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SOME GLOBAL CHARACTERIZATIONS OF THE SPHERE IN E^4

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1. Let M be a surface in the 4-dimensional Euclidean space E^4. Let $\{U_\alpha\}$ be a covering of M such that in any domain U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ such that $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$, $T(M)$, $N(M)$ being the tangent and the normal bundle of M, respectively. Then we have

$$(1) \qquad \begin{aligned} dM &= \omega^1 v_1 + \omega^2 v_2 , \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , \quad dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 , \quad dv_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 ; \end{aligned}$$

$$(2) \qquad d\omega^i = \omega^j \wedge \omega_j^i , \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j , \qquad \omega_i^j + \omega_j^i = 0 , \quad \omega^3 = \omega^4 = 0 .$$

Using the exterior differentiation and applying Cartan's lemma, we get from (2) the existence of real functions a_i, b_i $(i = 1, 2, 3)$; α_i, β_i $(i = 1, 2, 3, 4)$; A_i, B_i, C_i, D_i, E_i $(i = 1, 2)$ in each U_α such that

$$(3) \qquad \begin{aligned} \omega_1^3 &= a_1\omega^1 + a_2\omega^2 , \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2 , \\ \omega_1^4 &= b_1\omega^1 + b_2\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 ; \end{aligned}$$

$$(4) \qquad \begin{aligned} & da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 = \alpha_1\omega^1 + \alpha_2\omega^2 , \\ & da_2 + (a_1 - a_3)\,\omega_1^2 - b_2\omega_3^4 = \alpha_2\omega^1 + \alpha_3\omega^2 , \\ & da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 = \alpha_3\omega^1 + \alpha_4\omega^2 , \\ & db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2 , \\ & db_2 + (b_1 - b_3)\,\omega_1^2 + a_2\omega_3^4 = \beta_2\omega^1 + \beta_3\omega^2 , \\ & db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 = \beta_3\omega^1 + \beta_4\omega^2 ; \end{aligned}$$

$$(5)\qquad \begin{aligned}
&d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4 = A_1\omega^1 + (B_1 - a_2K - \tfrac12 b_1k)\,\omega^2\,,\\
&d\alpha_2 + (\alpha_1 - 2\alpha_3)\,\omega_1^2 - \beta_2\omega_3^4 = (B_1 + a_2K + \tfrac12 b_1k)\,\omega^1 +\\
&\qquad + (C_1 + a_1K - \tfrac12 b_2k)\,\omega^2\,,\\
&d\alpha_3 + (2\alpha_2 - \alpha_4)\,\omega_1^2 - \beta_3\omega_3^4 = (C_1 + a_3K + \tfrac12 b_2k)\,\omega^1 +\\
&\qquad + (D_1 + a_2K - \tfrac12 b_3k)\,\omega^2\,,\\
&d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4 = (D_1 - a_2K + \tfrac12 b_3k)\,\omega^1 + E_1\omega^2\,,\\
&d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 = A_2\omega^1 + (B_2 - b_2K + \tfrac12 a_1k)\,\omega^2\,,\\
&d\beta_2 + (\beta_1 - 2\beta_3)\,\omega_1^2 + \alpha_2\omega_3^4 = (B_2 + b_2K - \tfrac12 a_1k)\,\omega^1 +\\
&\qquad + (C_2 + b_1K + \tfrac12 a_2k)\,\omega^2\,,\\
&d\beta_3 + (2\beta_2 - \beta_4)\,\omega_1^2 + \alpha_3\omega_3^4 = (C_2 + b_3K - \tfrac12 a_2k)\,\omega^1 +\\
&\qquad + (D_2 + b_2K + \tfrac12 a_3k)\,\omega^2\,,\\
&d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4 = (D_2 - b_2K - \tfrac12 a_3k)\,\omega^1 + E_2\omega^2\,,
\end{aligned}$$

where

$$K = a_1a_3 - a_2^2 + b_1b_3 - b_2^2\,, \quad k = (a_1 - a_3)\,b_2 - (b_1 - b_3)\,a_2\,.$$

Denote further as usual

$$H = (a_1 + a_3)^2 + (b_1 + b_3)^2\,.$$

The invariants K, H are the Gauss and the mean curvature, respectively.

