1 < b_1 < b$ and

$$\int_a^{a_1} |\varphi| \, dx + \int_{b_1}^b |\varphi| \, dx < \varepsilon \,.$$

Further, there exists positive integer n such that the implication

$$|x_1 - x_2| < \frac{1}{n}(b_1 - a_1) \Rightarrow |g(x_1, t) - g(x_2, t)| < \frac{\varepsilon}{b_1 - a_1}$$

holds for $x_{1,2} \in \langle a_1, b_1 \rangle$, $t \in K$.

[2]) Cauchy's principal value of the integral is

$$\int_{E_r} f \, dx = \lim_{A \to +\infty} \int_{\prod_{k=1}^r \langle -A, A \rangle} f \, dx \,.$$

If the set $(\alpha, \beta) \cap (a_1, b_1)$ is empty, it is evident that

$$\left| \int_\alpha^\beta g(x, t) \sin A(x - \gamma) \, dx \right| < \varepsilon ;$$

in the opposite case let us divide the interval $(\alpha, \beta) \cap (a_1, b_1)$ into n parts of equal length by the points α_k $(k = 0, 1, \ldots, n)$. Then

$$\left| \int_\alpha^\beta g(x, t) \sin A(x - \gamma) \, dx \right| \leqq \int_a^{a_1} |\varphi| \, dx + \int_{b_1}^b |\varphi| \, dx +$$

$$+ \sum_{k=1}^n \left| \int_{\alpha_{k-1}}^{\alpha_k} (g(x, t) - g(\alpha_k, t)) \sin A(x - \gamma) \, dx \right| +$$

$$+ \sum_{k=1}^n \left| \int_{\alpha_{k-1}}^{\alpha_k} g(\alpha_k, t) \sin A(x - \gamma) \, dx \right| < \varepsilon + \frac{\varepsilon}{b_1 - a_1} \, .$$

$$. \sum_{k=1}^n \int_{\alpha_{k-1}}^{\alpha_k} |\sin A(x - \gamma)| \, dx + M \, . \sum_{k=1}^n \left| \int_{\alpha_{k-1}}^{\alpha_k} \sin A(x - \gamma) \, dx \right| < 2\varepsilon + Mn \frac{2}{A} \, .$$

Remark. If we put $\gamma = \bar{\gamma} - \pi/2A$ in (2) or (3), we get a similar lemma, where the function $\sin x$ is replaced by the function $\cos x$.

Lemma 2. *Property BV is invariant with respect to a translation of the origin of the Cartesian coordinate system of the space* E_r.

Proof. We can see this, if we translate the origin to the point $(a, 0, 0, \ldots, 0)$, where $a > 0$, and for $j = 1, 2$; $y \in E_{r-1}$ define the functions $g_j(x)$ in the following manner:

$$g_j(x_1, y) = f_j(x_1 + a, y) \quad \text{for} \quad x_1 \leqq -a \, ,$$
$$g_j(x_1, y) = f_1(0, y) + f_2(0, y) - f_{3-j}(x_1 + a, y) \quad \text{for} \quad -a \leqq x_1 \leqq 0 \, ,$$
$$g_j(x_1, y) = f_j(x_1 + a, y) + (f_1(0, y) + f_2(0, y) - f_1(a, y) - f_2(a, y)) (1 + x_1^2)^{-1}$$
$$\text{for} \quad x_1 \geqq 0 \, .$$

Lemma 3. *Let f be a function on* E_r, *which has everywhere the continuous derivative* $f_{x_1, x_2, \ldots, x_r}$ [3]) *and which has a Lebesgue integral on* E_r *and* $\lim_{\|x\| \to +\infty} f(x) = 0$.

Then the function f has the property BV.

Proof. It suffices to prove this only for the real part of the function f. For $x \in \in \prod_{k=1}^r \langle 0, +\infty)$ let us write

$$F_1(x) = \int_{\prod_{k=1}^r \langle x_k, +\infty)} (f_{x_1, x_2, \ldots, x_r})^+ (\xi) \, d\xi \, , \quad F_2(x) = \int_{\prod_{k=1}^r \langle x_k, +\infty)} (f_{x_1, x_2, \ldots, x_r})^- (\xi) \, d\xi \, .$$

[3]) We write $f_{x_1, x_2, \ldots, x_r} = \dfrac{\partial}{\partial x_r} \left(\dfrac{\partial}{\partial x_{r-1}} \left(\cdots \left(\dfrac{\partial}{\partial x_1} f \right) \right) \right)$.

The functions F_j $(j = 1, 2)$ are evidently non-negative, monotone on the set $\prod_{k=1}^{r} \langle 0, +\infty)$ and $\lim_{\|x\| \to +\infty} F_j(x) = 0$ $(j = 1,2)$.

$$F_1(x) - F_2(x) = \int_{\prod_{k=1}^{r} \langle x_k, +\infty)} [(f_{x_1,x_2,\ldots,x_r})^+ (\xi) - (f_{x_1,x_2,\ldots,x_r})^- (\xi)] \, d\xi =$$

$$= \int_{\prod_{k=1}^{r} \langle x_k, +\infty)} f_{x_1,x_2,\ldots,x_r}(\xi) \, d\xi = (-1)^r f(x) .$$

Let $K_1, K_2, \ldots, K_{2^r}$ be a sequence of all closed octants of E_r. On each octant K_l $(l = 1, 2, \ldots, 2^r)$ there are non-negative and monotone functions f_{jl} $(j = 1, 2)$ such that $\lim_{\|x\| \to +\infty} f_{jl}(x) = 0$ $(j = 1, 2)$ and $f(x) = f_{1l}(x) - f_{2l}(x)$ for $x \in K_l$.

For $x \in \sum_{l=1}^{2^r} H(K_l)$, let us put $\varphi_j(x) = \max_{E(l;\, x \in K_l)} f_{jl}(x)$ $(j = 1, 2)$.

