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WEILIAN PROLONGATIONS OF ACTIONS OF SMOOTH CATEGORIES

Ivan Kolář

Abstract. First of all, we find some further properties of the characterization of fiber product preserving bundle functors on the category of all fibered manifolds in terms of an infinite sequence A of Weil algebras and a double sequence H of their homomorphisms from [5]. Then we introduce the concept of Weilian prolongation $W^A_H S$ of a smooth category S over $\mathbb{N}$ and of its action D. We deduce that the functor (A,H) transforms D-bundles into $W^A_H D$-bundles.

In [4] we clarified that every fiber product preserving bundle functor on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fiber preserving morphisms with local diffeomorphisms as base maps is of finite order and can be identified with a triple (A_m, H_m, t_m), where A_m is a Weil algebra, $H_m\colon G^r_m \to \operatorname{Aut} A_m$ is a group homomorphism of the r-th jet group in dimension m into the group of all algebra automorphisms of A_m and $t_m\colon \mathbb{D}^r_m \to A_m$ is an equivariant algebra homomorphism, $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m,\mathbb{R})$. Our next result from [5] can be formulated as follows. Write

$$A = (A_1, \dots, A_m, \dots) \tag{1}$$

for an infinite sequence of Weil algebras,

$$\operatorname{Hom} A = \big(\operatorname{Hom}(A_m, A_n)\big) \tag{2}$$

for the double sequence of the algebra homomorphisms and

$$L^r = (L^r_{m,n})\,, \quad L^r_{m,n} = J^r_0(\mathbb{R}^m,\mathbb{R}^n)_0 \tag{3}$$

for the skeleton of the category of r-jets. Then the fiber product preserving bundle functors F on the category $\mathcal{FM}$ of all fibered manifold morphisms of the base order r are in bijection with the pairs (A,H) of a sequence (1) and of a functor

$$H\colon L^r \to \operatorname{Hom} A\,, \quad H_{m,n}\colon L^r_{m,n} \to \operatorname{Hom}(A_m, A_n)\,. \tag{4}$$

In the first two sections of the present paper, we deduce certain new results concerning F and an arbitrary fiber product preserving bundle functor on $\mathcal{FM}_m$, that are to be used in the sequel. In Section 3 we consider a smooth category S over

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integers and we describe how the pair (A,H) of (1) and (4) induces a category $W^A_H S$ over $\mathbb{N}$. Then we consider an action D of S on a sequence $Z=(Z_1,\dots,Z_m,\dots)$ of manifolds and we deduce that $W^A_H S$ determines canonically an action $W^A_H D$ on the sequence

$$T^A Z = (T^{A_1}Z_1,\dots,T^{A_m}Z_m,\dots)\,.$$

In Section 5 we introduce the category of D-bundles and we prove that the functor $F=(A,H)$ transforms D-bundles into $W^A_H D$-bundles.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [3].

1. The case of $\mathcal{FM}_m$ in trivializations. In general, every Weil algebra homomorphism $\mu\colon B\to C$ defines a natural transformation $\mu_M\colon T^B M\to T^C M$ of the corresponding Weil bundles over every manifold M, [2], [3], [6]. For a fibered manifold $p\colon Y\to M$, we define the vertical Weil bundle $V^B Y\subset T^B Y$ as the space of all B-velocities in the individual fibers of Y. Then μ_Y restricts and corestricts into a map

$$\mu^V_Y\colon V^B Y\to V^C Y\,,$$

that is a natural transformation of the bundle functors V^B and V^C on $\mathcal{FM}$.

