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ANISOTROPIC SOBOLEV INEQUALITIES

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Summary. By anisotropy of the Sobolev space we mean membership of $D_i u$ in L_{p_i} with generally different $p_i's$. There is proved an imbedding theorem in the form of the Sobolev inequality estimating L_q-mixed norm of a function by corresponding L_{p_i} norms of its first order derivatives, including known results (Krbec, Kruzhkov and Kolodii, Rákosník), further, it is generalized to higher order spaces as well.

Keywords: anisotropic Sobolev space, mixed norm L_p-space, Sobolev inequality.

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1. INTRODUCTION

In its most basic form, Sobolev's inequality provides an estimate for the L^q norm of a smooth function u, compactly supported in $\mathbb{R}^n$, in terms of the L^p norm of the gradient of u. Specifically, if $1 \leqq p < n$ then

$$\|u\|_q \leqq K \sum_{j=1}^{n} \|D_j u\|_p \tag{1}$$

where $q = np/(n - p)$, that is

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \tag{2}$$

Here, of course, D_j denotes the partial differential operator $\partial/\partial x_j$ and $\|\cdot\|_p$ denotes the norm in the space $L^p(\mathbb{R}^n)$. The constant K is valid for all compactly supported functions u for which $D_j u$ makes sense as a distribution, but it is common to assert the inequality for smooth such functions. Throughout this paper we assume $u \in C_0^\infty(\mathbb{R}^n)$, the space of infinitely often differentiable functions with compact support in $\mathbb{R}^n$. (Note that (1) does not, for example, hold for constant functions.) Best constants K for Sobolev's inequality are known, (see, for instance, Duff [3] or Talenti [11]), but will not concern us here.

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Our purpose is to generalize Sobolev's inequality to the anisotropic case where the various derivatives $D_j u$ belong to different L^p spaces, and then to extend the first order anisotropic version to higher orders.

It is well known that no inequality of type (1) is possible for all $u \in C_0^\infty(\mathbb{R}^n)$ unless q satisfies (2). To see this, observe that for given $u \in C_0^\infty(\mathbb{R}^n)$, the dilated function u_λ defined by $u_\lambda(x) = u(\lambda x)$, $(\lambda > 0,\ x \in \mathbb{R}^n)$, also belongs to $C_0^\infty(\mathbb{R}^n)$ and satisfies

$$\|u_\lambda\|_q = \lambda^{-n/q}\|u\|_q, \quad \|D_j u_\lambda\|_p = \lambda^{1-n/p}\|D_j u\|_p$$

so that if (1) holds we must have, for all $\lambda > 0$,

$$\lambda^{-(n/q)+(n/p)-1} \leqq K \frac{\sum_{j=1}^{n} \|D_j u\|_p}{\|u\|_q}.$$

This is not possible unless the exponent of λ is zero. This uniqueness of q extends, as we shall see, to the anisotropic case also.

Sobolev [10] originally proved (1) by using potential theoretic arguments based on convolution with the kernel $|x|^{-n}$. Because this kernel is not integrable, Sobolev's proof only works if $p > 1$. The case $p = 1$ was proved by Gagliardo [5] and Nirenberg [8]. The case of general p follows easily from the case $p = 1$. Gagliardo used a combinatorial argument to obtain the case $p = 1$ from a refinement of Hölder's inequality as applied to a product of n functions each of which is independent of one of the variables. As observed by Fournier in [4], Gagliardo's method can be mechanized by the use of Hölder's inequality in the context of mixed norm L^p spaces. We will obtain our anisotropic versions of Sobolev's inequality via elementary mixed-norm estimates similar to those in [4]. As a result we will also obtain an $(n-1)$-parameter family of mixed-norm anisotropic Sobolev inequalities.

