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Some new classes of topological vector spaces with closed graph theorems

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Abstract. In this note, we investigate non-locally-convex topological vector spaces for which the closed graph theorem holds. In doing so, we introduce new classes of topological vector spaces. Our study includes a direct extension of Pták duality to the non-locally-convex situation.

Keywords: inverse seminorm, Mackey seminorm, nearly-semi-continuous, semi-barrelled, semi-B-complete, semi-infra-(s), semi-Mackey

Classification: 46A30, 47A05

1. Introduction.

This note investigates non-locally-convex (non-LC) situations for which the Closed Graph Theorem (CGT) holds. In doing so, we introduce new classes on non-LC topological vector spaces (TVS's) which complement those given by Adasch [1]–[6], Iyahen [9], [10], Robertson [19] and Tomášek [23], [24]. Our study which includes a direct and natural extension of Pták duality to the non-LC situation, allows for the use of duality arguments and is different from that developed by various authors including Adasch [1]–[6], Iyahen [9], and W. Robertson and A. Robertson [19], [20]. In Section 2, we introduce the notions of *semi-continuous* maps and *inverse seminorms* in order to develop a duality theory (Section 3) which we use to extend (Theorem 7) the Pták CGT [13], [17], [20]. In Section 4, we extend the notion of a *semi-B-complete* space by introducing *semi-infra-(s)* spaces which we show to be maximal for the CGT for *semi-barrelled* domain spaces and semi-continuous maps (Theorem 12). Here, we generalize (Theorem 11) what we call the Kōmura–Adasch–Valdivia CGT (Theorem 10). In Section 5, we define *semi-bornological* spaces and give an extension of Powell's CGT [16], [8] and in Section 6, we examine briefly the notion of a *semi-Mackey* space. Throughout this note, we use the simplifying properties of seminorms, which allow us to obviate the need of having to deal with the more abstract concept of a quotient space. In addition to obtaining generalizations to the non-LC situation, our approach provides alternative derivations as well as the simplification of corresponding results (see, e.g., [3], [13], [14], [22], [27]) when adapted to the LC case.

2. Preliminaries.

Throughout this note, E and F will be real Hausdorff TVS's and $E'(F')$ and $E^*(F^*)$ will denote their respective topological and algebraic duals. $T : E \to F$ will be a linear map which is said to have closed graph whenever the set $\{(x, Tx) \in$

$E \times F : x \in E\}$ is closed in $E \times F$. We say that E, F and T has the "Closed-Graph Property" (CGP), if T is continuous whenever it has closed graph.

We will use the following from [21; §2]: For each seminorm P on E, write $E_P^* := \{a \in E^* : a \le P \text{ on } E\}$ and define E to be *semi-B-complete*, if each subspace L in E' is $\sigma(E', E)$-closed whenever $L \cap E_P^*$ is $\sigma(E', E)$-compact for every continuous seminorm P on E. We say that E is *semi-barrelled*, if every lower-semi-continuous (LSC) seminorm on E is continuous. Also, T is defined to be *adequate* if, for all $x \in E$ and all $a \in E'$, $\langle x, T^t(\Delta)\rangle = \{o\}$ and $\langle \operatorname{Ker} T, a\rangle = \{o\}$ imply $\langle x, a\rangle = o$, where $\Delta := \{d \in \operatorname{cl}(T(E))' : d \circ T \in E'\}$ and $T^t d := d \circ T (d \in \Delta)$, and we say that T is *semi-open* if, for each continuous seminorm P on E, the *quotient seminorm*, P/T, defined by $P/T(y) := \inf P(T^{-1}y)(y \in T(E))$, is continuous on $T(E)$.

We recall the following definition: A family of subsets, $\mathcal{U} = (U_n)(n \ge 1)$, of E is a *string*, if each U_n is balanced and absorbing, and $U_{n+1} + U_{n+1} \subset U_n$ for all n. If, also, each U_n is a neighbourhood of the origin, then $\mathcal{U}$ is said to be a *topological string*. Iyahen [9], who introduced this notion of a string, and Adasch [5], [6], define E to be *ultrabarrelled* whenever every closed string (i.e., one for which every U_n is closed) is a topological string.