Let us choose an octant, for example K_1. If we denote, for brevity, $\psi(x) = \varphi_1(x) - f_{11}(x)$, for $x \in H(K_1)$, it is evident that the function $F(x) = \min [\psi(0, x_2, x_3, \ldots, x_r), \psi(x_1, 0, x_3, \ldots, x_r), \ldots, \psi(x_1, x_2, \ldots, x_{r-1}, 0)]$, where $x \in K_1$, is non-negative and monotone on the octant K_1, $\lim_{\substack{\|x\| \to +\infty \\ x \in K_1}} F(x) = 0$ and for $x \in H(K_1)$ is $F(x) = \varphi_1(x) - f_{11}(x)$. If we put $f_j(x) = f_{j1} + F(x)$, for $x \in K_1$ $(j = 1, 2)$, then the identity $f_j(x) = f_{j1}(x) + \varphi_1(x) - f_{11}(x) = \varphi_j(x)$ $(j = 1, 2)$ holds for $x \in H(K_1)$. If we define the functions f_j $(j = 1, 2)$ on other octants in a similar manner, then the functions f_j $(j = 1, 2)$ have the required properties.

Definition 3. Let f be a complex function of r real variables. Then the difference

$$\Delta_h^i f(x) = f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_r) - \\ - f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r)$$

will be called the difference of 1st order of function f in the point x with respect to the i-th variable with the h distance (if the right-hand side is meaningful).

If we define the difference of n-th order $\Delta_{h_1,h_2,\ldots,h_n}^{i_1,i_2,\ldots,i_n} f(x)$, then the difference

$$\Delta_{h_1,h_2,\ldots,h_n,h_{n+1}}^{i_1,i_2,\ldots,i_n,i_{n+1}} f(x) = \Delta_{h_{n+1}}^{i_{n+1}} \Delta_{h_1,h_2,\ldots,h_n}^{i_1,i_2,\ldots,i_n} f(x)$$

will be called the difference of $(n + 1)$-th order of the function f.[4])

Lemma 4. *Let* $\begin{pmatrix} 1, & 2, & \ldots, n \\ p_1, & p_2, & \ldots, p_n \end{pmatrix}$ *be a permutation. Then*

$$\Delta_{h_1,h_2,\ldots,h_n}^{i_1,i_2,\ldots,i_n} f(x) = \Delta_{h_{p_1},h_{p_2},\ldots,h_{p_n}}^{i_{p_1},i_{p_2},\ldots,i_{p_n}} f(x) .$$

[4]) Usually the difference is defined in this manner: Let be $x \in E_r$, $h \in E_r$, then $\Delta_h f(x) = f(x + h) - f(x)$. But the special case in which we demand that the point $h \in E_r$ has only one coordinate different from zero is adequate here.

The proof is evident.

Lemma 5. *Let us have a vector* $h = (h_1, h_2, \ldots, h_r)$ *and a function* f *defined on the interval* $\prod_{k=1}^{r} \langle x_k, x_k + h_k \rangle$.

Then

$$f(x + h) = f(x) + \sum_{i=1}^{r} \Delta_{h_i}^{i} f(x) + \frac{1}{2!} \sum_{i,j=1;\, i \neq j}^{r} \Delta_{h_i, h_j}^{i,j} f(x) + \ldots + \tag{4}$$

$$+ \frac{1}{(r-1)!} \sum_{\substack{i_k = 1;\, k = 1, 2, \ldots, (r-1) \\ i_k \neq i_l (k \neq l)}}^{r} \Delta_{h_{i_1}, h_{i_2}, \ldots, h_{i_{r-1}}}^{i_1, i_2, \ldots, i_{r-1}} f(x) + \Delta_{h_1, h_2, \ldots, h_r}^{1, 2, \ldots, r} f(x).$$

The proof can be carried out by induction.

Lemma 6. *Let us have* n *positive integers* $i_1, i_2, \ldots, i_n$ $(i_k \leqq r;\ k = 1, 2, \ldots, n)$, n *real numbers* $h_{i_1}, h_{i_2}, \ldots, h_{i_n}$ *and a point* $x \in E_r$. *Let* M *be a convex hull of points*

$$(x_1, x_2, \ldots, x_r),\ (x_1, x_2, \ldots, x_{i_k - 1}, x_{i_k} + h_{i_k}, x_{i_k + 1}, \ldots, x_r) \quad (k = 1, 2, \ldots, n).$$

Let a function f *have a partial derivative* $f_{x_{i_1}, x_{i_2}, \ldots, x_{i_n}}$ *on the set* M.[5])

Then there is a point $\xi \in M$ *such that*

$$\Delta_{h_{i_1}, h_{i_2}, \ldots, h_{i_n}}^{i_1, i_2, \ldots, i_n} f(x) = h_{i_1} h_{i_2} h_{i_3} \ldots h_{i_n} f_{x_{i_1}, x_{i_2}, \ldots, x_{i_n}}(\xi).$$

The proof follows with the successive use of the mean value theorem for functions of one real variable.

Lemma 7. *Let us have* r *real numbers* $h_1, h_2, \ldots, h_r$ *and a point* $x \in E_r$. *Let a function* f *have a finite derivative* $f_{x_1, x_2, \ldots, x_r} \in L(J)$ *on the interval* $J = \prod_{k=1}^{r} \langle x_k, x_k + h_k \rangle$. *(A similar note about the points on* H(J) *applies here as in footnote*[5]).)

Then

$$\Delta_{h_1, h_2, \ldots, h_r}^{1, 2, \ldots, r} f(x) = \int_J f_{x_1, x_2, \ldots, x_r}(u)\, du. \tag{5}$$

The proof follows with the successive use of Fubini's theorem.