Some attention has been given in recent years to establishing imbeddings of various anisotropic spaces (Sobolev spaces, Besov spaces and their generalizations). See, for instance, the papers by Kruzhkov and Kolodii [7], Krbec [6], Rakosnik [9], as well as the Monograph [2] of Besov, Il'in and Nikolskii. Such imbedding generally involve inequalities similar to Sobolev's inequality, but including some L^p norm of u on the right side so that the inequalities can be be proved for suitably regular subdomains of $\mathbb{R}^n$, and for functions without compact support. The mixed-norm method used here can also be used to obtain some such imbeddings.

2. MIXED-NORM L^p SPACES ON $\mathbb{R}^n$

The general setting for considering mixed-norm spaces is in a Cartesian product X of sigma-finite measure spaces X_k. (See [1] or [4].) Although everything we say in this section applies in the general case we will be concerned only with functions defined on $\mathbb{R}^n$ and so will phrase our discussion in that context.

Given a measurable function u on $\mathbb{R}^n$, and an index vector $p = (p_1, p_2, \ldots, p_n)$, where $0 < p_j \leqq \infty$ for each j, we can calculate the numbers $\|u\|_p$ by first calculating the L^{p_1}-norm of $u(x_1, \ldots, x_n)$ with respect to x_1, and then the L^{p_2}-norm of the result with respect to x_2, and so on finishing with the L^{p_n}-norm with respect to x_n:

$$\|u\|_p = \|\ldots\| \, \|u\|_{L^{p_1}(dx_1)}\|_{L^{p_2}(dx_2)} \cdots \|_{L^{p_n}(dx_n)}$$

where

$$\|f\|_{L^p(dt)} = \begin{cases} \left[\int_{-\infty}^{\infty} |f(\ldots, t, \ldots)|^p \, dt\right]^{1/p} & \text{if } \; 1 < p < \infty \\ \operatorname{ess\,sup}_t |f(\ldots, t, \ldots)| & \text{if } \; p = \infty \, . \end{cases}$$

Of course, $\|\cdot\|_{L^p(dt)}$ is not a *norm* unless $p \geqq 1$. For instance, if all the numbers p_j are finite, then

$$\|u\|_p = \left[\int_{-\infty}^{\infty} \cdots \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |u(x_1, \ldots, x_n)|^{p_1} \, dx_1\right]^{p_2/p_1} dx_2\right]^{p_3/p_2} \ldots dx_n\right]^{1/p_n} .$$

We will denote by $L^p = L^p(\mathbb{R}^n)$ the set of (equivalence classes of almost everywhere equal) functions u for which $\|u\|_p < \infty$. Provided all $p_j \geqq 1$ this is a Banach space with norm $\|\cdot\|_p$. The reader is referred to Benedek and Panzone [1] for general information on spaces L^p. For our purposes we need only two elementary results about such mixed norms, Hölder's inequality and an inequality concerning the effect on mixed norms of permuting the order in which the L^{p_j} norms are evaluated.

2.1 **Hölder's Inequality.** *Let $0 < p_j \leqq \infty$, $0 < q_j \leqq \infty$ for $1 \leqq j \leqq n$. If $u \in L^p$ and $v \in L^q$ then $uv \in L^r$ where*

(3) $$\frac{1}{r_j} = \frac{1}{p_j} + \frac{1}{q_j}, \quad 1 \leqq j \leqq n \, ,$$

and also

(4) $$\|uv\|_r \leqq \|u\|_p \, \|v\|_q \, .$$

Hölder's inequality (4) can be proved by n successive applications of the ordinary Hölder inequality applied one variable at a time. Note that p_j and q_j are allowed to be less than 1. Then n equations (3) are usually summarized