In the LC situation, the notions of semi-B-completeness and semi-barrelledness coincide with those of B-complete and barrelled spaces, respectively, from Pták theory (see, e.g., [8], [11], [13], [14], [15], [17], [20], [27]). Raikov [18] gave a similar extension to the non-LC setting of the notion of B-completeness as that given by Adasch and the concept of a non-LC barrelled TVS was first introduced by Robertson in [19]. See also Tomášek [23]. Since semi-B-complete spaces need not be complete [21; §2], our extension of the notion of a B-completeness to the non-LC situation is different from that given by Adasch [4], [5], [6] where a "B-complete" TVS is necessarily complete. Also, we note that every ultrabarrelled [9], [19] topology is always semi-barrelled, whereas the converse need not be true [21; §2] and that in the LC case T is adequate if and only if it is weakly singular [13], [14], [21; §3].

3. Semi-barrelled and semi-B-complete spaces and the Pták CGT.

Definition. T is *nearly-semi-continuous* if, for every continuous seminorm Q on $T(E)$, there exists a continuous seminorm P on E such that $b \circ T \in E'_P$ whenever $b \in T(E)'_Q$ and $b \circ T \in E'$.

We note that in the case E and F are LC, T is nearly-semi-continuous if and only if T is *nearly-continuous*, that is, for every neighbourhood V of the origin in F, $\operatorname{cl}(T^{-1}(V))$ is a neighbourhood of the origin in E [14], [21; §4].

Definition. T is *semi-continuous* if, for every continuous seminorm Q on $T(E)$, the *inverse seminorm*, Q/T^{-1}, defined on E by $Q/T^{-1} := Q \circ T$, is continuous.

In the case T is injective, T is semi-continuous if and only if T^{-1} is semi-open. Any continuous map is semi-continuous and if F at least is LC, since $\{x \in E : Q/T^{-1}(x) < 1\} = T^{-1}\{y \in T(E) : Q(y) < 1\}$, T is semi-continuous if and only if T is continuous.

Lemma 1 below can be found in [21; Lemma 1]; Lemma 2 appears in [4; §2], [12], [27; 1, §4].

Lemma 1. *Let P be a seminorm on E; then, $b \in T(E^*)_{P/T}$ if and only if $b \circ T \in E^*_P$.*

Lemma 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be fundamental systems of neighbourhoods of the origin in E and $(F, \mathcal{T})$, respectively, and let $\mathcal{T}_T$ denote the topology on F formed by taking as its fundamental system of neighbourhoods of the origin the sets $\{T(U) + V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Then, if T has closed graph, $\mathcal{T}_T$, which is coarser than the initial topology $\mathcal{T}$, is Hausdorff and T is continuous into $(F, \mathcal{T}_T)$.*

Lemma 3. *For any seminorm Q on $T(E)$, $b \in T(E)^*_Q$ if and only if $b \circ T \in E^*_{Q/T^{-1}}$.*

PROOF: Let Q be a seminorm on $T(E)$; then, for $b \in T(E)^*$, $b \leq Q$ if and only if $b \circ T \leq Q \circ T := Q/T^{-1}$. □

In the sequel, we will abbreviate $(F, \mathcal{T})'$ to F' and take $\mathcal{T}_T$ to be as defined in Lemma 2 where $\mathcal{T}_T^{\circ\circ}$ will denote the associated LC topology of $\mathcal{T}_T$, that is, the topology formed by taking, as its base of neighbourhoods of origin, the convex and balanced $\mathcal{T}_T$-neighbourhoods. We recall [21; §2] that the associated LC topology is coarser than the given topology and is barrelled, if that topology is semi-barrelled.

Lemma 4. *Let T have closed graph and be nearly-semi-continuous and let* Id *denote the identity map from $(F, \mathcal{T})$ onto $(F, \mathcal{T}_T^{\circ\circ})$. Then, for each continuous seminorm Q on $(F, \mathcal{T})$, there exists a continuous seminorm R on $(F, \mathcal{T}_T^{\circ\circ})$ such that $b \in (F, \mathcal{T}_T^{\circ\circ})'_R$ whenever $b \in (F, \mathcal{T}_T^{\circ\circ})'$ and $b \circ \mathrm{Id} \in F'_Q$.*