Now, let us introduce some elementary remarks necessary in the following.

As mentioned in [1], a normal vector field $X = xv_3 + yv_4$ is parallel, if

$$(6)\qquad dx - y\omega_3^4 = 0\,, \quad dy + x\omega_3^4 = 0\,.$$

In this case we can choose orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ in each U_α in such a way that

$$k = 0\,.$$

Further, let $v_1, v_2 \in T(M)$ generate an orthogonal conjugate net of lines on M. Then it is easy to see that

$$a_2 = 0\,, \quad b_2 = 0\,, \quad k = 0$$

on M. Hence, because of (4), we see that ω_1^2 is the mean 1-form and there are real functions ϱ, σ such that

$$(7)\qquad \begin{aligned}
&\omega_1^2 = \varrho\omega^1 + \sigma\omega^2\,,\\
&\alpha_2 = \varrho(a_1 - a_3)\,, \quad \alpha_3 = \sigma(a_1 - a_3)\,,\\
&\beta_2 = \varrho(b_1 - b_3)\,, \quad \beta_3 = \sigma(b_1 - b_3)\,.
\end{aligned}$$

Now, let us revert to the object of our consideration.

Let $f : M \to \mathscr{R}$ be a function. Its covariant derivatives f_i, f_{ij} to U_α for each α with respect to the frames $\{M; v_1, v_2, v_3, v_4\}$ are defined by the formulas

$$(8) \qquad df = f_1\omega^1 + f_2\omega^2 ,$$

$$df_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2 , \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2 .$$

In all following proofs we use the maximum principle in this form:

Let M be a surface in E^4 and ∂M its boundary. Let f be a function on M and f_i, f_{ij} its covariant derivatives. Let (i) $f \geqq 0$ on M; (ii) $f = 0$ on ∂M; (iii) f satisfy in U_α the equation

$$a_{11}f_{11} + 2a_{12}f_{12} + a_{22}f_{22} + a_1f_1 + a_2f_2 + a_0f = a$$

with $a_{ij}x^ix^j$ positive definite, $a_0 \leqq 0$ and $a \geqq 0$. Then $f = 0$ on M.

2. In the following consider the mean curvature vector field

$$(9) \qquad \xi = (a_1 + a_3)\, v_3 + (b_1 + b_3)\, v_4$$

on M. Further, $v_1, v_2 \in T(M)$ being the tangent orthonormal vector fields, define normal vector fields ξ_i, ξ_{ij} $(i, j = 1, 2)$ by

$$(10) \qquad \xi_1 = (v_1\xi)^N , \quad \xi_2 = (v_2\xi)^N ;$$

$$(11) \qquad \xi_{11} = (v_1\xi_1)^N , \quad \xi_{12} = (v_1\xi_2)^N , \quad \xi_{21} = (v_2\xi_1)^N , \quad \xi_{22} = (v_2\xi_2)^N ,$$

where $(X)^N$ denotes the field of normal components of X. Under this notation introduce the fields

$$(12) \qquad v_{11} = (v_1v_1)^N , \quad v_{22} = (v_2v_2)^N .$$

Now, we are going to get another proof of the assertion mentioned in [2]:

Theorem 1. *Let M be a surface in E^4. Let*

(i) *$K > 0$ on M;*
(ii) *ξ be parallel in $N(M)$;*
(iii) *∂M consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. On M, consider the function

$$(13) \qquad f = H - 4K = (a_1 - a_3)^2 + (b_1 - b_3)^2 + 4a_2^2 + 4b_2^2 .$$

Relations (8) yield for f especially, by virtue of (4), (5),

$$(14)\qquad f_{11} = -2[(a_1 - a_3)a_3 + (b_1 - b_3)b_3 - 4(a_2^2 + b_2^2)]K - [k + 4(a_1b_2 - a_2b_1)]k + 2(\alpha_1 - \alpha_3)^2 + 2(\beta_1 - \beta_3)^2 + 8(\alpha_2^2 + \beta_2^2) + 2(a_1 - a_3)(A_1 - C_1) + 2(b_1 - b_3)(A_2 - C_2) + 8(a_2B_1 + b_2B_2),$$

$$f_{22} = 2[(a_1 - a_3)a_1 + (b_1 - b_3)b_1 + 4(a_2^2 + b_2^2)]K - [k + 4(a_2b_3 - a_3b_2)]k + 2(\alpha_2 - \alpha_4)^2 + 2(\beta_2 - \beta_4)^2 + 8(\alpha_3^2 + \beta_3^2) + 2(a_1 - a_3)(C_1 - E_1) + 2(b_1 - b_3)(C_2 - E_2) + 8(a_2D_1 + b_2D_2).$$