Theorem 1. *Let a function* f, *defined on* E_r, *be either integrable on* E_r *or have the property BV. For* $x \in E_r$, $A > 0$ *let us write*

$$J_A(x) = \left(\frac{1}{2\pi}\right)^r \int_{\substack{|\xi_k| < A \\ k = 1, 2, \ldots, r}} \left(\int_{E_r} (f(u + x)\, e^{i(u, \xi)}\, du \right) d\xi. \tag{6}$$

Then

$$J_A(x) = \left(\frac{1}{\pi}\right)^r \cdot \int_{E_r} f(u + x) \cdot \prod_{k=1}^{r} \frac{\sin A u_k}{u_k}\, du. \tag{7}$$

---

[5]) At the points on the boundary H(M), we mean always one-hand derivative.

Proof. I. $f \in L(E_r)$. If in accordance with Fubini's theorem we change the order of integration in (6) and then compute the internal integral, we get formula (7).

II. There are functions f_j $(j = 1, 2, 3, 4)$ which are non-negative and monotone on each closed octant and

$$f = (f_1 - f_2) + i(f_3 - f_4)\,; \quad \lim_{\|x\| \to +\infty} f_j(x) = 0 \quad (j = 1, 2, 3, 4)\,.$$

Let us prove formula (7) on the octant $\prod_{k=1}^{r} \langle 0, +\infty)$ for the function f_1. Let $0 < a < A$, $0 < B$; then

$$(8) \qquad \left(\frac{1}{2}\right)^r \cdot \int_{\substack{a < |\xi_k| < A \\ k=1,2,\ldots,r}} \left(\int_{\substack{0 < u_k < B \\ k=1,2,\ldots,r}} f_1(u) \,.\, e^{i(u-x,\xi)} \,\mathrm{d}u \right) \mathrm{d}\xi =$$

$$= \int_{\substack{0 < u_k < B \\ k=1,2,\ldots,r}} f_1(u) \,.\, \prod_{k=1}^{r} (u_k - x_k)^{-1} \,.\, [\sin A(u_k - x_k) - \sin a(u_k - x_k)] \,\mathrm{d}u\,.$$

Because the function f_1 is monotone on the set $\prod_{k=1}^{r} \langle 0, +\infty)$ and $\lim_{\|x\| \to +\infty} f_1(x) = 0$, the integral $\int_{\substack{0 < u_k < +\infty \\ k=1,2,\ldots,r}} f_1(u)\, e^{i(u-x,\xi)}\,\mathrm{d}u$ exists uniformly with respect to $\xi \in \prod_{k=1}^{r} (\langle -A\ -a \rangle \cup \langle a, A \rangle)$ and is also bounded on this set. Thus we can let $B \to +\infty$ in equation (8).

In a similar manner we can show that the integral

$$\int_{\substack{0 < u_k < +\infty \\ k=1,2,\ldots,r}} f_1(u) \prod_{k=1}^{r} \frac{\sin a(u_k - x_k)}{u_k - x_k} \,\mathrm{d}u$$

exists uniformly with respect to $a \in E_1$ and that it has the limit zero as $a \to 0+$.

CHAPTER I

In accordance with theorem 1 the Fourier integral of a function f satisfying the requirements in definition 2 exists if and only if the limit

$$(9) \qquad \lim_{A \to +\infty} J_A(x)$$

exists. In this chapter we shall find sufficient conditions for the existence of this limit for integrable functions. Functions which have the property BV will be treated in the next chapter.

Theorem 2. *Let us have a function* $f \in L(E_r)$, *a point* $x \in E_r$, *a real number* $\delta > 0$ *and an integer* n, $(0 < n < r)$. *Pulling, for brevity,* $y = (x_1, x_2, \ldots, x_n)$, $v = (u_{n+1}, u_{n+2}, \ldots, u_r)$, *let the function* $f(y, v)$ *be integrable as a function of* v *on*

E_{r-n} *and let the following integrals converge*:

$$(10)\qquad \int_{\substack{|u_{i_k}|<\delta;\,k=1,2,\dots,s\\ v\in E_{r-n}}} (u_{i_1}u_{i_2}\dots u_{i_s})^{-1}\,.\,\Delta^{i_1,i_2,\dots,i_s}_{u_{i_1},u_{i_2},\dots,u_{i_s}} f(y,v)\,du_{i_1}\,du_{i_2}\dots du_{i_s}\,dv\,,$$

where the indices $(i_1, i_2, \dots, i_s)$ *assume every combination of the numbers* $1, 2, \dots, n$, *taken* s *at a time* $(s = 1, 2, \dots, n)$.

Then

$$(11)\qquad \lim_{A\to+\infty} \int_{\substack{|u_p|<\delta,\,(1\leqq p\leqq n)\\ |u_q|>\delta,\,(n<q\leqq r)}} f(u+x)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du = 0\,.$$

Proof. Without loss of generality, let us put $x_1 = x_2 = \dots = x_r = 0$. If we denote, for brevity, $D = \prod_{k=1}^{n}(-\delta,\delta)\times\prod_{k=n+1}^{r}[(-\infty,-\delta)\cup(\delta,+\infty)]$, we have by (4)

$$(12)\qquad \int_D f(u)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du = \int_D f(0,v)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du +$$

$$+\sum_{i=1}^{n}\int_D \Delta^{i}_{u_i} f(0,v)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du + \dots + \int_D \Delta^{1,2,\dots,n}_{u_1,u_2,\dots,u_n} f(0,v)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du\,.$$

All the terms of the right-hand side of (12) tend to zero as $A\to+\infty$. Let us show this, for example, for the integral

$$(13)\qquad \int_D \Delta^{1,2,\dots,s}_{u_1,u_2,\dots,u_s} f(0,v)\prod_{k=1}^{r}\frac{\sin Au_k}{u_k}\,du = \prod_{k=s+1}^{n}\int_{-\delta}^{\delta}\frac{\sin Au_k}{u_k}\,du_k\,.$$

$$.\int_{\substack{|u_p|<\delta,(1<p\leqq s)\\ |u_q|>\delta,(n<q\leqq r)}} \Big(\prod_{1<k\leqq s}\sin Au_k\Big)\Big(\prod_{n<k\leqq r}\frac{\sin Au_k}{u_k}\Big).$$

$$.\left(\int_{-\delta}^{\delta}(u_1u_2\dots u_s)^{-1}\,.\,\Delta^{1,2,\dots,s}_{u_1,u_2,\dots,u_s} f(0,v)\,.\,\sin Au_1\,du_1\right)du_2\,du_3\dots du_s\,dv\,.$$