PROOF: By Lemma 2, since $\mathcal{T}_T^{\circ\circ} \subset \mathcal{T}_T \subset \mathcal{T}, \mathrm{Id} \circ T : E \to (F, \mathcal{T}_T^{\circ\circ})$ is continuous. Let Q be a continuous seminorm on $(F, \mathcal{T})$ and $b \in (F, \mathcal{T}_T^{\circ\circ})'$ be such that $b \circ \mathrm{Id} \in F'_Q$. Then, $b \circ \mathrm{Id} \in T(E)'_{Q|T(E)}$ and, since $\mathrm{Id} \circ T$ is continuous, $b \circ \mathrm{Id} \circ T \in E'$. Hence, since T is nearly-semi-continuous, there exists a continuous seminorm P on E such that $b \circ \mathrm{Id} \circ T \in E'_P$ and therefore $b \in H := (\mathrm{Id} \circ T)^{t(-1)}(E'_P)$. Here, H is an equicontinuous subset of $(F, \mathcal{T}_T^{\circ\circ})'$, if it is contained in the polar of some neighbourhood of the origin in $(F, \mathcal{T}_T^{\circ\circ})$. But this is clear since, as is easily verified, $H \subset [(\mathrm{Id} \circ T)\{x \in E : P(x) \leq 1\}]^\circ$ (where "$^\circ$" denotes the operation of polarity), where $T\{x \in E : P(x) \leq 1\}$ is a convex and balanced $\mathcal{T}_T$-neighbourhood of the origin. Since it is equicontinuous and $(F, \mathcal{T}_T^{\circ\circ})$ is LC, H is determined by some continuous seminorm R on $(F, \mathcal{T}_T^{\circ\circ})$ for which we conclude that $b \in (F, \mathcal{T}_T^{\circ\circ})'_R$. □

Lemma 5. *Let T and* Id *be as in Lemma* 4 *and let $(F, \mathcal{T})$ be semi-B-complete. Then, $\mathrm{Id}^t(F, \mathcal{T}_T^{\circ\circ})'$ is $\sigma(F', F)$-closed in F'.*

PROOF: Since $(F, \mathcal{T})$ is semi-B-complete, it suffices to show that $\mathrm{Id}^t(F, \mathcal{T}_T^{\circ\circ})' \cap F^*_Q$ is $\sigma(F', F)$-compact for each continuous seminorm Q on $(F, \mathcal{T})$. The seminorm $S : (F, \mathcal{T}_T^{\circ\circ}) \to \mathbb{R}$ defined by $S := \sup\{b \in (F, \mathcal{T}_T^{\circ\circ})' : b \circ \mathrm{Id} \in F'_Q\}$ is continuous since, by Lemma 4, $S \leq R$ for some continuous seminorm R on $(F, \mathcal{T}_T^{\circ\circ})$. Hence, by Lemma 1, $S = \sup\{b \in (F, \mathcal{T}_T^{\circ\circ})'_{Q/\mathrm{Id}}\}$ and therefore $(F, \mathcal{T}_T^{\circ\circ})'_S = (F, \mathcal{T}_T^{\circ\circ})'_{Q/\mathrm{Id}}$, from which we deduce that $(F, \mathcal{T}_T^{\circ\circ})'_{Q/\mathrm{Id}}$ is $\sigma((F, \mathcal{T}_T^{\circ\circ})', (F, \mathcal{T}_T^{\circ\circ}))$-compact. Since

$\mathrm{Id}^t(F,\mathcal{T}_T^{\circ\circ})'\cap F_Q^* = \{b\circ \mathrm{Id} : b\in (F,\mathcal{T}_T^{\circ\circ})', b\circ\mathrm{Id}\in F_Q'\} = \{b\circ\mathrm{Id} : b\in (F,\mathcal{T}_T^{\circ\circ})'_{Q/\mathrm{Id}}\}$, the claim follows by the $\sigma((F,\mathcal{T}_T^{\circ\circ})', (F,\mathcal{T}_T^{\circ\circ}))$ - $\sigma(F^*, F)$ continuity of the map $b\to b\circ\mathrm{Id}$. □

We now give the main results of this section.

Theorem 6. *Let T have closed graph and be nearly-semi-continuous and let $(F,\mathcal{T})$ be semi-B-complete. Then T is semi-continuous.*