Adding these equations under the condition (ii) which implies $k = 0$ on M, we get

$$(15)\qquad f_{11} + f_{22} - 2fK = 2V + 2\Phi + 8\varphi + 8(a_2^2 + b_2^2)K$$

where

$$(16)\qquad V = (\alpha_1 - \alpha_3)^2 + (\alpha_2 - \alpha_4)^2 + (\beta_1 - \beta_3)^2 + (\beta_2 - \beta_4)^2 + 4(\alpha_2^2 + \alpha_3^2) + 4(\beta_2^2 + \beta_3^2),$$

$$(17)\qquad \Phi = (a_1 - a_3)(A_1 - E_1) + (b_1 - b_3)(A_2 - E_2),$$
$$\varphi = a_2(B_1 + D_1) + b_2(B_2 + D_2).$$

Now, we have from (ii) using (4), (6), (9)

$$(18)\qquad \alpha_1 + \alpha_3 = 0, \quad \alpha_2 + \alpha_4 = 0,$$
$$\beta_1 + \beta_3 = 0, \quad \beta_2 + \beta_4 = 0.$$

By exterior differentiation of these equations we obtain

$$(A_1 + C_1 + a_3K)\omega^1 + (B_1 + D_1)\omega^2 + (\alpha_2 + \alpha_4)\omega_1^2 + (\beta_1 + \beta_3)\omega_3^4 = 0,$$
$$(B_1 + D_1)\omega^1 + (C_1 + E_1 + a_1K)\omega^2 - (\alpha_1 + \alpha_3)\omega_1^2 + (\beta_2 + \beta_4)\omega_3^4 = 0,$$
$$(A_2 + C_2 + b_3K)\omega^1 + (B_2 + D_2)\omega^2 + (\beta_2 + \beta_4)\omega_1^2 - (\alpha_1 + \alpha_3)\omega_3^4 = 0,$$
$$(B_2 + D_2)\omega^1 + (C_2 + E_2 + b_1K)\omega^2 - (\beta_1 + \beta_3)\omega_1^2 - (\alpha_2 + \alpha_4)\omega_3^4 = 0$$

and hence using (18)

$$A_1 + C_1 + a_3K = 0, \quad C_1 + E_1 + a_1K = 0, \quad B_1 + D_1 = 0,$$
$$A_2 + C_2 + b_3K = 0, \quad C_2 + E_2 + b_1K = 0, \quad B_2 + D_2 = 0.$$

By means of these relations we finally have $\varphi = 0$ and

$$\Phi = [(a_1 - a_3)^2 + (b_1 - b_3)^2]K.$$

Thus the equation (15) reduces to

$$f_{11} + f_{22} - 4fK = 2V$$

and the maximum principle yields our assertion.

3. The following theorems are generalizations of this basic result. One of them is

Theorem 2. *Let M be a surface in E^4. Let*
(i) $K > 0$ *on* M;
(ii) $v_1, v_2 \in T(M)$ *generate an orthogonal conjugate net on* M;
(iii) $\xi_1, \xi_2 \in N(M)$ *be parallel in* $N(M)$;
(iv) ∂M *consist of umbilical points.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. Recall that the condition (ii) implies the relations (7) and

$$a_2 = 0\,, \quad b_2 = 0\,, \quad k = 0$$

on M. Thus the equation (15) has the form

$$f_{11} + f_{22} - 2fK = 2V + 2\Phi$$

where V, Φ are the functions introduced in (16), (17) respectively.
Now, we get from (9), (10) using (4)

$$\xi_1 = (\alpha_1 + \alpha_3)\, v_3 + (\beta_1 + \beta_3)\, v_4\,, \qquad (19)$$
$$\xi_2 = (\alpha_2 + \alpha_4)\, v_3 + (\beta_2 + \beta_4)\, v_4\,.$$

