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ON A CERTAIN MODIFICATION OF THE THEOREM ON THE CONTINUOUS DEPENDENCE ON A PARAMETER

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Theorems modified with respect to [4] on the existence of generalized Perron's integral and on the continuous dependence on a parameter are proved.

In the papers [1] and [2], the assumption of convergence of the sum

$$\sum_{n=1}^{\infty} 2^n \omega_1\left(\frac{\sigma}{2^n}\right) \omega_2\left(\frac{\sigma}{2^n}\right) \tag{1}$$

was investigated in the theorem on the continuous dependence on a parameter (cf. [4], theorem 4,2,1). It was shown that this assumption cannot be weakened, if at least one of the following conditions is fulfilled:

i) There exist $\alpha > 0$, $d > 0$ such that the function $\eta^{-\alpha}\,\omega_i(\eta)$ is non-decreasing (for $i = 1, 2$) and $\omega_2(\eta) \leqq d\omega_1(\eta)$ for all $\eta \in \langle 0, \sigma \rangle$;

ii) $\omega_1(\eta) = \omega_2(\eta)$ on the interval $\langle 0, \sigma)$.

Let us now draw attention to a special case in which neither of these conditions is fulfilled.

Consider a sequence of ordinary differential equations

$$\frac{dx}{dt} = a_k(t)\,x + b_k(t)\,, \quad x(0) = \xi \tag{2}$$

for $k = 0, 1, 2, \ldots$, where a_k, b_k are continuous on $\langle 0, T \rangle$. Let $A_k(t) = \int_0^t a_k(\tau)\,d\tau$, $B_k(t) = \int_0^t b_k(\tau)\,d\tau$, $A_k(t) \to A_0(t)$, $B_k(t) \to B_0(t)$, uniformly on $\langle 0, T \rangle$. Let the functions A_k, B_k fulfil the inequalities

$$|A_k(t_2) - A_k(t_1)| \leqq L|t_2 - t_1|\,,$$
$$|B_k(t_2) - B_k(t_1)| \leqq \omega(|t_2 - t_1|)$$

($k = 0, 1, 2, \ldots$) for some constant L and some function ω, $\omega(0) = 0$, for arbitrary $t_1, t_2 \in \langle 0, T \rangle$.

If we put $\omega_1(\eta) = K\,\omega(\eta)$, $\omega_2(\eta) = L\eta$, then the functions $F_k(x, t) = A_k(t)\,x + B_k(t)$ fulfil the assumptions of the theorem mentioned above on a set $t \in \langle 0, T \rangle$, $|x| \leqq M$.

As the functions A_k fulfil the Lipschitz condition with a constant L independent of k, the functions $a_k(t)$ are uniformly bounded. Therefore the functions $f_k(x, t) = a_k(t)\, x + b_k(t)$ are equi-continuous in x. The sequence of equations (2) fulfils all the assumptions of theorem 1 from [3] (see also [4], theorem 0,1). According to this theorem, the sequence of solutions $x_k(t)$ of (2) converges uniformly to $x_0(t)$ with $k \to \infty$. This convergence does not depend on the behaviour of the sum (1).

This result is correct even for the generalized differential equations which have been introduced by J. KURZWEIL in [4]. Of course, the proof cannot be carried through by means of the theorem mentioned above, because we know nothing about the behaviour nor even about the existence of the functions $f_k(x, t)$. Instead of this, we can start from the formula of variation of constants. Then we obtain the required result by means of integration by parts (cf. [6]).

This fact leads us to the question, whether, in similar cases, the assumption of convergence of the sum (1) cannot be weakened.

In the present paper we introduce a new assumption to replace that of convergence of (1), if $\omega_2(\eta) = \eta\, \varphi(\eta)$, where $\varphi(\eta) \to \infty$ with $\eta \to 0$. The new criterion gives better results in some cases. Roughly speaking, this happens if $\eta^\alpha \varphi(\eta) \to 0$ with $\eta \to 0$ for all $\alpha > 0$.