Proof: As in Lemma 4, let Id denote the identity map from $(F,\mathcal{T})$ onto $(F,\mathcal{T}_T^{\circ\circ})$. Since Id is continuous and $(F,\mathcal{T}_T^{\circ\circ})$ is LC, it is adequate [21;§2]. Let Q be a continuous seminorm on $(F,\mathcal{T})$ and $b\in (F,\mathcal{T}_T^{\circ\circ})^*_{Q\circ\mathrm{Id}^{-1}}$; then, by Lemma 3, $b\circ\mathrm{Id}\in F_Q^*(\subset F')$. Let $x\in F$ be such that $\langle x, \mathrm{Id}^t(F,\mathcal{T}_T^{\circ\circ})'\rangle = \{o\}$; then, since $\langle\mathrm{Ker\,Id}, b\circ\mathrm{Id}\rangle = \{o\}$ and Id is adequate, $\langle x, b\circ\mathrm{Id}\rangle = o$. Since, by Lemma 5, $\mathrm{Id}^t(F,\mathcal{T}_T^{\circ\circ})'$ is $\sigma(F',F)$-closed, it follows by the separation theorem that $b\circ\mathrm{Id}^t(F,\mathcal{T}_T^{\circ\circ})'$ and hence $b\in(F,\mathcal{T}_T^{\circ\circ})'$. From this, and since (by Hahn–Banach theorem) $Q\circ\mathrm{Id}^{-1}$ is the supremum of linear functionals it dominates, $Q\circ\mathrm{Id}^{-1} = \sup\{b\in(F,\mathcal{T}_T^{\circ\circ})'_{Q\circ\mathrm{Id}^{-1}}\}$. It follows, by Lemma 3, that $Q\circ\mathrm{Id}^{-1} = \sup\{b : b\in(F,\mathcal{T}_T^{\circ\circ})', b\circ\mathrm{Id}\in F_Q'\}$. This shows that $Q\circ\mathrm{Id}^{-1}$ is continuous since now, by Lemma 4, $Q\circ\mathrm{Id}^{-1}\le R$ for some seminorm R continuous on $(F,\mathcal{T}_T^{\circ\circ})$. From the proof in Lemma 4, we know that $\mathrm{Id}\circ T$ is continuous and hence, since $Q/T^{-1} = Q\circ T = (Q\circ\mathrm{Id}^{-1})\circ(\mathrm{Id}\circ T)$, the proof is complete. □

Theorem 7. *Let T have closed graph, E be semi-barrelled and F be semi-B-complete. Then, T is semi-continuous.*

Proof: Any map from a semi-barrelled space is nearly-semi-continuous: Let Q be a continuous seminorm on $T(E)$ and choose P to be the LSC seminorm $\sup\{b\circ T : b\circ T\in E', b\in T(E)_Q'\}$. The result follows from Theorem 6. □

Since equicontinuous sets remain equicontinuous for finer topologies, an easy consequence of Theorem 7 is: Let T have closed graph, E be semi-barrelled and F be semi-B-complete. Then, T is continuous for any finer LC topology, $\mathcal{T}_1$, on F such that $(F,\mathcal{T}_1)' = F'$. We also note that Theorem 7 is valid, if either E or F is LC, that is, if E is barrelled or F is B-complete. For the case both E and F are LC, an easy consequence of Theorem 7 is (cf. [13], [14], [17], [20]):

Corollary 8 (Pták's CGT)**.** *Let T have closed graph, E be barrelled and F be B-complete. Then, T is continuous.*

Remarks. The proofs given to obtain Pták's CGT can be simplified, if we assume the spaces—in particular, the range—to be LC. In this case, we have an alternative and simple derivation of the Pták CGT. The standard proofs of this important result (see, e.g., [11; 11.1.7], [22; IV, §8.4]) require the use of Collins' theorem [22; IV, §8.2] which states that closed subspaces of B-complete spaces remain B-complete. Our proof (when adapted to the LC case) is also different from those found in [11; 11.1.7], [13; §11] and [14; §34.6(7)] which use the more abstract concept of a quotient space and the application of the Hahn–Banach theorem to the product space $(E \times F)$.

4. Semi-infra-(s) spaces and the Kōmura–Adasch–Valdivia CGT.

Semi-barrelled spaces share many properties with barrelled LC spaces; in particular, Lemma 9 below shows that the (unrestricted) inductive limit topology with respect to a family of semi-barrelled spaces is itself semi-barrelled (cf. [22; II, §7.2]).

Lemma 9. *Let* $(E, \mathcal{T}) = \mathrm{ind}_\alpha(E_\alpha, \mathcal{T}_\alpha, T_\alpha : \alpha \in I)$ *be the inductive limit with respect to the family of semi-barrelled spaces* $\{(E_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ *and maps* $(T_\alpha : \alpha \in I)$*. Then,* $(E, \mathcal{T})$ *is also semi-barrelled.*

Proof: Let P be LSC on $(E, \mathcal{T})$; then, $P \circ T_\alpha$ is LSC on $(E_\alpha, \mathcal{T}_\alpha)(\alpha \in I)$ and therefore continuous. Since $T_\alpha^{-1}\{y \in E : P(y) \le 1\} = \{x \in E_\alpha : P \circ T_\alpha(x) \le 1\}$, it follows that $\{y \in E : P(y) \le 1\}$ is a neighbourhood and therefore that P is continuous. □

Example. In view of Lemma 9, the finest vector topology on any given TVS $(E, \mathcal{T})$ is always semi-barrelled.