In [4], we deduced that every fiber product preserving bundle functor F_m on $\mathcal{FM}_m$ of the base order r is of the form $F_m=(A_m,H_m,t_m)$ specified in the introduction. The homomorphism H_m defines an action of G^r_m on $T^{A_m}Y$, $g\mapsto H_m(g)_Y$. So we can construct the associated bundle $P^rM[T^{A_m}Y,H_{m\,Y}]$. Then we have

$$F_mY=\big\{\{u,X\}\in P^rM[T^{A_m}Y];\,t_{m\,M}u=T^{A_m}p(X)\big\}\,,$$

where $t_{m\,M}\colon T^r_mM\to T^{A_m}M$ and we use the inclusion $P^rM\subset T^r_mM$. Let $\bar p\colon \bar Y\to\bar M$ be another $\mathcal{FM}_m$-object and $f\colon Y\to\bar Y$ be an $\mathcal{FM}_m$-morphism with the base map $\underline{f}\colon M\to\bar M$. Since $T^{A_m}f\colon T^{A_m}Y\to T^{A_m}\bar Y$ is a G^r_m-equivariant map, we can construct the induced morphism of associated bundles

$$P^r\underline{f}[T^{A_m}f]\colon P^rM[T^{A_m}Y]\to P^r\bar M[T^{A_m}\bar Y]\,. \tag{5}$$

Clearly, (5) maps F_mY into $F_m\bar Y$. This defines F_mf.

In the case of a product $p_1\colon M\times N\to M$, the condition $t_{m\,M}u=T^{A_m}p_1(X_1,X_2)$, $(X_1,X_2)\in T^{A_m}M\times T^{A_m}N$ yields $t_{m\,M}u=X_1$. Hence

$$F_m(M\times N)=P^rM[T^{A_m}N]\,.$$

Every $\mathcal{FM}_m$-morphism $f\colon M\times N\to\bar M\times\bar N$ is of the form $f=(\underline{f},\widetilde{f})$, $\underline{f}\colon M\to\bar M$, $\widetilde{f}\colon M\times N\to\bar N$. Then

$$T^{A_m}f=(T^{A_m}\underline{f},T^{A_m}\widetilde{f})\colon T^{A_m}M\times T^{A_m}N\to T^{A_m}\bar M\times T^{A_m}\bar N\,.$$

For $\{u,X\}\in P^rM[T^{A_m}N]$, we have to consider

$$\big\{u,(t_{m\,M}u,X)\big\}\in P^rM[T^{A_m}M\times T^{A_m}N]\,.$$

Then we obtain

$$P^r\underline{f}[T^{A_m}f]\big(\{u,(t_{m\,M}u,X)\}\big) = \\ \big\{P^r\underline{f}(u),(t_{m\,\bar{M}}P^r\underline{f}(u),T^{A_m}\widetilde{f}(t_{m\,M}u,X))\big\}\,.$$

This implies

Proposition 1. *For* $\{u,X\}\in P^rM[T^{A_m}N]$, *we have*

$$F_m(\underline{f},\widetilde{f})\big(\{u,X\}\big)=\big\{P^r\underline{f}(u),T^{A_m}\widetilde{f}(t_{m\,M}u,X)\big\}\,.$$

In the case $M=\mathbb{R}^m=\bar{M}$, we use the injection $\varrho_m\colon\mathbb{R}^m\to P^r\mathbb{R}^m$, $\varrho_m(x)=j^r_0\tau_x$, where $\tau_x\colon\mathbb{R}^m\to\mathbb{R}^m$ is the translation $y\mapsto y+x$. This defines an identification $F_m(\mathbb{R}^m\times N)\approx\mathbb{R}^m\times T^{A_m}N$, $\big\{\varrho_m(x),X\big\}\mapsto(x,X)$. Consider $f=(\underline{f},\widetilde{f})\colon\mathbb{R}^m\times N\to\mathbb{R}^m\times\bar{N}$ and write $\varphi\colon\mathbb{R}^m\to G^r_m$ for the map defined by $P^r\underline{f}\big(\varrho_m(x)\big)=\varrho_m\big(\underline{f}(x)\big)\varphi(x)$. Then we obtain the following formula for F_mf

$$\begin{aligned}F_mf(x,X)&=\big\{P^r\underline{f}\big(\varrho_m(x)\big),T^{A_m}\widetilde{f}(t_{m\,M}\varrho_m(x),X)\big\}\\ &=\big(\underline{f}(x),H_m(\varphi(x))_{\bar{N}}T^{A_m}\widetilde{f}(t_{m\,M}\varrho_m(x),X)\big)\,.\end{aligned}\tag{6}$$