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} .$$

Iteration of (4) leads to a version for the product of k functions:

$$\|\prod_{j=1}^{k} u_j\|_r \leqq \prod_{j=1}^{k} \|u_j\|_{p_j}$$

where

$$\frac{1}{r} = \sum_{j=1}^{n} \frac{1}{p_j}.$$

2.2 **Permuted Mixed Norms.** The definition of $\|u\|_p$ requires the successive L^{p_j} norms to be calculated in the order of appearance of the variables in the argument of u. This order can be changed by permuting the arguments and associated indices. If σ is a permutation of the set $\{1, 2, \ldots, n\}$ let $\sigma x = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$, and let $\sigma \boldsymbol{p}$ be defined similarly. If σu is defined by $\sigma u(\sigma x) = u(x)$, (so that $\sigma u(x) = u(\sigma^{-1} x)$,) then $\|\sigma u\|_{\sigma p}$ is called a *permuted mixed norm* of u. For example, if $n = 2$ and $\sigma\{1, 2\} = \{2,1\}$ then

$$\|u\|_p = \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |u(x_1, x_2)|^{p_1} \, dx_1\right]^{p_2/p_1} dx_2\right]^{1/p_2}$$

and

$$\|\sigma u\|_{\sigma p} = \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |u(x_1, x_2)|^{p_2} \, dx_2\right]^{p_1/p_2} dx_1\right]^{1/p_1}.$$

Note that $\|u\|_p$ and $\|\sigma u\|_{\sigma p}$ involve the same L^{p_j} norms with respect to the same variables; only the order of evaluation of those norms is changed.

2.3 **Permutation Inequality.** Given an index vector $\boldsymbol{p}$ let σ_* and σ^* be permutations of $\{1, 2, \ldots, n\}$ such that $\sigma_* \boldsymbol{p}$ and $\sigma^* \boldsymbol{p}$ have components in nondecreasing order and nonincreasing order respectively:

$$p_{\sigma_*(1)} \leqq p_{\sigma_*(2)} \leqq \cdots \leqq p_{\sigma_*(n)},$$

$$p_{\sigma^*(1)} \geqq p_{\sigma^*(2)} \geqq \cdots \geqq p_{\sigma^*(n)}.$$

Then for any permutation σ of $\{1, 2, \ldots, n\}$ and any function u we have

$$\|\sigma_* u\|_{\sigma_* p} \leqq \|\sigma u\|_{\sigma p} \leqq \|\sigma^* u\|_{\sigma^* p}. \tag{5}$$

Since any permutation can be decomposed into a product of special permutations each of which transposes two adjacent elements and leaves the rest unmoved, proving (5) reduces to demonstrating the special case: if $p_1 \leqq p_2$ then

$$\left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |u|^{p_1} \, dx_1\right]^{p_2/p_1} dx_2\right]^{1/p_2} \leqq \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |u|^{p_2} \, dx_2\right]^{p_1/p_2} dx_1\right]^{1/p_1}.$$

But this is just Minkowski's inequality for integrals:

$$\left\|\int_{-\infty}^{\infty} |v(x_1, x_2)| \, dx_1\right\|_{L^r(dx_2)} \leqq \int_{-\infty}^{\infty} \|v(x_1, \cdot)\|_{L^r(dx_2)} \, dx_1$$

applied to $v = |u|^{p_1}$ with $r = p_2/p_1$.

An inequality of the form

$$\|u\|_q \leqq K \sum_{j=1}^{n} \|D_j u\|_{p_j} \tag{6}$$

is called an anisotropic Sobolev inequality because different L^p norms are used to estimate derivatives in different coordinate directions. We assume $p_j \geqq 1$ for $1 \leqq \leqq j \leqq n$, and shall show that such an inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$ for

$$\frac{1}{q} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_j} - \frac{1}{n} \tag{7}$$

provided that $\sum_{j=1}^{n} (1/p_j) > 1$.

Let us begin with a dilation argument to show that (7) defines the only possible value of q for which (6) can hold.

3.1 **Lemma.** *If there exists a constant K such that inequality* (6) *holds for all* $u \in C_0^\infty(\mathbb{R}^n)$ *then q must satisfy* (7).