It follows from this example that there is a coarsest semi-barrelled topology on E of all the semi-barrelled topologies finer that $\mathcal{T}$. We call this topology the *associated semi-barrelled* topology of $\mathcal{T}$ and denote it by $\mathcal{T}^t$.

The notion of an "infra-(s)" space in the LC case was introduced by Adasch in [2]. We use the following equivalent characterization [14; §34.9(3)] of this notion: $(E, \mathcal{T})$ is an LC *infra-(s)* space if it is LC and for every LC Hausdorff topology $\mathcal{T}_1$ coarser than $\mathcal{T}, \mathcal{T}_1^{ct} = \mathcal{T}^{ct}$, where $\mathcal{T}_1^{ct}$ and $\mathcal{T}^{ct}$ denote the coarsest barrelled topologies finer than $\mathcal{T}_1$ and $\mathcal{T}$, respectively. In extending this notion of an infra-(s) space to the non-LC situation, Adasch [4], [6], gives the following definition: $(E, \mathcal{T})$ is *infra-(s)* if, for every Hausdorff topology $\mathcal{T}_1$ coarser than $\mathcal{T}, \mathcal{T}_1^{ut} = \mathcal{T}^{ut}$, where $\mathcal{T}_1^{ut}$ and $\mathcal{T}^{ut}$ denote the coarsest ultrabarrelled topologies finer than $\mathcal{T}_1$ and $\mathcal{T}$, respectively. This leads us to the following:

Definition. $(E, \mathcal{T})$ is a *semi-infra-(s)* space if, for every Hausdorff topology $\mathcal{T}_1$ coarser than $\mathcal{T}, \mathcal{T}_1^t = \mathcal{T}^t$.

Since every ultrabarrelled topology is semi-barrelled as already noted, it is clear from the definitions that every semi-infra-(s) space is infra-(s). This is in contrast with the fact that semi-B-complete spaces are not necessarily B-complete in the sense of Adasch. The notion of a semi-infra-(s) space is, however, the correct one for our purpose of extending the CGT to where the domain is semi-barrelled (see Theorems 11 and 12 below). Indeed, the following Example shows that we cannot hope that the CGP holds between semi-barrelled and infra-(s) spaces.

Example. Consider the identity map from $(l^{1/2}, \|\cdot\|_{3/4})$ onto $(l^{1/2}, \|\cdot\|_{1/2})$, where $\|\cdot\|_{3/4}$ is the topology induced on $l^{1/2}$ from $l^{3/4}$ and $\|\cdot\|_{1/2}$ is the natural topology on $l^{1/2}$ (see [19; §7], [21; §2]). From [21; §2], we know that $(l^{1/2}, \|\cdot\|_{3/4})$ is semi-barrelled and from [6; §19] that $(l^{1/2}, \|\cdot\|_{1/2})$ is infra-(s). The identity, which has closed graph, cannot be continuous since $\|\cdot\|_{3/4}$ is strictly coarser than $\|\cdot\|_{1/2}$.

In the LC situation, Adasch [2] showed that the CGP holds, where the domain space is assumed to be barrelled and the range to be infra-(s). This generalized the Pták CGT in that every B-complete space is necessarily infra-(s) [2], [14; §34.9(7)] and was significant since infra-(s) spaces need not be complete [14; §34.9(9)]. In the LC case, the concept of an infra-(s) space is coincident with that of an "S-space" or a "Γ_r-space" as given by Valdivia in [25], [26], [27; I, §6] where the results on the CGT parallel some of those in [2]. Both authors rely on a principle given by Kōmura [15]. We state the following here for an easy reference (see, e.g., [2; §3], [8], [14; §34.9], [15], [25], [26], [27]).

Theorem 10 (Kōmura–Adasch–Valdivia CGT). *Let F be an* LC *infra-(s) space. Then, every linear map with closed graph from a barrelled space into F is continuous.*

In extending this result to the non-LC case, Adasch [4], [6] shows that the CGP holds, where the domain is ultrabarrelled and the range is infra-(s). We will show that with the domain space semi-barrelled, the CGP holds whenever the range is semi-infra-(s) (Theorem 11) and that semi-infra-(s) spaces are maximal for the CGT for semi-barrelled domain spaces (Theorem 12).