As ξ_1 is parallel according to the assumption (iii), we have from (6) using (5)

$$(A_1 + C_1 + a_3K)\, \omega^1 + (B_1 + D_1)\, \omega^2 + (\alpha_2 + \alpha_4)\, \omega_1^2 = 0\,,$$
$$(A_2 + C_2 + b_3K)\, \omega^1 + (B_2 + D_2)\, \omega^2 + (\beta_2 + \beta_4)\, \omega_1^2 = 0\,.$$

Multiply these equations by $a_1 - a_3$, $b_1 - b_3$ respectively. Then using (7) we get in particular

$$(a_1 - a_3)(A_1 + C_1 + a_3K) + \alpha_2(\alpha_2 + \alpha_4) = 0\,, \qquad (20)$$
$$(b_1 - b_3)(A_2 + C_2 + b_3K) + \beta_2(\beta_2 + \beta_4) = 0\,.$$

In the same way we obtain from the condition of parallelness of ξ_2

$$(a_1 - a_3)(C_1 + E_1 + a_1K) - \alpha_3(\alpha_1 + \alpha_3) = 0\,, \qquad (21)$$
$$(b_1 - b_3)(C_2 + E_2 + b_1K) - \beta_3(\beta_1 + \beta_3) = 0\,.$$

Hence from (20), (21)

$$(a_1 - a_3)(A_1 - E_1) = (a_1 - a_3)^2 K - \alpha_3(\alpha_1 + \alpha_3) - \alpha_2(\alpha_2 + \alpha_4),$$
$$(b_1 - b_3)(A_2 - E_2) = (b_1 - b_3)^2 K - \beta_3(\beta_1 + \beta_3) - \beta_2(\beta_2 + \beta_4)$$

and

$$\Phi = fK - \alpha_3(\alpha_1 + \alpha_3) - \alpha_2(\alpha_2 + \alpha_4) - \beta_3(\beta_1 + \beta_3) - \beta_2(\beta_2 + \beta_4).$$

Thus we have

$$f_{11} + f_{22} - 4fK =$$
$$= 2V - 2[\alpha_3(\alpha_1 + \alpha_3) + \alpha_2(\alpha_2 + \alpha_4) + \beta_3(\beta_1 + \beta_3) + \beta_2(\beta_2 + \beta_4)]$$

V being the function (16), and further

$$f_{11} + f_{22} - 4fK = \tfrac{7}{2}(\alpha_2^2 + \alpha_3^2 + \beta_2^2 + \beta_3^2) +$$
$$+ 2[(\alpha_1 - \tfrac{3}{2}\alpha_3)^2 + (\alpha_4 - \tfrac{3}{2}\alpha_2)^2 + (\beta_1 - \tfrac{3}{2}\beta_3)^2 + (\beta_4 - \tfrac{3}{2}\beta_2)^2]$$

so that by means of the maximum principle $f = 0$ on M. This completes our proof.

4. A generalization of the characterization of the sphere in E^4 is formulated in the following

Theorem 3. *Let M be a surface in E^4. Let*

(i) $K > 0$ *on* M;
(ii) $v_1, v_2 \in T(M)$ *generate an orthogonal conjugate net on* M;
(iii) (a) $\langle \xi_{11} + S(\xi_{12} - \xi_{21}), v_{11} - v_{22} \rangle \geqq 0$ *on M where $S : M \to \mathscr{R}$ is a function satisfying* $|S| \leqq 4\sqrt{(2)} - 5$ *and*
(b) $\xi_2 \in N(M)$ *be parallel in* $N(M)$;

or

(iii') (a') $\langle -\xi_{22} + S(\xi_{12} - \xi_{21}), v_{11} - v_{22} \rangle \geqq 0$ *on M where $S : M \to \mathscr{R}$ is a function such that* $|S| \leqq 4\sqrt{(2)} - 5$ *and*
(b') $\xi_1 \in N(M)$ *be parallel in* $N(M)$;
(iv) *each point of ∂M be umbilical.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. We are going to prove the case of the assumption (iii), the proof of the theorem under the condition (iii') being analogous.