The internal integral on the right-hand side of (13) tends to zero as $A\to+\infty$ for almost all $v\in E_{r-n}$, $(u_2, u_3, \dots, u_s)\in\prod_{k=2}^{s}(-\delta,\delta)$ and according to (10) the integrand on the right-hand side of (13) has the integrable majorante $\delta^{n-r}\,.\,|u_1u_2\dots u_s|^{-1}\,.\,.\,|\Delta^{1,2,\dots,s}_{u_1,u_2,\dots,u_s} f(0,v)|$.

Thus we can take the limit as $A\to+\infty$ within the integral sign.

Theorem 3. *Let us have a function* $f\in L(E_r)$ *and a point* $x\in E_r$. *Let a real number* $\delta>0$ *exist such that the assumptions of the theorem 2 are fulfilled for* $n = 1, 2, \dots, (r-1)$ *and for all permutations of the coordinates. Let the integral*

$$(14)\qquad \int_{\substack{|u_k|<\delta\\ k=1,2,\dots,r}} (u_1u_2\dots u_r)^{-1}\,\Delta^{1,2,\dots,r}_{u_1,u_2,\dots,u_r} f(x)\,du$$

be convergent. Let the limit (9) exist and be equal to the value $f(x)$ for all functions which we get from the function $f(u)$ if we fix the variables $u_{i_k} = x_{i_k}$ $(k = 1, 2, \ldots, s)$, where the indices $i_1, i_2, \ldots, i_s$ go through all the combinations of the numbers $1, 2, \ldots, r$, taken s at a time, for $s = 1, 2, \ldots, (r - 1)$.

Then the Fourier integral of the function f is convergent at the point x and equal to $f(x)$.

Proof. Let us put again $x_1 = x_2 = \ldots = x_r = 0$. According to theorem 2 it will do to prove that

$$\lim_{A+\to\infty} \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} f(u) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du = \pi^r . f(0) . \tag{15}$$

$$\int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} f(u) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du = \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} f(0) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du + \tag{16}$$

$$+ \sum_{i=1}^{r} \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} \Delta_{u_i}^{i} f(0) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du + \sum_{\substack{i,j=1 \\ i<j}}^{r} \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} \Delta_{u_i,u_j}^{i,j} f(0) .$$

$$. \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du + \ldots + \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} \Delta_{u_1,u_2,\ldots,u_r}^{1,2,\ldots,r} f(0) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du .$$

Obviously

$$\lim_{A\to+\infty} \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} f(0) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} du = \pi^r . f(0) .$$

All the other addends in (16) have the limit zero. Let us show this for

$$\int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,r}} \Delta_{u_1,u_2,\ldots,u_n}^{1,2,\ldots,n} f(0) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} = du = \tag{17}$$

$$= \prod_{k=n+1}^{r} \int_{-\delta}^{\delta} \frac{\sin Au_k}{u_k} du_k . \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,n}} \Delta_{u_1,u_2,\ldots,u_n}^{1,2,\ldots,n} f(0) \prod_{k=1}^{n} \frac{\sin Au_k}{u_k} du .$$

If on the last integral on the right side of (17) we compute the integral of each addends in the difference separately, we get

$$\lim_{A\to+\infty} \pi^{-n} . \int_{\substack{|u_k|<\delta \\ k=1,2,\ldots,n}} \Delta_{u_1,u_2,\ldots,u_n}^{1,2,\ldots,n} f(0) \prod_{k=1}^{n} \frac{\sin Au_k}{u_k} du = \sum_{k=0}^{n} (-1)^k \binom{n}{k} . f(0) = 0 .$$

Remark 1. The assumptions in theorem 3 can be a little weaker. Let us show this for the case of two variables. Let be $f \in L(E_2)$, $(x_1, x_2) \in E_2$ and let exist a real number $\delta > 0$, complex number S and functions $g, h \in L(E_1)$ such that the following integrals are convergent:

$$\int_{\substack{|u_1|>\delta \\ |u_2|<\delta}} u_2^{-1} (f(x_1 + u_1, x_2 + u_2) - g(x_1 + u_1))\, du_1\, du_2 ;$$

$$\int_{\substack{|u_1|<\delta \\ |u_2|>\delta}} u_1^{-1} (f(x_1 + u_1, x_2 + u_2) - h(x_2 + u_2))\, du_1\, du_2 ;$$

$$\iint_{\substack{|u_1|<\delta \\ |u_2|<\delta}} (u_1 u_2)^{-1} (f(x_1 + u_1, x_2 + u_2) - g(x_1 + u_1) - h(x_2 + u_2) + S)\, du_1\, du_2$$

and

$$\lim_{A \to +\infty} \pi^{-1} . \int_{-\delta}^{\delta} g(x_1 + u_1) \frac{\sin Au_1}{u_1}\, du_1 = S ;$$

$$\lim_{A \to +\infty} \pi^{-1} \int_{-\delta}^{\delta} h(x_2 + u_2) \frac{\sin Au_2}{u_2}\, du_2 = S .$$

Then the Fourier integral of the function f is convergent at the point (x_1, x_2) and has the value S.