Theorem 11. *Let $(F,\mathcal{T})$ be a semi-infra-(s) space. Then, every linear map with closed graph from a semi-barrelled space into $(F,\mathcal{T})$ is continuous.*

PROOF: Let $(E,\mathcal{T}_1)$ be semi-barrelled and $T:(E,\mathcal{T}_1)\to(F,\mathcal{T})$ have closed graph. By Lemma 2, T is continuous into $(F,\mathcal{T}_T)$ where $\mathcal{T}_T$ is Hausdorff and coarser than $\mathcal{T}$. Then, since T is continuous into $(F,\mathcal{T}_T)$ and $(E,\mathcal{T}_1)$ is semi-barrelled, it is easily shown using a standard transfinite construction as given in [14; §34(9)] that $T:(E,\mathcal{T}_1^t)\to(F,\mathcal{T}_T^t)$ is continuous. Now, since $\mathcal{T}_T$ is Hausdorff and because $(F,\mathcal{T})$ is semi-infra-(s), we deduce that $T:(E,\mathcal{T}_1)\to(F,\mathcal{T})$ is continuous. □

In particular, Theorem 11 is valid, if the domain was LC and/or the range is LC since our definition of a semi-infra-(s) space includes those with an LC initial topology. In this case, Theorem 11 gives Theorem 10. We note here that the generalizations of the CGT to the non-LC situation given by Adasch [4], [6] do not reduce to Theorem 10 for the LC case.

We now show that the semi-infra-(s) spaces are maximal for the CGP for semi-barrelled domain spaces.

Theorem 12. *Let $(F,\mathcal{T})$ be such that every linear map with closed graph from a semi-barrelled space into $(F,\mathcal{T})$ is continuous. Then, $(F,\mathcal{T})$ is semi-infra-(s).*

PROOF: Consider the identity from $(F,\mathcal{T}_1)$ onto $(F,\mathcal{T})$, where $\mathcal{T}_1$ is a Hausdorff topology coarser than $\mathcal{T}$. The identity is closed and remains closed as a map from $(F,\mathcal{T}_1^t)$ onto $(F,\mathcal{T})$ since $\mathcal{T}_1^t$ is finer than $\mathcal{T}_1$; hence, by assumption, it is continuous. Thus, $\mathcal{T}_1^t\supset\mathcal{T}$, from which we conclude that $\mathcal{T}_1^t=\mathcal{T}^t$ and $(F,\mathcal{T})$ is semi-infra-(s). □

Remarks. It is clear from the definition that if $(F,\mathcal{T})$ is semi-infra-(s), then $(F,\mathcal{T}_1)$ is semi-infra-(s) for any Hausdorff topology $\mathcal{T}_1$ coarser than $\mathcal{T}$. From this it follows that there exist semi-infra-(s) spaces that are not semi-B-complete.

5. Semi-bornological spaces and the Powell CGT.

We recall [8], [14] that a TVS is *bornological*, if it is LC and if every bornivorous set (i.e., one that absorbs all bounded sets) which is absolutely convex is a neighbourhood of the origin. To extend this, Iyahen [9] and Adasch [1], [6] introduced the following concept of a non-LC bornological space: E is *ultrabornological* whenever every bounded linear map (i.e., one that maps bounded sets into bounded sets) from E into any TVS is continuous. Equivalently, E is ultrabornological whenever every bornivorous string (i.e., one for which every U_n is bornivorous) is a topological string [1], [6], [9]. (We point out here that ultrabornological TVS should not be confused with the "ultrabornological" spaces from the LC theory [11].)

Definition. E is *semi-bornological*, if every bounded seminorm on E is continuous.

Equivalently, E is semi-bornological whenever every bornivorous and absolutely convex set is a neighbourhood of the origin; here, we do *not* assume that E has a neighbourhood base of the origin consisting of absolutely convex sets. It is clear, however, that every bornological space is semi-bornological. This notion should be compared with that of an "M-bornological" space given by Tomášek [24].

Examples. Every pseudometrizable (and hence, every locally-bounded) space is semi-bornological; the finest linear topology on any vector space is semi-bornological.

Proposition 13. *Every ultrabornological space is semi-bornological.*

Proof: Let E be ultrabornological and define $U_P := \{x \in E : P(x) \leq 1\}$, where P is a given bounded seminorm on E. Clearly, $\mathcal{U} := \{2^{-(n-1)} \cdot U_P\}(n \geq 1)$ is a string which, since P is bounded, is also bornivorous. It follows that U_P is a neighbourhood of the origin and, therefore, that P is continuous. □

In particular, every LC ultrabornological space is semi-bornological. Examples of bornological spaces that are not ultrabornological can be found in [6], [9]. The following provides an example of a non-LC semi-bornological space which is not ultrabornological and which, together with Proposition 13 above, shows that the class of ultrabornological spaces is a proper subclass of the class of semi-bornological spaces.