First of all, we get from (19) by means of (5), having in mind that $k = 0$ on M,

$$d\xi_1 = [(A_1 + C_1 + a_3K) v_3 + (A_2 + C_2 + b_3K) v_4] \omega^1 +$$
$$+ [(B_1 + D_1) v_3 + (B_2 + D_2) v_4] \omega^2 + \xi_2 \omega_1^2,$$

$$d\xi_2 = [(C_1 + E_1 + a_1K)\, v_3 + (C_2 + E_2 + b_1K)\, v_4]\, \omega^2 +$$
$$+ [(B_1 + D_1)\, v_3 + (B_2 + D_2)\, v_4]\, \omega^1 - \xi_1\omega_1^2 \quad (\text{mod } v_1, v_2)$$

and hence, using (7) implied by (ii) we get from (11)

$$\begin{aligned} \xi_{11} &= (A_1 + C_1 + a_3K)\, v_3 + (A_2 + C_2 + b_3K)\, v_4 + \varrho\xi_2\,, \\ \xi_{12} &= (B_1 + D_1)\, v_3 + (B_2 + D_2)\, v_4 - \varrho\xi_1\,, \\ \xi_{21} &= (B_1 + D_1)\, v_3 + (B_2 + D_2)\, v_4 + \sigma\xi_2\,, \\ \xi_{22} &= (C_1 + E_1 + a_1K)\, v_3 + (C_2 + E_2 + b_1K)\, v_4 - \sigma\xi_1\,. \end{aligned} \tag{22}$$

Further, we have from (12) directly

$$v_{11} - v_{22} = (a_1 - a_3)\, v_3 + (b_1 - b_3)\, v_4\,. \tag{23}$$

Assumption (iii) (b) yields immediately, see (21),

$$(a_1 - a_3)(C_1 + E_1 + a_1K) + (b_1 - b_3)(C_2 + E_2 + b_1K) =$$
$$= \alpha_3(\alpha_1 + \alpha_3) + \beta_3(\beta_1 + \beta_3)\,.$$

From (22), (23) using (7) we obtain

$$(a_1 - a_3)(A_1 + C_1 + a_3K) + (b_1 - b_3)(A_2 + C_2 + b_3K) =$$
$$= \langle \xi_{11} + S(\xi_{12} - \xi_{21}),\, v_{11} - v_{22} \rangle - \alpha_2(\alpha_2 + \alpha_4) - \beta_2(\beta_2 + \beta_4) +$$
$$+ S[\alpha_2(\alpha_1 + \alpha_3) + \alpha_3(\alpha_2 + \alpha_4) + \beta_2(\beta_1 + \beta_3) + \beta_3(\beta_2 + \beta_4)]\,.$$

Hence the relation (17) has the form

$$\Phi = \langle \xi_{11} + S(\xi_{12} - \xi_{21}),\, v_{11} - v_{22} \rangle + fK -$$
$$- \alpha_3(\alpha_1 + \alpha_3) - \alpha_2(\alpha_2 + \alpha_4) - \beta_3(\beta_1 + \beta_3) - \beta_2(\beta_2 + \beta_4) +$$
$$+ S[\alpha_2(\alpha_1 + \alpha_3) + \alpha_3(\alpha_2 + \alpha_4) + \beta_2(\beta_1 + \beta_3) + \beta_3(\beta_2 + \beta_4)]$$

and

$$f_{11} + f_{22} - 4fK = 2\langle \xi_{11} + S(\xi_{12} - \xi_{21}),\, v_{11} - v_{22} \rangle + 2W \tag{24}$$

where

$$W = V - \alpha_3(\alpha_1 + \alpha_3) - \alpha_2(\alpha_2 + \alpha_4) - \beta_3(\beta_1 + \beta_3) - \beta_2(\beta_2 + \beta_4) + \tag{25}$$
$$+ S[\alpha_2(\alpha_1 + \alpha_3) + \alpha_3(\alpha_2 + \alpha_4) + \beta_2(\beta_1 + \beta_3) + \beta_3(\beta_2 + \beta_4)]\,.$$