In many cases we can choose

$$g(u_1) = \tfrac{1}{2}(f(u_1, x_2 + 0) + f(u_1, x_2 - 0)) ,$$
$$h(u_2) = \tfrac{1}{2}(f(x_1 + 0, u_2) + f(x_1 - 0, u_2)) ;$$
$$S = \tfrac{1}{4}(f(x_1 + 0, x_2 + 0) + f(x_1 + 0, x_2 - 0) + f(x_1 - 0, x_2 + 0) + + f(x_1 - 0, x_2 - 0)) .$$

Remark 2. According to lemma 6 condition (14) holds if the function f has the derivative $f_{x_1, x_2, \ldots, x_r}$ bounded on the set $\prod_{k=1}^{r} (-\delta, \delta)$.

Remark 3. For the convergence of the integral

$$\int_{\substack{|u_k|<\delta;\, k=1,2,\ldots,n \\ v \in E_{r-n}}} (u_1 u_2 \ldots u_n)^{-1} . \Delta^{1,2,\ldots,n}_{u_1, u_2, \ldots, u_n} f(0, v)\, du_1\, du_2 \ldots du_n\, dv$$

from (10) it is sufficient to show that for the function

$$F(x_1, x_2, \ldots, x_n) = \int_{v \in E_{r-n}} |f_{x_1, x_2, \ldots, x_n}(x_1, x_2, \ldots, x_n, v)|\, dv$$

there exist numbers $C > 0$, $\alpha > 0$ such that $F(x_1, x_2, \ldots, x_n) \leqq C|x_1 x_2 \ldots x_n|^{\alpha - 1}$ for $|x_i| < \delta$ $(i = 1, 2, \ldots, n)$ and derivative $f_{x_1, x_2, \ldots, x_n}(x_1, x_2, \ldots, x_n, v)$ is bounded when $|x_i| < \delta$ $(i = 1, 2, \ldots, n)$ for each $v \in E_{r-n}$.

The proof is obvious from the Fubini's theorem and from lemma 7.

Example. Integrability of the partial derivatives of the function f is not sufficient for the integrability of integrals (10) and (14). To show this, let us put

$$f(x, y) = (1 - \lg |y|)^{-1} . e^{-|x|} \quad \text{for} \quad x \in E_1,\ 0 < |y| < 1 ;$$

$$f(x, y) = e^{1-|x|-|y|} \quad \text{for} \quad x \in E_1,\ |y| \geqq 1 ; \quad f(x, 0) = 0 \quad \text{for} \quad x \in E_1 .$$

The function f is then continuous and f, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\, \partial y \in L(E_2)$. If, however, we choose $0 < \delta \leqq 1$, then

$$\int_{\substack{0<y<\delta \\ x \in E_1}} y^{-1} . \Delta_y^2 f(x, 0)\, dx\, dy = \int_{-\infty}^{+\infty} \left(\int_0^\delta y^{-1} \left(f(x, y) - f(x, 0)\, dy\right) \right) dx =$$

$$= \int_{-\infty}^{+\infty} e^{-|x|}\, dx . \int_0^\delta y^{-1}(1 - \lg y)^{-1}\, dy = +\infty .$$

Theorem 4. *Let $f \in L(E_r)$ be a continuous function on E_r and let exist r functions $\varphi_k \in L(E_1)$ such that*

$f(x)| \leqq \varphi_k(x_k)$ for $(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r) \in E_{r-1}$ and for $k = 1, 2, \ldots, r$.

Let $J_k \subset E_1$ $(k = 1, 2, \ldots, r)$ be open intervals and $J = \prod_{k=1}^{r} J_k$. If we put, for brevity, $y = (x_1, x_2, \ldots, x_n)$, $v = (u_{n+1}, u_{n+2}, \ldots, u_r)$, let

(18)

$$\lim_{\delta \to 0+} \int_{\substack{|u_{i_k}| < \delta;\, k=1,2,\ldots,s \\ v \in E_{r-n}}} |u_{i_1} u_{i_2} \ldots u_{i_s}|^{-1} . |\Delta^{i_1, i_2, \ldots, i_s}_{u_{i_1}, u_{i_2}, \ldots, u_{i_s}} f(y, v)|\, du_{i_1}\, du_{i_2} \ldots du_{i_s}\, dv = 0$$

almost uniformly on the set $\prod_{k=1}^{n} J_k$ for all the combinations $(i_1, i_2, \ldots, i_s)$ of the numbers $1, 2, \ldots, n$, taken s at a time $(s = 1, 2, \ldots, n)$, where $n = 1, 2, \ldots, r$, for all possible combinations of the variables.

Then the Fourier integral of the function f is almost uniformly convergent on J to the function f.

Proof. Let us choose $\varepsilon > 0$ and compact sets $K_k \subset J_k$ $(k = 1, 2, \ldots, r)$ and put $K = \prod_{k=1}^{r} K_k$. There is a $\delta > 0$ such that all integrals in (18) are smaller than ε on the set K. Let this δ be stable, then for all $x \in K$

$$\left| \int_{\substack{|u_k| < \delta \\ k=1,2,\ldots,r}} f(u + x) \prod_{k=1}^{r} \frac{\sin A u_k}{u_k}\, du - \int_{\substack{|u_k| < \delta \\ k=1,2,\ldots,r}} f(x) \prod_{k=1}^{r} \frac{\sin A u_k}{u_k}\, du \right| \leqq$$

$$\leqq \sum_{i=1}^{r} \left| \int_{\substack{|u_k| < \delta \\ k=1,2,\ldots,r}} \Delta^i_{u_i} f(x) \prod_{k=1}^{r} \frac{\sin A u_k}{u_k}\, du \right| + \ldots +$$

$$+ \left| \int_{\substack{|u_k| < \delta \\ k=1,2,\ldots,r}} \Delta^{1,2,\ldots,r}_{u_1, u_2, \ldots, u_r} f(x) \prod_{k=1}^{r} \frac{\sin A u_k}{u_k}\, du \right| \leqq (2^r - 1) . \varepsilon .$$