Let $\omega_1(\eta)$, $\omega_2(\eta)$ be continuous increasing functions of η on $\langle 0, \sigma \rangle$ $(\sigma > 0)$, $\omega_1(0) = \omega_2(0) = 0$. Let $\omega_2(\eta) = \eta\, \varphi(\eta)$, where $\varphi(\eta)$ is a continuous decreasing function, $\lim\limits_{\eta \to 0+} \varphi(\eta) = \infty$.

Theorem 1. *Let the function $U(\tau, t)$ be defined and continuous on a square $Q = \langle \tau_*, \tau^* \rangle \times \langle \tau_*, \tau^* \rangle$. Let the inequality $2^{n_0} \geqq \varphi(\sigma)$ hold for some positive integer n_0. If we denote by ψ the function inverse to φ, let*

$$(3) \qquad \sum_{n=n_0}^{\infty} 2^n \omega_1(\psi(2^n)) < \infty .$$

Let the inequality

$$(4) \quad |U(\tau_1, t_1) - U(\tau_1, t_2) - U(\tau_2, t_1) + U(\tau_2, t_2)| \leqq \omega_1(|\tau_2 - \tau_1|)\, \omega_2(|t_2 - t_1|)$$

hold for all $\tau_1, \tau_2, t_1, t_2 \in Q$, $|\tau_2 - \tau_1| \leqq \sigma$, $|t_2 - t_1| \leqq \sigma$. Let $\tau_* \leqq \lambda_1 < \lambda_2 \leqq \tau^*$.

Then the integral (cf. [4])

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U(\tau, t)$$

exists and the inequalities

$$(5) \qquad \left| \int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U(\tau, t) - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1) \right| \leqq \lambda\, \Psi(\lambda),$$

$$(6) \qquad \left| \int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U(\tau, t) - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1) \right| \leqq \lambda\, \Psi(\lambda)$$

hold with $\lambda = |\lambda_2 - \lambda_1|$ and with a function Ψ depending only on ω_1, ω_2, for which

(7) $$\lim_{\eta \to 0+} \Psi(\eta) = 0 .$$

Proof. Let $\varphi(\eta) \geqq 4$ on an interval $\langle 0, \sigma_1 \rangle$. Let us choose $\eta \in \langle 0, \sigma_1 \rangle$. Then there exists a positive integer $N = N(\eta)$ such that

(8) $$2^{N+2} > \varphi(\eta) \geqq 2^{N+1}$$

and evidently

(9) $$\lim_{\eta \to 0+} N(\eta) = \infty .$$

Let us form a sequence of positive integers k_n such that $k_1 = 1$, $k_{n+1} = k_n r_n$, where

(10) $$k_n r_n > \frac{\eta}{\psi(2^{N+n+1})} \geqq k_n(r_n - 1) .$$

Then, evidently,

$$\frac{\eta}{c\,\psi(2^{N+1})} \geqq k_1 > \frac{\eta}{\psi(2^{N+1})}$$

holds with $0 < c \leqq \frac{1}{2}$, according to (8). Moreover, there is for all $n > 1$

(11) $$\frac{\eta}{\psi(2^{N+n})} < k_n \leqq 2\frac{\eta}{\psi(2^{N+n})} ,$$

(12) $$r_{n-1} > 1 .$$

Let us first prove this for $n = 2$. Then according to (8), there holds

$$1 \leqq \frac{\eta}{\psi(2^{N+2})} < k_1 r_1 = r_1 ,$$

$$\frac{\eta}{\psi(2^{N+2})} < k_2 = k_1 r_1 \leqq 2k_1(r_1 - 1) \leqq \frac{2\eta}{\psi(2^{N+2})} .$$

Now, assume that (11), (12) hold for some $n \geqq 2$. Then, according to the definition of k_n,

$$k_n r_n > \frac{\eta}{\psi(2^{N+n+1})} \geqq k_n(r_n - 1) .$$

As $\eta\,\varphi(\eta) \to 0$ monotonously with $n \to \infty$, also $\lim_{n \to \infty} 2^n\,\psi(2^n) = 0$ monotonously. Hence