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $0 < \lambda_j < \infty$ for all j. Choose a function $u \in C_0^\infty(\mathbb{R}^n)$ for which both sides of (6) are positive, and let u_λ be the anisotropic dilation

$$u_\lambda(x) = u(\lambda_1 x_1, \ldots, \lambda_n x_n).$$

It is readily shown that

$$\|u_\lambda\|_q = (\lambda_1 \lambda_2, \ldots, \lambda_n)^{-1/q} \|u\|_q,$$

$$\|D_j u_\lambda\|_{p_j} = \lambda_j (\lambda_1 \lambda_2, \ldots, \lambda_n)^{-1/p_j} \|D_j u\|_{p_j}.$$

If $t > 0$ we can choose λ so that

$$\lambda_j (\lambda_1 \lambda_2, \ldots, \lambda_n)^{-1/p_j} = t^{-1}, \quad (1 \leqq j \leqq n).$$

It then follows that

$$\lambda_1 \lambda_2, \ldots, \lambda_n = t^{n/(\sum_{j=1}^{n}(1/p_j) - 1)}.$$

Since all the dilations u_λ belongs to $C_0^\infty(\mathbb{R}^n)$ we must have, by (6),

$$t^{-n/[q(\sum_{j=1}^{n}(1/p_j) - 1)] + 1} \leqq K \frac{\sum_{j=1}^{n} \|D_j u\|_{p_j}}{\|u\|_q}. \tag{8}$$

Observe that t is large if $|\lambda|$ is large and small if $|\lambda|$ is small. Therefore the exponent of t on the left side of (8) must vanish, and (7) follows. □

3.2 **Theorem.** *Suppose $p_j \geqq 1$ for $1 \leqq j \leqq n$, and $\sum_{j=1}^{n} (1/p_j) > 1$. If*

$$\frac{1}{q} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_j} - \frac{1}{n}$$

then there exists a constant K such that the anisotropic Sobolev inequality

$$\|u\|_q \leqq K \sum_{j=1}^{n} \|D_j u\|_{p_j}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Throughout this proof (and subsequent ones) K denotes a generic constant independent of u. Its value may change with each usage; we make no attempt to keep track of its value.

Let $s_j \geqq 1$ for $1 \leqq j \leqq n$. Since u has compact support we have

$$|u(x_1, \ldots, x_n)|^{s_j} = \int_{-\infty}^{x_j} D_j |u(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)|^{s_j} \, dt ,$$

so that

$$\sup_{x_j} |u(x_1, \ldots, x_n)|^{s_j} \leqq \int_{-\infty}^{\infty} D_j |u(x_1, \ldots, x_n)|^{s_j} \, dx_j$$

and integration with respect to the other $n - 1$ components of x leads to the permuted mixed norm estimate

$$\begin{aligned} \|\sigma_j |u|^{s_j}\|_{(\infty, 1, 1, \ldots, 1)} &\leqq \|D_j |u|^{s_j}\|_1 = \\ = s_j \| |u|^{s_j - 1} D_j u\|_1 &\leqq K \|u\|_{(s_j-1)p_j'}^{s_j-1} \|D_j u\|_{p_j} \end{aligned}$$

where σ_j is a permutation of $\{1, 2, \ldots, n\}$ for which $\sigma_j(1) = j$, and $p_j' = p_j/(p_j - 1)$, (or $p_j' = \infty$ if $p_j = 1$), is the exponent conjugate to p_j. For each j let $v_j = (1, 1, \ldots, \infty, \ldots, 1)$ be the index vector with j'th component infinite and all other components equal to 1. By the permutation inequality

$$\| |u|^{s_j} \|_{v_j} \leqq \|\sigma_j |u|^{s_j}\|_{(\infty, 1, \ldots, 1)} \leqq K \|u\|_{(s_j-1)p_j'}^{s_j-1} \|D_j u\|_{p_j} . \tag{9}$$

Let $s = s_1 + \ldots + s_n$ and let $1/r = \sum_{j=1}^{n} 1/v_j$, so that $r_j = 1/(n - 1)$ for each j and $\|\cdot\|_r = \|\cdot\|_{1/(n-1)}$. Using Hölder's inequality for mixed norms we obtain

$$\begin{aligned} \|u\|_{s/(n-1)}^s = \| |u|^s \|_{1/(n-1)} = \| |u|^{s_1 + \ldots + s_n} \|_r &\leqq \\ \leqq \prod_{j=1}^{n} \| |u|^{s_j} \|_{v_j} \leqq K \prod_{j=1}^{n} \|u\|_{(s_j-1)p_j'}^{s_j-1} \|D_j u\|_{p_j} &. \end{aligned}$$