If we substitute (4), with n instead of r, then in the integral

$$\int_{\substack{|u_p|<\delta\,(0<p\leqq n)\\ |u_q|>\delta\,(n<q\leqq r)}} f(u + x)\,.\prod_{k=1}^{r} \frac{\sin Au_k}{u_k}\,du\,,$$

we get 2^n addends which all tend to zero with $A \to +\infty$ uniformly on K. Let us show this for

$$\left|\int_{\substack{|u_p|<\delta\,(0<p\leqq n)\\ |u_q|>\delta\,(n<q\leqq r)}} \Delta^{1,2,\ldots,s}_{u_1,u_2,\ldots,u_s} f(y, v + z)\,.\prod_{k=1}^{r} \frac{\sin Au_k}{u_k}\,du\right| \leqq \varepsilon +$$

$$+ (2\pi)^{n-s}\,.\left|\int_{\substack{\Delta<|u_p|<\delta\,(0<p\leqq s)\\ \delta<|u_q|<c\,(n<q\leqq r)}} \Delta^{1,2,\ldots,s}_{u_1,u_2,\ldots,u_s} f(y, v + z) \prod_{\substack{1\leqq k\leqq s\\ n<k\leqq r}} \frac{\sin Au_k}{u_k}\,du_1\,du_2\ldots du_s\,dv\right| < 2\varepsilon$$

where, for brevity, we put $y = (x_1, x_2, \ldots, x_n)$, $z = (x_{n+1}, x_{n+2}, \ldots, x_r)$, $v := (u_{n+1}, u_{n+2}, \ldots, u_r)$ for a sufficiently small number $\Delta > 0$ sufficiently great $C > 0$ and sufficiently great $A > 0$.

Remark 4. Equation (18) holds, if the function

$$F(x_1, x_2, \ldots, x_n) = \int_{E_{r-n}} |f_{x_{i_1},x_{i_2},\ldots,x_{i_s}}(y, v)|\,dv$$

is almost uniformly bounded on the interval $\prod_{k=1}^{n} J_k$ and for each $v \in E_{r-n}$ the derivative $f_{x_{i_1},x_{i_2},\ldots,x_{i_s}}(y, v)$ is also almost uniformly bounded on the $\prod_{k=1}^{n} J_k$.

CHAPTER II

Throughout this chapter we shall assume that the given function f has the property BV and for such a function we shall seek a sufficient condition for the existence of limit (9).

Lemma 8. *For* $-\infty \leqq a_k < b_k \leqq +\infty$ $(k = 1, 2, \ldots, r)$, *let* $\lambda_k(t)$ $(k = 1, 2, \ldots, r)$ *be functions which satisfy the inequalities*

$$0 \leqq \int_{a_k}^{\xi_k} \lambda_k(t)\,dt \leqq c_k \quad \textit{for} \quad \xi_k \in \langle a_k, b_k\rangle \quad (k = 1, 2, \ldots, r)\,.$$

Let the monotone function f be non-negative on the interval $I = \prod_{k=1}^{r} \langle a_k, b_k\rangle$.

Then the integral $\int_I f(x) \prod_{k=1}^{r} \lambda_k(x_k)\,dx$ *is convergent and*

$$0 \leqq \int_I f(x) \prod_{k=1}^{r} \lambda_k(x_k)\,dx \leqq \max_{x\in I} f(x)\,.\prod_{k=1}^{r} c_k\,. \tag{19}$$

Proof. For $r = 1$ inequality (19) is obvious from the 2nd mean value theorem. Further for $r > 1$ the proof follows with mathematical induction.

Remark 1. The assumptions of lemma (8) hold for the function $\lambda(t) = t^{-1} . \sin t$ for $t \neq 0$, $\lambda(0) = 1$ on the interval $\langle 0, +\infty)$. That is,

$$(20) \qquad 0 \leqq \int_0^{\xi} t^{-1} \sin t \, dt \leqq \pi \quad \text{for} \quad \xi \geqq 0 .$$

Lemma 9. *Let f be a non-negative, monotone and bounded function on the interval* $I = \prod_{k=1}^{r} \langle 0, +\infty)$.

Then

$$(21) \qquad \lim_{A \to +\infty} \left(\frac{2}{\pi}\right)^r . \int_I f(x) \prod_{k=1}^{r} \frac{\sin Ax_k}{x_k} \, dx = f(0+, 0+, \ldots, 0+) .$$

Proof. The integral

$$\int_I f(x) \prod_{k=1}^{r} \frac{\sin Ax_k}{x_k} \, dx = \int_I f\left(\frac{x}{A}\right) \prod_{k=1}^{r} \frac{\sin x_k}{x_k} \, dx$$

is uniformly convergent on I so we take the limit within the integral.

Theorem 5. *Let a function f have the property BV and $x \in E_r$. Then*

$$(22) \quad \lim_{A \to +\infty} \pi^{-r} . \int_{E_r} f(u + x) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k} \, du = 2^{-r} . \sum f(x_1 \pm 0, x_2 \pm 0, \ldots, x_r \pm 0) .$$

Proof. Theorem 5 is the immediate consequence of both lemmata 2,9.

Example. The function $f(x, y) = 0$ for $x . y = 0$, $f(x, y) = x^{-2} y^{-2} \sin x^3 \sin y^3$ for $xy \neq 0$ fulfils the assumptions of theorem 3 at each point $(x, y) \in E_2$, but it does not fulfil the assumptions of theorem 5 because the function f has on each unbounded interval the variation $+\infty$ with respect to each of its variable.

Remark 2. According to lemma 3 the assumptions of theorem 5 are fulfiled when the function f is continuous on E_r, has the continuous derivative $f_{x_1, x_2, \ldots, x_r}$, which has Lebesgue integral on E_r and $\lim_{\|x\| \to +\infty} f(x) = 0$.

Theorem 6. *Let a function f have the property BV and let the functions f_j ($j = 1, 2, 3, 4$) be continuous in some open set $G \subset E_r$.*

Then the Fourier integral of the function f is almost uniformly convergent on the set G to the function f.