Now it is easy to see that

$$W = (\alpha_1 - \tfrac{3}{2}\alpha_3 + \tfrac{1}{2}S\alpha_2)^2 + (\alpha_4 - \tfrac{3}{2}\alpha_2 + \tfrac{1}{2}S\alpha_3)^2 + \tag{26}$$
$$+ (\beta_1 - \tfrac{3}{2}\beta_3 + \tfrac{1}{2}S\beta_2)^2 + (\beta_4 - \tfrac{3}{2}\beta_2 + \tfrac{1}{2}S\beta_3)^2 +$$
$$+ \tfrac{1}{4}[(7 - S^2)\,\alpha_2^2 + 20S\alpha_2\alpha_3 + (7 - S^2)\,\alpha_3^2] +$$
$$+ \tfrac{1}{4}[(7 - S^2)\,\beta_2^2 + 20S\beta_2\beta_3 + (7 - S^2)\,\beta_3^2]\,.$$

The two last terms of (26) are non-negative for each α_i, β_i $(i = 2, 3)$ because of (iii) (a). The assumption (iii) (a) and the maximum principle complete again the proof.

As a special case of this assertion, we introduce

Corollary 1. *Let M be a surface in E^4. Assume* (i), (ii), (iv) *and let*

(iii) (a) $\xi_{11} + S(\xi_{12} - \xi_{21}) = 0$ *on M, $S : M \to \mathscr{R}$ being a function such that* $|S| \leqq 4\sqrt{(2)} - 5$ *on M and*

(b) ξ_2 *be parallel in $N(M)$*

or

(iii′) (a′) $-\xi_{22} + S(\xi_{12} - \xi_{21}) = 0$ *on M where $S : M \to \mathscr{R}$ is a function satisfying* $|S| \leqq 4\sqrt{(2)} - 5$ *on M and*

(b′) ξ_1 *be parallel in $N(M)$.*

Then M is a part of a 2-dimensional sphere in E^4.

Other trivial consequences can be obtained by putting $S = 0$ in Theorem 3 and Corollary 1.

5. Finally, we are going to prove a more general version of Theorem 3.

Theorem 4. *Let M be a surface in E^4. Let*

(i) $K > 0$ *on M;*

(ii) $v_1, v_2 \in T(M)$ *generate an orthogonal conjugate net of lines on M;*

(iii) $\langle \xi_{11} - \xi_{22} + S(\xi_{12} - \xi_{21}), v_{11} - v_{22} \rangle \geqq 0$ *on M where $S : M \to \mathscr{R}$ is a function such that* $|S| \leqq 4\sqrt{(2)} - 5$ *on M;*

(iv) *each point of ∂M be umbilical.*

Then M is a part of a 2-dimensional sphere in E^4.

Proof. From (17), (22) and (23) we have immediately

$$\Phi = \langle \xi_{11} - \xi_{22} + S(\xi_{12} - \xi_{21}), v_{11} - v_{22} \rangle + fK -$$
$$- \alpha_3(\alpha_1 + \alpha_3) - \alpha_2(\alpha_2 + \alpha_4) - \beta_3(\beta_1 + \beta_3) - \beta_2(\beta_2 + \beta_4) +$$
$$+ S[\alpha_2(\alpha_1 + \alpha_3) + \alpha_3(\alpha_2 + \alpha_4) + \beta_2(\beta_1 + \beta_3) + \beta_3(\beta_2 + \beta_4)]$$

and hence

$$f_{11} + f_{22} - 4fK = 2\langle \xi_{11} - \xi_{22} + S(\xi_{12} - \xi_{21}), v_{11} - v_{22} \rangle + 2W$$

where W is the function introduced by (25). Thus we get again the relation (26) and, according to the condition (iii), M is a part of a sphere in E^4 by virtue of the maximum principle.

Again we can formulate

Corollary 2. *Let M be a surface in E^4 satisfying the assumptions* (i), (ii), (iv) *of Theorem* 4. *Let*

(iii) $\xi_{11} - \xi_{22} + S(\xi_{12} - \xi_{21}) = 0$ *on* M, $S : M \to \mathcal{R}$ *being a function such that* $|S| \leqq 4\sqrt{(2)} - 5$ *on* M.

Then M is a part of a 2-dimensional sphere in E^4.

We have got trivial consequences of Theorem 4 and Corollary 2 for $S = 0$ on M.

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