Example. Following Iyahen [10], a subset A of E is said to be *semiconvex*, if $A + A \subset \lambda A$ for some $\lambda > 0$; and E is said to be a *semiconvex space*, if it has a base of neighbourhoods of the origin consisting of balanced semiconvex sets. If $\tau(E, E^*), \mathcal{T}^{fsc}$ and $\mathcal{T}^f$ denote the Mackey, finest-semiconvex and finest topologies on E, respectively, then, since $\tau(E, E^*)$ is the finest convex topology on E and every LC topology is clearly semiconvex, $\tau(E, E^*) \subset \mathcal{T}^{fsc} \subset \mathcal{T}^f$. If, however, the (algebraic) dimension of E is uncountable, then $\tau(E, E^*), \mathcal{T}^{fsc}$ and $\mathcal{T}^f$ are all, in fact, distinct [10]. Since $\tau(E, E^*)$ and $\mathcal{T}^f$ share the same bounded sets (since they induce the same topology on finite-dimensional spaces), each $\mathcal{T}^{fsc}$-bornivorous set is easily seen to be $\tau(E, E^*)$-bornivorous; hence, since the Mackey topology is bornological (indeed, every seminorm on E is $\tau(E, E^*)$-continuous), each $\mathcal{T}^{fsc}$-bornivorous and absolutely convex set is a $\tau(E, E^*)$-neighbourhood of the origin.

From this, it follows that $(E, \mathcal{T}^{fsc})$ is semi-bornological. $(E, \mathcal{T}^{fsc})$ is, however, not ultrabornological since the identity from $(E, \mathcal{T}^{fsc})$ onto $(E, \mathcal{T}^{f})$ which is clearly bounded, cannot be continuous.

We can, in fact, give the following characterization: E is semi-bornological if and only if every bounded linear map from E into any LC space is continuous (cf. [8], [11], [14]). This is easily verified: Let E be semi-bornological, T be a bounded linear map from E into any LC space F, and Q be any continuous seminorm on F. From the identity $\{x \in E : Q \circ T(x) < 1\} = T^{-1}\{y \in F : Q(y) < 1\}$, and since $Q \circ T$ is a bounded seminorm on E, it follows that T is continuous. Conversely, suppose that every bounded linear map from E into any LC space is continuous. Let P be a bounded seminorm on E and consider the identity from E onto $(E, \mathcal{T}_P)$, where $\mathcal{T}_P$ is the LC topology on E generated by P. The identity is easily seen to be bounded (since P is bounded) and hence is continuous. It follows that $\{x \in E : P(x) < 1\}$ is open in E and P is continuous.

We can now state a generalization of Powell's CGT [16], [8] in Theorem 14 below. Powell employs Kōmura's principle as used in the proof of Theorem 11. The proof of Theorem 14 is similar to that given for this theorem and will therefore be omitted. Here, for the given topology $\mathcal{T}$ on F, $\mathcal{T}^x$ will denote the coarsest semi-bornological topology on F finer than $\mathcal{T}$. That such a topology exists, it is clear by an adaptation of Lemma 9 for semi-bornological spaces, and since we have already noted that the finest linear topology on any vector space is semi-bornological.

Theorem 14. *Let T have closed graph, E be semi-bornological and $(F, \mathcal{T})$ be such that $\mathcal{T}_1^x = \mathcal{T}^x$ for any Hausdorff topology $\mathcal{T}_1$ coarser than $\mathcal{T}$. Then, T is continuous.*

6. Semi-Mackey spaces.

We conclude this note by investigating briefly a notion of a Mackey space for the non-LC situation, which will complement the notions of semi-barrelled and semi-bornological spaces already given.

Definition. P is a *Mackey seminorm* whenever $E_P^* \subset E'$.

Definition. E is a *semi-Mackey* space, if every Mackey seminorm on E is continuous.

We note that if E is a (Hausdorff) LC space, then P is Mackey seminorm if and only if P is continuous with respect to the Mackey topology $\tau(E, E')$. Hence, an LC semi-Mackey space is a Mackey space.