Proof. Let $K \subset G$ be a compact set, then the functions f_j ($j = 1, 2, 3, 4$) are uniformly continuous on the set K. There is a number M such that $|f_j(x)| \leqq M$ for $x \in E_r$ ($j = 1, 2, 3, 4$). For each $x \in K$ there exist according to the lemma 2 the functions g_j^x $(j = 1, 2, 3, 4)$ which are non-negative and monotone on each closed octant with the origin at the point x and for which

$$\lim_{\|\xi\| \to +\infty} g_j^x(\xi) = 0\,; \quad |g_j^x(\xi)| \leqq 2^r . M\,, \quad \text{for} \quad \xi \in \mathrm{E}_r \quad (j = 1, 2, 3, 4)\,;$$

$$f(u + x) = (g_1^x(u) - g_2^x(u)) + i(g_3^x(u) - g_4^x(u)) = g^x(u)\,, \quad \text{for} \quad u \in \mathrm{E}_r\,.$$

Furthermore

(23)

$$\int_{\mathrm{E}_r} f(u + x) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k}\, du = \int_{\mathrm{E}_r} g^x(u) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k}\, du = \int_{\mathrm{E}_r} g^x\left(\frac{u}{A}\right) \prod_{k=1}^{r} \frac{\sin u_k}{u_k}\, du\,.$$

In a similar way as in lemma 9 we can show that the last integral in (23) is uniformly convergent, independently with respect to $x \in \mathrm{K}$ and on each bounded set the integrand has the integrable majorante $2^r . M$. So we can carry out the limiting cross

$$\lim_{A \to +\infty} \int_{\mathrm{E}_r} f(u + x) \prod_{k=1}^{r} \frac{\sin Au_k}{u_k}\, du = \lim_{A \to +\infty} \int_{\mathrm{E}_r} g^x\left(\frac{u}{A}\right) \prod_{k=1}^{r} \frac{\sin u_k}{u_k}\, du =$$
$$= \pi^r . g^x(0) = \pi^r . f(x)\,.$$

CHAPTER III

Till now we have taken the external integral in (1) in the sense of Cauchy. In this chapter we take it in the sense of Fejér, which means

(24) $$\int_{\mathrm{E}_r} F(x)\, dx = \lim_{A \to +\infty} \int_{\substack{|x_k| < A \\ k = 1, 2, \ldots, r}} \prod_{k=1}^{r} \left(1 - \frac{|x_k|}{A}\right) . F(x)\, dx\,.$$

We shall search for the conditions for the convergence of this Fourier integral (1).

Lemma 10. *Let*

(25) $$\lim_{A \to +\infty} \int_{\substack{|x_k| < A \\ k = 1, 2, \ldots, r}} f(x)\, dx = \mathrm{I}\,.$$

Then also

(26) $$\lim_{A \to +\infty} \int_{\substack{|x_k| < A \\ k = 1, 2, \ldots, r}} \prod_{k=1}^{r} \left(1 - \frac{|x_k|}{A}\right) . f(x)\, dx = \mathrm{I}\,.$$

Proof. Let us take instead of the function f the function $F(x) = \sum f(\pm x_1, \pm x_2 \ldots, \pm x_r)$. (This means the summa 2^r addends by all possible combinations of signs.) The function F is even with respect to all its variables and

$$\int_{\substack{|x_k|<A\\ k=1,2,\dots,r}} f(x)\,\mathrm{d}x = \int_{\substack{0<x_k<A\\ k=1,2,\dots,r}} F(x)\,\mathrm{d}x\,,$$

$$\int_{\substack{|x_k|<A\\ k=1,2,\dots,r}} f(x)\prod_{k=1}^{r}\left(1-\frac{|x_k|}{A}\right)\mathrm{d}x = \int_{\substack{0<x_k<A\\ k=1,2,\dots,r}} F(x)\prod_{k=1}^{r}\left(1-\frac{x_k}{A}\right)\mathrm{d}x\,.$$

It is sufficient to prove lemma 10 for the function F on the octant $\prod_{k=1}^{r}\langle 0,+\infty)$.

For any number $\varepsilon>0$, there is a number $A_0>0$ such that for any $A>A_0$

$$\left|\int_{\substack{0<x_k<A\\ k=1,2,\dots,r}} F(x)\,\mathrm{d}x - \mathrm{I}\right| < \varepsilon\,.$$

Then

$$\left|\int_{\substack{0<x_k<A\\ k=1,2,\dots,r}} F(x)\prod_{k=1}^{r}\left(1-\frac{x_k}{A}\right)\mathrm{d}x - \mathrm{I}\right| \leqq \left|\int_{\substack{0<x_k<A\\ k=1,2,\dots,r}} F(x)\,\mathrm{d}x - \mathrm{I}\right| +$$

$$+\left|\int_{\substack{0<x_k<A_0\\ k=1,2,\dots,r}} F(x)\left(1-\prod_{k=1}^{r}\left(1-\frac{x_k}{A}\right)\right)\mathrm{d}x\right| + \left|\int_{\substack{0<x_k<A\\ \max x_k>A_0\\ k=1,2,\dots,r}} F(x)\,\mathrm{d}x\right| +$$

$$+\left|\int_{\substack{0<x_k<A\\ \max x_k>A_0\\ k=1,2,\dots,r}} F(x)\prod_{k=1}^{r}\left(1-\frac{x_k}{A}\right)\mathrm{d}x\right| = \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4\,.$$

Obviously $\mathrm{I}_1<\varepsilon$, $\lim_{A\to+\infty}\mathrm{I}_2=0$, $\mathrm{I}_3<2\varepsilon$. Let us fix $A>A_0$ such that $\mathrm{I}_2<\varepsilon$. If we make up the indicated multiplication at I_4 and estimate according to the 2nd mean value theorem each addends, we get the estimate $\mathrm{I}_4 \leqq 2^r\,.\,2\varepsilon$.