(13) $$\psi(2^{N+n+1}) < \tfrac{1}{2}\,\psi(2^{N+n}) .$$

According to this and to (11), we have

$$1 \leqq \frac{2\eta}{k_n\,\psi(2^{N+n})} < \frac{\eta}{k_n\,\psi(2^{N+n+1})} ,$$

and therefore

$$r_n > \frac{\eta}{k_n\,\psi(2^{N+n+1})} > 1\,.$$

Then

$$\frac{\eta}{\psi(2^{N+n+1})} < k_{n+1} = k_n r_n \leqq 2k_n(r_n - 1) \leqq \frac{2\eta}{\psi(2^{N+n+1})}\,.$$

Let us define

$$\Psi(\eta) = \sum_{n=1}^{\infty} \varphi\left(\frac{\eta}{k_{n+1}}\right) \omega_1\left(\frac{\eta}{k_n}\right).$$

If we write (10) in the form

$$\tfrac{1}{2}\,\psi(2^{N+n}) \leqq \frac{\eta}{k_n} < \psi(2^{N+n})\,,$$

we see that

$$\varphi\left(\frac{\eta}{k_n}\right) \leqq \varphi(\tfrac{1}{2}\psi(2^{N+n}))\,.$$

But according to (13), the inequality

$$\varphi(\tfrac{1}{2}\psi(2^{N+n})) < 2^{N+n+1}$$

holds. Hence

$$\Psi(\eta) \leqq \sum_{n=1}^{\infty} 2^{N+n+2}\omega_1(\psi(2^{N+n})) = 4 \sum_{n=N+1}^{\infty} 2^n\omega_1(\psi(2^n))\,.$$

Since the sum (3) converges, on the right side of this inequality we have the remainder of a convergent sum. Consequently, (7) holds.

Let us approximate the functions $U(\tau, t)$ by a sequence of functions $U_k(\tau, t)$ having the following properties (cf. [5]):

1. $U_k \to U$ uniformly on Q;
2. U_k have continuous partial derivatives of the second order;
3. for every $\vartheta > 0$ there exists a number $K(\vartheta)$ such that for all $k > K(\vartheta)$ and $\tau_1, \tau_2, t_1, t_2 \in \langle \tau_* + \vartheta, \tau^* - \vartheta \rangle$, $|\tau_2 - \tau_1| \leqq \sigma$, $|t_2 - t_1| \leqq \sigma$, the functions $U_k(\tau, t)$ fulfil the inequality (4).

According to lemma 1 from [5], the integral $\int_{\varrho_1}^{\varrho_2} \mathrm{D}_t U_k(\tau, t)$ exists if $\vartheta > 0$, $\varrho_1, \varrho_2 \in \langle \tau_* + \vartheta, \tau^* - \vartheta \rangle$, $|\varrho_2 - \varrho_1| \leqq \sigma_1$, $k > K(\vartheta)$.

Let us denote

$$S(U_k, n) = U_k\left(\varrho_1, \varrho_1 + \frac{\varrho}{n}\right) - U_k(\varrho_1, \varrho_1) + U_k\left(\varrho_1 + \frac{\varrho}{n}, \varrho_1 + \frac{2\varrho}{n}\right) -$$

$$- U_k\left(\varrho_1 + \frac{\varrho}{n}, \varrho_1 + \frac{\varrho}{n}\right) + \ldots + U_k\left(\varrho_1 + \frac{n-1}{n}\varrho, \varrho_2\right) - U_k\left(\varrho_1 + \frac{n-1}{n}\varrho,\right.$$

$$\left.\varrho_1 + \frac{n-1}{n}\varrho\right) = \sum_{i=0}^{n-1}\left[U_k\left(\varrho_1 + \frac{i}{n}\varrho, \varrho_1 + \frac{i+1}{n}\varrho\right) - U_k\left(\varrho_1 + \frac{i}{n}\varrho, \varrho_1 + \frac{i}{n}\varrho\right)\right.$$

where $\varrho = \varrho_2 - \varrho_1$, and analogously

$$Z(U_k, n) = \sum_{i=0}^{n-1} \left[U_k\left(\varrho_1 + \frac{i+1}{n}\varrho, \varrho_1 + \frac{i+1}{n}\varrho\right) - U_k\left(\varrho_1 + \frac{i+1}{n}\varrho, \varrho_1 + \frac{i}{n}\varrho\right)\right].$$