Consider the case t_m is the zero homomorphism $\mathcal{O}$. Then we have $t_{m\,M}u=j^{A_m}\hat{x}$, $u\in P^r_xM$, where $\hat{x}$ is the constant map of $\mathbb{R}^k$ into x, $k=$ the width of A_m. This implies directly

$$F_mY=P^rM[V^{A_m}Y,H^V_{m\,Y}]\,.$$

Then F_mf can be written as

$$F_mf=P^r\underline{f}[V^{A_m}f]\,.$$

In the case of $M=\mathbb{R}^m=\bar{M}$, (6) yields

$$F_mf(x,X)=\big(\underline{f}(x),H_m(\varphi(x))_{\bar{N}}(V^{A_m}f(x,X))\big)\,.\tag{7}$$

2. The case of $\mathcal{FM}$. Consider a fiber product preserving bundle functor F on $\mathcal{FM}$ and write F_m for its restriction to $\mathcal{FM}_m$. By [5], F is determined by the sequences A and H from (1) and (4) as follows. If we restrict $H_{m,m}$ to the open subset $G^r_m\subset L^r_{m,m}$, we obtain a group homomorphism $H_m\colon G^r_m\to\operatorname{Aut}A_m$. Then $F_m=(A_m,H_m,\mathcal{O})$. Hence $F_mY=P^rM[V^{A_m}Y]$. Further, let $f\colon Y\to\bar{Y}$ be an $\mathcal{FM}$-morphism over $\underline{f}\colon M\to\bar{M}$, $\dim\bar{M}=n$. For every $u\in P^r_xM$ and $v\in P^r_{\underline{f}(x)}\bar{M}$, we can write $j^r_x\underline{f}$ in the form $j^r_x\underline{f}=\{u,v,g\}$, $g=v^{-1}\circ j^r_x\underline{f}\circ u\in L^r_{m,n}$. Then our construction of $Ff\colon FY\to F\bar{Y}$ of [5] can be expressed in the form

$$Ff\big(\{u,X\}\big)=\big\{v,H(g)^V_{\bar{Y}}(V^{A_m}f(X))\big\}\,,\quad X\in V^{A_m}_xY\,.\tag{8}$$

In the case $Y=\mathbb{R}^m\times N$ and $\bar{Y}=\mathbb{R}^n\times\bar{N}$, we consider $\varrho_m\colon\mathbb{R}^m\to P^r\mathbb{R}^m$ and $\varrho_n\colon\mathbb{R}^n\to P^r\mathbb{R}^n$. Then $j^r\underline{f}$ defines a map $\varphi\colon\mathbb{R}^m\to L^r_{m,n}$ by

$$\varphi(x)=\varrho_n\big(\underline{f}(x)\big)^{-1}\circ j^r_x\underline{f}\circ\varrho_m(x)\,.$$

In the corresponding identifications

$$F(\mathbb{R}^m\times N)=\mathbb{R}^m\times T^{A_m}N\,,\quad F(\mathbb{R}^n\times\bar{N})=\mathbb{R}^n\times T^{A_n}\bar{N}\,,$$

(8) is of the form

$$Ff(x,X) = \big(\underline{f}(x), H(\varphi(x))_{\bar N}(V^{A_m} f(x,X))\big)\,, \quad X \in T^{A_m}N\,. \tag{9}$$

Clearly, (7) is a special case of (9).

3. The category $W^A_H S$. Investigating the prolongation of principal bundles with respect to the functor $F_m = (A_m, H_m, t_m)$ on $\mathcal{FM}_m$, M. Doupovec and the author used essentially the fact that every Lie group G induces the semidirect group product

$$W^{A_m}_{H_m} G = G^r_m \rtimes T^{A_m} G$$

with the group composition

$$(g_2, C_2)(g_1, C_1) = (g_2 \circ g_1, H_m(g_1^{-1})_G(C_2) \bullet C_1)\,,$$

where $\bullet$ denotes the induced group composition in $T^{A_m}G$. Replacing A_m by the sequence A and H_m by the double sequence H, we extend this construction to a smooth category S over $\mathbb{N}$.