Theorem 7. *Let f be a function, defined on E_r, which is either integrable on E_r or has the property BV. Let us put for $x\in\mathrm{E}_r$, $A>0$*

$$\mathrm{I}_A(x) = (2\pi)^{-r}\,.\left|\int_{\substack{|\xi_k|<A\\ k=1,2,\dots,r}} \prod_{k=1}^{r}\left(1-\frac{|\xi_k|}{A}\right).\left(\int_{\mathrm{E}_r} f(u+x)\,.\,e^{i(u,\xi)}\,\mathrm{d}u\right)\mathrm{d}\xi\,.\right.$$

Then

$$\mathrm{I}_A(x) = (2\pi)^{-r}\,.\int_{\mathrm{E}_r} f\left(x+\frac{u}{A}\right)\prod_{k=1}^{r}\left(\frac{\sin\frac{1}{2}u_k}{\frac{1}{2}u_k}\right)^2\mathrm{d}u\,. \tag{27}$$

The proof is similar to the proof of the theorem 1.

Lemma 11. *Let f be a function, defined and bounded on E_r. For a function $\lambda\in\mathrm{L}(\mathrm{E}_r)$ on each octant K let $\int_{\mathrm{K}}\lambda(u)\,\mathrm{d}u = 2^{-r}$.*

Then

$$\lim_{A \to +\infty} \int_{E_r} f\left(x + \frac{u}{A}\right) \lambda(u)\,\mathrm{d}u = 2^{-r} \,.\, \sum f(x \pm 0)\,, \tag{28}$$

for each point $x \in E_r$ at which exist the limits $f(x \pm 0)$. If moreover the function f is continuous in an open set $G \subset E_r$, then (28) holds almost uniformly on G.

Proof. I. The integral $\int_{E_r} f(x + u/A)\,\lambda(u)\,\mathrm{d}u$ is uniformly convergent since f is a bounded function, and the integrable majorante is $|\lambda(u)| \,.\, \max_{x \in E_r} |f(x)|$. So we can take the limit $A \to +\infty$ beyond the integration sign.

II. Let $K \subset G$ be a compact set, then the function f is uniformly continuous on K. Let us choose a bounded interval I so that $K \subset I$ and $\int_{E_r - I} |\lambda(u)|\,\mathrm{d}u < \varepsilon \,.\, (2M)^{-1}$, where $M = \max_{x \in E_r} |f(x)|$. Then for sufficiently great A and for all $x \in K$

$$\left|\int_{E_r} f\left(x + \frac{u}{A}\right) \lambda(u)\,\mathrm{d}u - f(x)\right| \leqq \left|\int_I \left(f\left(x + \frac{u}{A}\right) - f(x)\right) \lambda(u)\,\mathrm{d}u\right| + \\ + 2M \,.\, \int_{E_r - I} |\lambda(u)|\,\mathrm{d}u < 2\varepsilon\,.$$

Theorem 8. *Let f be a complex function, defined and bounded on E_r, which is either integrable on E_r or has property BV. Then the Fourier integral of the function f is convergent in the sense of Fejér at each point $x \in E_r$ at which both the limits $f(x \pm 0)$ and the sum $2^{-r} \,.\, \sum f(x \pm 0)$ exist.*

If moreover the function f is continuous on an open set $G \subset E_r$, then the convergence of Fourier integral is almost uniform on G.

Proof. It is sufficient to put $\lambda(u) = (2\pi)^{-r} \,.\, \prod_{k=1}^{r} ((\sin \frac{1}{2}u_k)/\frac{1}{2}u_k)^2$ in lemma 11.

Theorem 9. *Let f be a complex function, defined and bounded on E_r, which is either integrable on E_r or has the property BV. Then at each point $x \in E_r$, at which there exist the limits $f(x \pm 0)$ and the external integral (1) is taken in the sense of Cauchy (it is fulfiled especially when the Fourier image of the function f is integrable), the Fourier integral of the function f is convergent to the value $2^{-r} \,.\, \sum f(x \pm 0)$.*

Proof is the immediate consequence of the lemma 10.

Remark. Let the function f, defined on E_r, be bounded and have the improper Riemann integral on E_r, then it is almost everywhere (in the sense of Lebesgue) continuous. Then according to the theorem 8 the Fourier integral of f is convergent in the sense of Fejér almost everywhere to f.

References

[1] *V. Jarník*: Integrální počet. Praha 1955.

[2] *L. Tonelli*: Serie trigonometriche. Bologna 1928.

[3] *S. Bochner*: Vorlesungen über Fouriersche Integrale. Leipzig 1932. (There are also the English and Russian translations: Lectures on Fourier Integrals, Princeton, 1959; Лекции об интегралах Фурье, Moscow 1962.)

[4] *S. Bochner, K. Chandrasekharan*: Fourier Transforms. Princeton 1949.

Výtah

MNOŽNÝ FOURIERŮV INTEGRÁL

JAN KUČERA, Praha

V článku je formulí (1) definován Fourierův integrál pro funkce f více proměnných, které jsou buďto integrovatelné nebo mají v jistém smyslu konečnou variaci. Jestliže konvergují všechny integrály (10) pro všechny možné permutace $(i_1, i_2, \ldots, i_s)$ a integrál (14) a jestliže konverguje Fourierův integrál pro všechny funkce, které dostaneme zafixujeme-li u funkce f některé proměnné, potom Fourierův integrál funkce f konverguje a má hodnotu f.

Резюме

КРАТНЫЙ ИНТЕГРАЛ ФУРЬЕ

ЯН КУЧЕРА (Jan Kučera), Прага

В статье определяется формулой (1) интеграл Фурье для функций f многих переменных, которые или интегрируемы или имеют в определенном смысле ограниченное изменение. Если сходятся все интегралы (10) для всех возможных перестановок $(i_1, i_2, \ldots, i_s)$, далее интеграл (14) и, наконец, интеграл Фурье для всех функций, полученных путем закрепления у функции f некоторых переменных, то сходится и интеграл Фурье функции f к значению f.