Now choose the numbers s_j so that

$$(10) \qquad (s_1 - 1)\, p_1' = (s_2 - 1)\, p_2' = \ldots = (s_n - 1)\, p_n' = \frac{s}{n-1} = q\,,$$

where q is defined to be the common value of the other $n + 1$ expressions in (10). (If any $p_j = 1$ then the corresponding $s_j = 1$ also.) Assuming this done, we will have

$$\|u\|_q^s \leqq K \prod_{j=1}^{n} \|u\|_q^{s_j - 1} \, \|D_j u\|_{p_j}$$

whence, by cancellation,

$$\|u\|_q \leqq K\Big(\prod_{j=1}^{n} \|D_j u\|_{p_j}\Big)^{1/n} \leqq K \sum_{j=1}^{n} \|D_j u\|_{p_j}\,.$$

It remains to find the common value q of the expressions in (10). If $p_j > 1$ we have

$$q = (s_j - 1)\, p_j' = (s_j - 1)\, \frac{p_j}{p_j - 1}$$

so that

$$s_j = 1 + q\left[1 - \frac{1}{p_j}\right],$$

and this latter formula evidently holds if $p_j = 1$ also. Therefore

$$(n - 1)\, q = s = s_1 + \ldots + s_n = n + nq - q \sum_{j=1}^{n} \frac{1}{p_j}\,.$$

It follows that $1/q = \big(\sum_{j=1}^{n} (1/p_j) - 1\big)/n$ as required. □

3.3 Remark. Of course Theorem 3.2 gives the (isotropic) Sobolev inequality if $p_j = p$ for all j, and $1 \leqq p < n$. Observe that p in the isotropic version is replaced by the harmonic mean of the n indices $p_1, \ldots, p_n$ in the anisotropic version.

3.4 Remark. Using the result of Theorem 3.2 the mixed-norm estimates (9) can be rewritten as

$$\|u\|_{s_j v_j}^{s_j} \leqq K \|u\|_q^{s_j - 1} \, \|D_j u\|_{p_j} \leqq K Q^{s_j - 1} \, \|D_j u\|_{p_j} \leqq K Q^{s_j}$$

where

$$Q = \sum_{j=1}^{n} \|D_j u\|_{p_j}\,.$$

These estimates lead, in turn, to an $(n - 1)$-parameter family of mixed-norm, anisotropic Sobolev inequalities as follows. Let $\lambda_1, \ldots, \lambda_n$ be positive numbers such that $\lambda_1 + \ldots + \lambda_n = 1$. We have

$$\|\, |u|^{\lambda_j} \|_{s_j v_j / \lambda_j} = \|u\|_{s_j v_j}^{\lambda_j} \leqq K Q^{\lambda_j}\,.$$

If $1/q = \sum_{j=1}^{n} \lambda_j/(s_j v_j)$ we have, by Hölder's inequality,

$$\|u\|_q = \| |u|^{\lambda_1 + \ldots + \lambda_n} \|_q \leqq \prod_{j=1}^{n} \| |u|^{\lambda_j} \|_{s_j v_j/\lambda_j} \leqq KQ$$

that is,

$$\|u\|_q \leqq \sum_{j=1}^{n} \|D_j u\|_{p_j} . \tag{11}$$

The components of q are given by

$$\frac{1}{q_k} = \sum_{j=1}^{n} \frac{\lambda_j}{s_j} - \frac{\lambda_k}{s_k}, \quad (1 \leqq k \leqq n),$$

where

$$s_j = 1 + q\left(1 - \frac{1}{p_j}\right), \quad (1 \leqq j \leqq n),$$

$$\frac{1}{q} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_j} - \frac{1}{n} .$$

Imbedding inequalities analogous to the special case $n = 2$ of (11) were obtained by Krbec in [6].