In the same manner as in the lemma mentioned above it is easily shown that

$$\int_{\varrho_1}^{\varrho_2} \mathrm{D}_t U(\tau, t) = \lim_{n\to\infty} S(U_k, n) = \lim_{n\to\infty} Z(U_k, n).$$

Now estimate the difference $|S(U_k, k_{n+1}) - S(U_k, k_n)|$. We have $k_{n+1} = k_n r_n$ and, consequently,

$$|S(U_k, k_{n+1}) - S(U_k, k_n)| = \left| \sum_{i=0}^{k_n - 1} \left\{ \sum_{p=0}^{r_n - 1} \left[U_k\left(\varrho_1 + \left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right)\varrho, \varrho_1 + \right.\right.\right.\right.$$

$$+ \left(\frac{i}{k_n} + \frac{p+1}{k_{n+1}}\right)\varrho\Big) - U_k\left(\varrho_1 + \left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right)\varrho, \varrho_1 + \left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right)\varrho\right) -$$

$$- \sum_{p=0}^{r_n-1} U_k\left(\varrho_1 + \frac{i}{k_n}\varrho, \varrho_1 + \left(\frac{i}{k_n} + \frac{p+1}{k_{n+1}}\right)\varrho\right) - U_k\left(\varrho_1 + \frac{i}{k_n}\varrho, \varrho_1 + \right.$$

$$\left.\left.\left.\left. + \left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right)\varrho\right)\right]\right\}\right|.$$

If we put

$$\tau_1 = \varrho_1 + \frac{i}{k_n}\varrho, \qquad \tau_2 = \varrho_1 + \varrho\left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right),$$

$$t_1 = \varrho_1 + \varrho\left(\frac{i}{k_n} + \frac{p}{k_{n+1}}\right), \quad t_2 = \varrho_1 + \varrho\left(\frac{i}{k_n} + \frac{p+1}{k_{n+1}}\right),$$

then according to (4),

$$|S(U_k, k_{n+1}) - S(U_k, k_n)| \leqq \sum_{i=0}^{k_n-1} \sum_{p=0}^{r_n-1} \omega_1\left(\frac{\varrho}{k_n}\right)\omega_2\left(\frac{\varrho}{k_{n+1}}\right) = \varrho\omega_1\left(\frac{\varrho}{k_n}\right)\varphi\left(\frac{\varrho}{k_{n+1}}\right).$$

Further,

(14) $$\left|\int_{\varrho_1}^{\varrho_2} \mathrm{D}_t U_k(\tau, t) - U_k(\varrho_1, \varrho_2) + U_k(\varrho_1, \varrho_1)\right| =$$

$$= |\lim_{n\to\infty} S(U_k, k_n) - S(U_k, k_1)| \leqq \sum_{n=1}^{\infty} |S(U_k, k_{n+1}) - S(U_k, k_n)| \leqq \varrho\, \Psi(\varrho).$$

Similarly we find that

(15) $$\left|\int_{\varrho_1}^{\varrho_2} \mathrm{D}_t U_k(\tau, t) - U_k(\varrho_2, \varrho_2) + U_k(\varrho_2, \varrho_1)\right| \leqq \varrho\, \Psi(\varrho).$$

Let $0 < \delta < \sigma_1$ and let us denote by A a decomposition $\{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ of the interval $\langle\lambda_1, \lambda_2\rangle$, $\tau_* + \vartheta < \lambda_1 < \lambda_2 < \tau^* - \vartheta$, for which there is

$$\tau_j - \alpha_{j-1} < \delta\,, \quad \alpha_j - \tau_j < \delta \tag{16}$$

for $j = 1, 2, 3, \ldots, s$. If

$$R(V, A) = \sum_{j=1}^{s} [V(\tau_j, \alpha_j) - V(\tau_j, \alpha_{j-1})]\,,$$

then

$$\left|\int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U_k - R(U_k, A)\right| \leqq$$