Definition 1. A smooth category S over $\mathbb{N}$ is a category over $\mathbb{N}$ such that each set $S_{m,n}$ is a smooth manifold and every composition map

$$\varkappa_{m,n,p}\colon S_{n,p} \times S_{m,n} \to S_{m,p}$$

is a smooth map.

Having in mind the description (9) of $F = (A, H)$, we define

$$(W^A_H S)_{m,n} = L^r_{m,n} \times T^{A_n} S_{m,n}\,. \tag{10}$$

For every $(g_1, C_1) \in L^r_{m,n} \times T^{A_n} S_{m,n}$ and $(g_2, C_2) \in L^r_{n,p} \times T^{A_p} S_{n,p}$, we define their composition by

$$(g_2, C_2)(g_1, C_1) = \big(g_2 \circ g_1, C_2 \bullet H(g_2)_{S_{m,n}}(C_1)\big)\,, \tag{11}$$

where $\bullet$ denotes the induced map

$$T^{A_p}\varkappa_{m,n,p}\colon T^{A_p} S_{n,p} \times T^{A_p} S_{m,n} \to T^{A_p} S_{m,p}\,.$$

Proposition 2. *$W^A_H S$ is a smooth category over $\mathbb{N}$, that will be called the Weilian (A, H)-prolongation of S.*

Proof. It suffices to verify explicitly the associativity of (11), for the remaining steps of the proof are trivial. By the associativity in S and the functoriality of H, we obtain (with omitting the subscripts of H)

$$(g_3, C_3)\big(g_2 \circ g_1, C_2 \bullet H(g_2)(C_1)\big) = \big(g_3 \circ g_2 \circ g_1, C_3 \bullet H(g_3)(C_2) \bullet H(g_3 \circ g_2)(C_1)\big)\,.$$

□

Further, if $\bar S = (\bar S_{m,n})$ is another smooth category over $\mathbb{N}$ and $\varphi\colon S \to \bar S$ is a smooth functor, i.e. all maps $\varphi_{m,n}\colon S_{m,n} \to \bar S_{m,n}$ are smooth, then the rule

$$(W^A_H \varphi)_{m,n} = \mathrm{id}_{L^r_{m,n}} \times T^{A_n} \varphi_{m,n}$$

defines a smooth functor $W^A_H \varphi\colon W^A_H S \to W^A_H \bar S$.

4. The action $W^A_H D$. Consider a sequence $Z = (Z_1, \dots, Z_m, \dots)$ of manifolds.

Definition 2. An action D of S on Z is a double sequence $D_{m,n} \colon S_{m,n} \times Z_m \to Z_n$ of smooth maps such that $D_{m,m}(e_m, y) = y$, $e_m =$ the unit of $D_{m,m}$, $y \in Z_m$, and

$$D_{n,p}\big(s_2, D_{m,n}(s_1, y)\big) = D_{m,p}\big(\varkappa_{m,n,p}(s_2, s_1), y\big)\,, \quad s_1 \in S_{m,n},\ s_2 \in S_{n,p}\,.$$

In [1], we deduced that every action of a Lie group G on a manifold Q induces an action of $W^{A_m}_{H_m} G$ on $T^{A_m} Q$. Analogously we introduce the action $W^A_H D$ of $W^A_H S$ on the sequence

$$T^A Z = (T^{A_1} Z_1, \dots, T^{A_m} Z_m, \dots)\,.$$

For $(g, C) \in L^r_{m,n} \times T^{A_n} S_{m,n}$ and $B \in T^{A_m} Z_m$, we define

$$(W^A_H D)_{m,n}\big((g, C), B\big) = C \bullet H(g)_{Z_m}(B)\,, \tag{12}$$

where $\bullet$ denotes the map $T^{A_n} D_{m,n} \colon T^{A_n} S_{m,n} \times T^{A_n} Z_m \to T^{A_n} Z_n$.

