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A New Semi-Orthogonal Relation for the Laguerre Polynomials

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Abstract. A new semi-orthogonal relation for the Laguerre polynomials is given with an elementary weight function.

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1. Introduction

The Laguerre polynomials are orthogonal polynomials [1, p.183, (16) and (17)] over the interval $(0, \infty)$ with respect to the weight function $e^{-x}x^a$, if $\mathrm{Re}\,a > -1$.

In this paper, we present a new semi-orthogonal relation for the Laguerre polynomials over the interval $(0, \infty)$ with respect to the weight function $e^{-x}x^{n-m+a-1}$, if $\mathrm{Re}\,a > m-n$. With the help of our semi-orthogonal relation, we obtain a Fourier-Laguerre expansion for an elementary function.

The Laguerre polynomials are defined by the relation [1, p.325, 6(a)]:

$$L_n^a(x) = \frac{(-1)^n}{n!}x^n \, {}_2F_0\left(-n, \; -n-a; \; -; \; -\frac{1}{x}\right) \tag{1.1}$$

2. The Semi-Orthogonal Relation

The semi-orthogonal relation to be established is

$$\int_0^\infty e^{-x}x^{n-m+a-1}L_m^a(x)\,L_n^a(x)\,\mathrm{d}x \tag{2.1}$$

$$= 0, \qquad \text{if } m < n \tag{2.1a}$$

$$= \frac{\Gamma(a)\,(a+1)_n}{n!}, \qquad \text{if } m = n \tag{2.1b}$$

$$= \frac{2\Gamma(a-1)\,(a+2)_n}{n!}, \qquad \text{if } m = n+1 \tag{2.1c}$$

where $\mathrm{Re}\,a > m-n$.

PROOF: In view of (1.1), the integral (2.1) can be written as

$$\frac{(-1)^{m+n}}{m!n!}\int_0^\infty e_{-x}x^{2n+a-1}\ {}_2F_0\left(-m,\ -m-a;\ -;\ -\frac{1}{x}\right)\times$$

$$\times\ {}_2F_0\left(-n,\ -n-a;\ -;\ -\frac{1}{x}\right)\mathrm{d}x=$$

$$=\frac{(-1)^{m+n}}{m!n!}\sum_{r=0}^{m}\frac{(-m)_r(-m-a)_r(-1)^r}{r!}\times$$

$$\times\sum_{u=0}^{n}\frac{(-n)_u(-n-a)_u(-1)^u}{u!}\times$$

$$\times\int_0^\infty e^{-x}x^{2n+a-1-r-u}\mathrm{d}x \tag{2.2}$$

Evaluating the last integral in (2.2) with the help of the definition of the gamma-function [1, p.335, (1)], then using the relation [1, p.275, (8)], viz. $\Gamma(a+1-n)=\frac{(-1)^n\Gamma(a+1)}{(-a)_n}$ and simplifying, the right hand side of (2.2) becomes

$$\frac{(-1)^{m+n}}{m!n!}\sum_{r=0}^{m}\frac{(-m)_r(-a-m)_r}{r!}(-1)^r\Gamma(2n+a-r)\ {}_2F_1\left[{-n,\ -n-a;\ 1 \atop 1-2n-a+r}\right] \tag{2.3}$$

Now applying Vandermode's theorem [1, p.283, 19(a)], viz.

$$F(-n,\ a;\ c,\ 1)=\frac{(c-b)_n}{(c)_n},\qquad n=0,\ 1,\ 2,\ \ldots \tag{2.4}$$

to (2.3) and using the relation $(1-n+r)_n=(-1)^n(-r)_n$, we have

$$\frac{(-1)^{m+n}}{m!n!}\sum_{r=0}^{m}\frac{(-m)_r(-r)_n(-a-m)_r\Gamma(2n+a-r)}{r!(1-2n-a+r)_n}(-1)^{r+n} \tag{2.5}$$

If $r<n$, the numerator of (2.5) vanishes, and since r runs from 0 to m, it follows that (2.5) also vanishes, when $m<n$. Now, it is clear that for $m<n$ all terms of (2.5) vanish, which proves (2.1a).

When $m=n$, using the standard result

$$(-r)_n=\begin{cases}\frac{(-1)^n r!}{(r-n)!}, & \text{if } 0\le n\le r\\ 0, & \text{if } n>r\end{cases} \tag{2.6}$$

and simplifying, we have

$$\int_0^\infty e^{-x}x^{a-1}\{L_n^a(x)\}^2\mathrm{d}x=\frac{\Gamma(a)(a+1)_n}{n!};\qquad \mathrm{Re}a>0, \tag{2.7}$$

which proves (2.1b).

In (2.5), putting $m = n + 1$, using (2.6) and adding the resulting two terms $(r = n,\, n + 1)$, and simplifying, we obtain

$$\int_0^\infty e^{-x} x^{a-2} L_{n+1}^a(x)\, L_n^a(x)\, dx =$$

$$= \frac{2\Gamma(a-1)(a+2)_n}{n!}, \qquad \text{Re}a > 1 \tag{2.8}$$

which proves (2.1c). □

Note. On continuing as above we can find the values of the integral (2.1) for $m = n + 2,\, n + 3,\, n + 4, \ldots$.

3. Fourier-Laguerre Expansion

Based on the relations (2.1a) and (2.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a finite series expansion of the Laguerre polynomials. Specially if $f(x)$ is a suitable function defined for all x, we consider for expansions of the general form

$$f(x) = \sum_{m=0}^{n} C_m x^{-m} L_m^a(x), \qquad 0 < x < \infty, \quad m \le n \tag{3.1}$$

where the Fourier coefficients C_m are given by

$$C_m = \frac{m!}{\Gamma(a)(a+1)_m} \int_0^\infty e^{-x} x^{m+a-1} f(x)\, L_m^a(x)\, dx \tag{3.2}$$

4. Fourier-Laguerre Expansion For x^{-n}

The Fourier-Laguerre expansion to be obtained is

$$f(x) = x^{-n} = \frac{1}{\Gamma(a)} \sum_{m=0}^{n} \frac{(-1)^m (-n)_m \Gamma(m-n+a)\, x^{-m}}{(a+1)_m} L_m^a(x) \tag{4.1}$$

where Re$a > n - m$.

PROOF: On using the following modified form of the integral [2, p.292, (1)]:

$$\int_0^\infty x^{b-1} e^{-x} L_m^a(x)\, dx = (-1)^n \frac{\Gamma(b)\Gamma(b-a)}{n!\Gamma(b-a-m)} \tag{4.2}$$

where Re$b > 0$.

and (3.1) and (3.2) with $f(x)$ given in (4.1), the Fourier-Laguerre expansion (4.1) is obtained. □

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