$$\leqq \sum_{j=1}^{s} \left|\int_{\alpha_{j-1}}^{\tau_j} \mathrm{D}_t U_k - U_k(\tau_j, \tau_j) + U_k(\tau_j, \alpha_{j-1}) + \int_{\tau_j}^{\alpha_j} \mathrm{D}_t U_k - U_k(\tau_j, \alpha_j) + U_k(\tau_j, \tau_j)\right| \leqq$$

$$\leqq \sum_{j=1}^{s} [(\tau_j - \alpha_{j-1})\, \Psi(\tau_j - \alpha_{j-1}) + (\alpha_j - \tau_j)\, \Psi(\alpha_j - \tau_j)] \leqq \lambda \sup_{0<\eta\leqq\delta} \Psi(\eta)\,.$$

For two decompositions of $\langle\lambda_1, \lambda_2\rangle$ which both fulfil (16) thence follows

$$|R(U_k, A_1) - R(U_k, A_2)| \leqq 2\lambda \sup_{0<\eta\leqq\delta} \Psi(\eta)\,.$$

As $U_k \to U$ uniformly, also

$$|R(U, A_1) - R(U, A_2)| \leqq 2\lambda \sup_{0<\eta\leqq\delta} \Psi(\eta)\,.$$

According to [4], definition and theorem 1,2,1, it follows that the integral

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U(\tau, t)$$

exists. The inequalities (5), (6) are obtained without difficulty from (14), (15), again taking into consideration the uniform convergence of U_k.

The constant $\vartheta > 0$ being arbitrarily small, evidently $\int_{\lambda_1}^{\lambda_2} \mathrm{D}_t U(\tau, t)$ exists for arbitrary $\lambda_1, \lambda_2 \in (\tau_*, \tau^*)$. The existence of this integral for $\lambda_1, \lambda_2 \in \langle\tau_*, \tau^*\rangle$ (and, of course, the inequalities (5), (6) also) follows from its uniform continuity as a function of its upper (lower respectively) bound, and from the continuity of $U(\tau, t)$ (cf. theorem 1,3,5 in [4]).

The theorem just proved is the starting point in the proof of the theorem on the continuous dependence on a parameter for solutions of differential equations. As we have the estimates (5), (6), this proof can be performed analogously to [4] (cf. [4], theorem 4,2,1). Therefore we shall only formulate the result for classical differential equations.

Let us denote by $\mathsf{F}(G, \omega_1, \omega_2, \sigma)$ a class of functions $F(x, t)$ which fulfil the conditions:

$F(x, t)$ is defined and continuous on an open set $G \subset E_{n+1}$, $F(x, t) \in E_n$;

$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|)$ for $|t_2 - t_1| \leqq \sigma$;

$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1) \leqq \|x_2 - x_1\| \, \omega_2(|t_2 - t_1|)$

for $\|x_2 - x_1\| \leqq 2\omega_1(\sigma)$, $|t_2 - t_1| \leqq \sigma$.

Theorem 2. *Let $\omega_1(\eta), \omega_2(\eta)$ be increasing continuous functions for $\eta \in \langle 0, \sigma \rangle$, $\omega_i(0) = 0$, $\omega_i(\eta) \geqq c\eta$ for $i = 1, 2$, where σ, c are positive constants. Let $\omega_2(\eta) = \eta \, \varphi(\eta)$, where $\varphi(\eta)$ is a decreasing function on $\langle 0, \sigma \rangle$, $\lim_{\eta \to 0+} \varphi(\eta) = \infty$.*

Let us have a sequence of ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_k(x, t), \quad x(\zeta) = \xi \tag{17}$$

for $k = 0, 1, 2, \ldots$ Let $F_k(x, t) = \int_\zeta^t f_k(x, \tau) \, \mathrm{d}\tau$, $F_k \in \mathsf{F}(G, \omega_1, \omega_2, \sigma)$, $F_k \to F_0$ uniformly with $k \to \infty$. For $k = 0$ let there exist a unique solution of (17). Finally, let (3) hold.