Proposition 3. *$W^A_H D$ is an action of $W^A_H S$ on $T^A Z$.*

Proof. For another $(\bar g, \bar C) \in (W^A_H D)_{n,p}$, the fact that D is an action and H is a functor yields directly

$$\bar C \bullet H(\bar g)\big(C \bullet H(g)(B)\big) = \big(\bar C \bullet H(\bar g)(C)\big) \bullet H(\bar g \circ g)(B)\,.$$

□

Let $\bar D$ be another action of S on $\bar Z = (\bar Z_1, \dots, \bar Z_m, \dots)$. An action morphism $\psi \colon D \to \bar D$ is a sequence $(\psi_m \colon Z_m \to \bar Z_m)$ of smooth maps such that

$$\bar D_{m,n}\big(s, \psi_m(y)\big) = \psi_n\big(D_{m,n}(s, y)\big)\,, \quad s \in S_{m,n},\ y \in Z_m\,.$$

Then it is easy to see that

$$T^A \psi = (T^{A_m} \psi_m \colon T^{A_m} Z_m \to T^{A_m} \bar Z_m)$$

is a morphism of the actions $W^A_H D$ and $W^A_H \bar D$.

5. (A, H) transforms D-bundles into $W^A_H D$-bundles. The elementary D-bundles are the products $M \times Z_m$. The elementary morphisms of D-bundles are the pairs $f_0 \colon M \to \bar M$ and $f_1 \colon M \to S_{m,n}$, that are interpreted as $\mathcal{FM}$-morphisms

$$f = (f_0, f_1) \colon M \times Z_m \to \bar M \times Z_n \quad f(x, y) = \big(f_0(x), f_1(x)(y)\big)\,, \tag{13}$$

where $f_1(x)(y) = D_{m,n}\big(f_1(x), y\big)$. Globally, the category $D\mathcal{B}$ of D-bundles is defined by the standard "gluing together" procedure. So the structure of D-bundle on a fibered manifold $p \colon Y \to M$ is determined by an open covering (U_α) of M and a family of local trivializations

$$\psi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times Z_m$$

such that all transition functions are the elementary $D\mathcal{B}$- morphisms. If $\bar p \colon \bar Y \to \bar M$ is another D-bundle, an $\mathcal{FM}$-morphism $f \colon Y \to \bar Y$ is said to be a $D\mathcal{B}$-morphism, if it is expressed by elementary $D\mathcal{B}$-morphisms in the generating trivializations of the $D\mathcal{B}$-structures on Y and $\bar Y$. We say that (13) is an admissible local expression of a $D\mathcal{B}$-morphism.

We recall that we have a canonical injection $i^B_M\colon M\to T^BM$ for every Weil bundle T^BM, $i^B_M(x)=j^B\hat{x}$, $x\in M$.

Consider a fiber product preserving bundle functor $F=(A,H)$ on $\mathcal{FM}$. By Section 1, we have $F(\mathbb{R}^m\times Z_m)=\mathbb{R}^m\times T^{A_m}Z_m$. Consider an elementary $D\mathcal{B}$-morphism $f=(f_0,f_1)\colon \mathbb{R}^m\times Z_m\to\mathbb{R}^n\times Z_n$ and write $f_{1x}\colon Z_m\to Z_n$ for the map $y\mapsto f_1(x)(y)$. By (9) and (13), we obtain

$$Ff(x,X)=\big(f_0(x),H(\varphi(x))\big)_{Z_n}\big((T^{A_m}f_{1x}(X))\big)\,,\quad X\in T^{A_m}Z_m\,. \tag{14}$$

But we have $T^{A_m}f_{1x}(X)=T^{A_m}D_{m,n}(j^{A_m}\widehat{f_1(x)},X)$ and

$$H\big(\varphi(x)\big)_{Z_n}\big(T^{A_m}D_{m,n}(j^{A_m}\widehat{f_1(x)},X)\big)=T^{A_n}D_{m,n}\big(j^{A_n}\widehat{f_1(x)},H(\varphi(x))_{Z_m}(X)\big)\,.$$

By (12) and (13), (14) is an elementary morphism of W^A_HD-bundles. Since our constructions are of functorial character, we have deduced

Proposition 4. *The functor (A,H) transforms D-bundles into W^A_HD-bundles.*

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