Example. Every TVS with a degenerate topological dual is semi-Mackey; this includes, for example, $\mathfrak{M}$, the (non-LC) space of all μ-a.e. equivalence classes of real-valued measurable functions on $[0, 1]$, where μ is the Lebesgue measure.

Proposition 15. *Every semi-barrelled space is semi-Mackey.*

Proof: By the Hahn–Banach theorem , every seminorm is the supremum of the linear functionals it dominates, any Mackey seminorm is LSC. Hence, semi-barrelled spaces are semi-Mackey. □

Proposition 16. *Every semi-bornological space is semi-Mackey.*

Proof: By the Alaoglu–Bourbaki theorem, for P a Mackey seminorm, E_P^* is $\sigma(E', E)$-compact. This gives P is continuous as in the proof for the LC case (see, e.g., [7]). □

References

[1] Adasch N., *Topologische Produkte gewisser topologischer Vectorräume*, Math. Ann. **186** (1970), 280–284.

[2] Adasch N., *Tonnelierte Räume und zwei Sätze von Banach*, Math. Ann. **186** (1970), 209–214.

[3] Adasch N., *Eine Bemerkung über den Graphensatz*, Math. Ann. **186** (1970), 327–333.

[4] Adasch N., *Der Graphensatz in topologischen Vectorräumen*, Math. Z. **119** (1971), 131–142.

[5] Adasch N., *Vollständigkeit und der Graphensatz*, J. reine angew. Math. **249** (1971), 217–220.

[6] Adasch N., Ernst B., Keim D., *Topological vector spaces, The theory without local convexity conditions*, Lecture Notes in Mathematics, **639**, Eds. A. Dodd and B. Eckmann, Springer–Verlag, Berlin, Heidelberg, New York, 1978.

[7] Beckenstein E., Narici L., *Topological vector spaces*, Marcel Dekker, Inc., New York and Basel, 1985.

[8] Horváth J., *Locally convex spaces*, Lecture Notes in Mathematics, **331**, Ed. L. Waelbroeck, Springer–Verlag, New York, Heidelberg, Berlin, 1973.

[9] Iyahen S.O., *On certain classes of topological spaces*, Proc. London Math. Soc. **(3)** 18 (1968), 285–307.

[10] Iyahen S.O., *Semiconvex spaces*, Glasgow J. Math. **9** (1968), 111–118.

[11] Jarchow H., *Locally convex spaces*, B.G. Tuebner, Stuttgart, 1981.

[12] Kelley J., Namioka I., *Linear topological spaces*, Van Nostrand, Princeton, 1963.

[13] Köthe G., *General linear transformations of locally convex spaces*, Math. Ann. **159** (1965), 309–328.

[14] Köthe G., *Topological vector spaces I,II*, Springer–Verlag, New York, Heidelberg, Berlin, 1979.

[15] Kōmura Y., *On linear topological spaces*, Kumamoto J. Sci., Ser.A **5** No. 3 (1962), 148–157.

[16] Powell M., *On Komura's closed-graph theorem*, Trans. AMS **211** (1975), 391–426.

[17] Pták V., *Completeness and the open mapping theorem*, Bull. Soc. Math. France **86** (1958), 41–74.

[18] Raikov D.A., *Closed graph theorem and completeness*, Proc. 4th All Union Math. Congress, Leningrad, 1961.

[19] Robertson W., *Completions of topological vector spaces*, Proc. London Math. Soc. **8** (1958), 242–257.

[20] Robertson A., Robertson W., *On the closed graph theorem*, Proc. Glasgow Math. Assoc. **3** (1956), 9–12.

[21] Rodrigues B., *On the Pták homomorphism theorem*, J. Aust. Math. Soc. Ser. A **47** (1989), 322-333.

[22] Schaefer H.H., *Topological vector spaces*, Springer–Verlag, New York, Heidelberg, Berlin, 1986.

[23] Tomášek S., *M-barrelled spaces*, Comment. Math. Univ. Carolinae **11** (1970), 185–204.

[24] Tomášek S., *M-bornological spaces*, Comment. Math. Univ. Carolinae **11** (1970), 235–248.

[25] Valdivia Ureña M., *El teorema general de la gráfica cerrada en los espacios vectoriales topológicos localmente convexos*, Rev. Real Acad. Ci. Exact. Fis. Nat. Madrid **62** (1968), 545–551.

[26] Valdivia M., *Sobre el teorema de la gráfica cerreda*, Collectanea Math. **22** (1971), 51–72.
[27] Valdivia M., *Locally convex spaces*, Mathematics Studies **67**, North-Holland, Amsterdam, New York, Oxford, 1982.

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