Then the sequence of solutions $x_k(t)$ of (17) converges to $x_0(t)$ uniformly with $k \to \infty$.

Note. The assumption (3) can be replaced by another. Let us estimate the integral

$$\int_{x_{n+1}}^{x_n} x^{-2} \, \omega_1\left(\psi\left(\frac{1}{x}\right)\right) \mathrm{d}x,$$

where $x_n = 2^{-n}$. As $\int_{x_{n+1}}^{x_n} x^{-2} \, \mathrm{d}x = 2^n = x_n^{-1}$, we have

$$\frac{1}{x_n} \omega_1\left(\psi\left(\frac{1}{x_{n+1}}\right)\right) \leqq \int_{x_{n+1}}^{x_n} x^{-2} \, \omega_1\left(\psi\left(\frac{1}{x}\right)\right) \mathrm{d}x \leqq \frac{1}{x_n} \omega_1\left(\psi\left(\frac{1}{x_n}\right)\right),$$

$$\tfrac{1}{2} 2^{n+1} \, \omega_1(\psi(2^{n+1})) \leqq \int_{x_{n+1}}^{x_n} x^{-2} \, \omega_1\left(\psi\left(\frac{1}{x}\right)\right) \mathrm{d}x \leqq 2^n \, \omega_1(\psi(2^n)).$$

The assumption (3) is therefore equivalent with the assumption of existence of the integral

$$\int_0^\alpha x^{-2} \, \omega_1\left(\psi\left(\frac{1}{x}\right)\right) \mathrm{d}x \tag{18}$$

for some positive α.

Similarly we can show that the convergence of (1) is equivalent with the existence of the integral

$$\int_0^\gamma x^{-2} \, \omega_1(x) \, \omega_2(x) \, \mathrm{d}x. \tag{19}$$

In fact,

$$\sum_{n=1}^{\infty} 2^n \omega_1 \left(\frac{1}{2^n}\right) \omega_2 \left(\frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \omega_1 \left(\frac{1}{2^n}\right) \varphi \left(\frac{1}{2^n}\right);$$

if we put $x_n = 2^{-n}$ again, then

$$x_n^{-1} \omega_1(x_{n+1}) \omega_2(x_{n+1}) \leqq \int_{x_{n+1}}^{x_n} x^{-2} \omega_1(x) \omega_2(x) \, dx \leqq x_n^{-1} \omega_1(x_n) \omega_2(x_n),$$

$$\frac{1}{2} \omega_1 \left(\frac{1}{2^{n+1}}\right) \varphi \left(\frac{1}{2^{n+1}}\right) \leqq \int_{x_{n+1}}^{x_n} x^{-1} \omega_1(x) \varphi(x) \, dx \leqq \omega_1 \left(\frac{1}{2^n}\right) \varphi \left(\frac{1}{2^n}\right).$$

If $\varphi'(x)$ exists and

$$\chi(x) = -\frac{\varphi(x)}{x \varphi'(x)}$$

is continuous, we can transform the integrals (18), (19) to

$$\int_0^\beta \omega_1(x) \varphi'(x) \, dx, \quad \int_0^\gamma \omega_1(x) \varphi'(x) \chi(x) \, dx$$

respectively. According to the definition of χ, we have

$$\varphi(x) = c \exp \int_x^1 \frac{dt}{t \chi(t)}, \quad x \varphi(x) = c \exp \int_x^1 \left(\frac{1}{t \chi(t)} - \frac{1}{t}\right) dt. \tag{20}$$

As $x \varphi(x) \to 0$ with $x \to 0$, we have $\chi(x) \geqq c' > 0$. Hence the new criterion never gives a worse result than the original one.

On the other hand, it is easy to show that in some cases we obtain a better result (*e. g.* for $\varphi(x) = -\log x$). This can happen if $\chi(x) \to \infty$ with $x \to 0$. From (20) it follows that then the function $\varphi(x)$ diverges more slowly than any negative power of x.

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Výtah

O JISTÉ MODIFIKACI VĚTY O SPOJITÉ ZÁVISLOSTI NA PARAMETRU

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S použitím věty o existenci zobecněného Perronova integrálu, obdobné větě 1 v [5], dokazuje se věta o spojité závislosti na parametru:

Označme $\mathbf{F}(G, \omega_1, \omega_2, \sigma)$ množinu všech funkcí $F(x, t)$, majících vlastnosti:

$$G \subset E_{n+1}\,, \quad F(x, t) \in E_n \quad \text{pro} \quad (x, t) \in G\,;$$
$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) \quad \text{pro} \quad |t_2 - t_1| \leqq \sigma\,,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\, \omega_2(|t_2 - t_1|)$$

pro $|t_2 - t_1| \leqq \sigma,\ \|x_2 - x_1\| \leqq 2\omega_1(\sigma)$.

Buďte $\omega_1(\eta)$, $\omega_2(\eta)$ spojité rostoucí funkce pro $\eta \in \langle 0, \sigma \rangle$; $\omega_i(0) = 0$, $\omega_i(\eta) \geqq c\eta$ pro $i = 1, 2$, kde σ, c jsou kladné konstanty. Buď dále $\omega_2(\eta) = \eta\, \varphi(\eta)$, kde φ je klesající v $\langle 0, \sigma \rangle$, $\lim_{\eta \to 0+} \varphi(\eta) = \infty$.

Buď dána posloupnost diferenciálních rovnic (17) pro $k = 0, 1, 2, \ldots$ Nechť $F_k(x, t) = \int_\zeta^t f_k(x, \tau)\, d\tau$, $F_k \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$, $F_k \to F_0$ stejnoměrně. Nechť existuje jediné řešení $x_0(t)$ rovnice (17) při $k = 0$. Konečně nechť platí (3).

Pak posloupnost řešení $x_k(t)$ rovnic (17) konverguje stejnoměrně k $x_0(t)$.

Резюме

ОБ ОДНОЙ МОДИФИКАЦИИ ТЕОРЕМЫ О НЕПРЕРЫВНОЙ ЗАВИСИМОСТИ ОТ ПАРАМЕТРА

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С помощью теоремы существования обобщенного интеграла Перрона, аналогичной теореме 1 в [5], доказывается теорема о непрерывной зависимости от параметра.

Обозначим через $\mathbf{F}(G, \omega_1, \omega_2, \sigma)$ множество всех функций $F(x, t)$, имеющих следующие свойства:

$$G \subset E_{n+1}, F(x, t) \in E_n \quad \text{для} \quad (x, t) \in G\,;$$
$$\|F(x, t_2) - F(x, t_1)\| \leqq \omega_1(|t_2 - t_1|) \quad \text{для} \quad |t_2 - t_1| \leqq \sigma\,,$$
$$\|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)\| \leqq \|x_2 - x_1\|\, \omega_2(|t_2 - t_1|)$$

для $|t_2 - t_1| \leqq \sigma,\ \|x_2 - x_1\| \leqq 2\omega_1(\sigma)$.

Пусть $\omega_1(\eta)$, $\omega_2(\eta)$ — непрерывные возрастающие функции для $\eta \in \langle 0, \sigma \rangle$; $\omega_i(0) = 0$, $\omega_i(\eta) \geqq c\eta$, для $i = 1, 2$, где σ, c — положительные постоянные. Пусть, далее, $\omega_2(\eta) = \eta\, \varphi(\eta)$, где φ — убывающая в $\langle 0, \sigma \rangle$, $\lim_{\eta \to 0+} \varphi(\eta) = \infty$.

Пусть задана последовательность дифференциальных уравнений (17) для $k = 0, 1, 2, \ldots$ Пусть $F_k(x, t) = \int_\zeta^t f_k(x, \tau)\, d\tau$, $F_k \in \mathbf{F}(G, \omega_1, \omega_2, \sigma)$, $F_k \to F_0$ равномерно. Пусть для $k = 0$ существует единственное решение $x_0(t)$ уравнения (17). Наконец, пусть выполняется (3).

Тогда последовательность решений $x_k(t)$ уравнений (17) сходится равномерно к $x_0(t)$.