4. HIGHER ORDER SOBOLEV INEQUALITIES

The isotropic Sobolev inequality of order m,

$$\|u\|_q \leqq K \sum_{|\alpha| = m} \|D^\alpha u\|_p \tag{12}$$

where $q = np/(n - mp)$, that is, where

$$\frac{1}{q} = \frac{1}{p} - \frac{m}{n},$$

holds for all u in $C_0^\infty(R^n)$ provided $mp < n$. (We are using standard multi-index notation: if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a vector of nonnegative integers then $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots \ldots D_n^{\alpha_n}$ is a differential operator of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$.) Inequality (12) is easily obtained by induction from the first order case, inequality (1).

A corresponding anisotropic version,

$$\|u\|_q \leqq K \sum_{|\alpha| = m} \|D^\alpha u\|_{p_\alpha}$$

where

$$\frac{1}{q} = \frac{1}{n^m} \sum_{|\alpha| = m} \begin{bmatrix} m \\ \alpha \end{bmatrix} \frac{1}{p_\alpha} - \frac{m}{n},$$

can be obtained inductively from Theorem 3.2 under suitable restrictions on p_α. (Here we have used the multinomial coefficient

$$\begin{bmatrix} m \\ \alpha \end{bmatrix} = \frac{m!}{\alpha_1!\,\alpha_2!\,\dots\,\alpha_n!}$$

where $|\alpha| = m$.) The restrictions must guarantee that any derivative of order less than m can be estimated in terms of derivatives of one higher order, so the obvious condition

$$\frac{1}{n^m}\sum_{|\alpha|=m}\begin{bmatrix} m \\ \alpha \end{bmatrix}\frac{1}{p_\alpha} > \frac{m}{n}$$

will not suffice in general. Conditions $mp_\alpha < n$ for all α satisfying $|\alpha| = m$ will suffice, but are stronger than necessary. The appropriate conditions are stated in terms of averages of certain n-tuples of the numbers p_α.

Before formulating the m'th order anisotropic Sobolev inequality we prepare some combinatorial necessities. Note that

$$\sum_{|\alpha|=m}\begin{bmatrix} m \\ \alpha \end{bmatrix} = n^m\,; \tag{13}$$

this is the multinomial theorem. If β is any multi-index and $j \in \{1, 2, \dots, n\}$ let $\beta[j] = (\beta_1, \dots, \beta_{j-1}, \beta_j + 1, \beta_{j+1}, \dots, \beta_n)$. Evidently $|\beta[j]| = |\beta| + 1$.

4.1 **Lemma.** *If numbers p_α are defined for all α satisfying $|\alpha| = m$ then*

$$\sum_{|\beta|=m-1}\begin{bmatrix} m-1 \\ \beta \end{bmatrix}\sum_{j=1}^{n}\frac{1}{p_{\beta[j]}} = \sum_{|\alpha|=m}\begin{bmatrix} m \\ \alpha \end{bmatrix}\frac{1}{p_\alpha}.$$

Proof. Given α with $|\alpha| = m$, for each j such that $\alpha_j > 0$ there exists β such that $\beta[j] = \alpha$; specifically $\beta = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_n)$. Accordingly

$$\sum_{|\beta|=m-1}\begin{bmatrix} m-1 \\ \beta \end{bmatrix}\sum_{j=1}^{n}\frac{1}{p_{\beta[j]}} = \sum_{|\alpha|=m}\frac{1}{p_\alpha}\sum_{\{j:\beta[j]=\alpha\}}\begin{bmatrix} m-1 \\ \beta \end{bmatrix} =$$

$$= \sum_{|\alpha|=m}\frac{1}{p_\alpha}\sum_{\{j:\alpha_j>0\}}\frac{(m-1)!}{\alpha_1!\dots\alpha_{j-1}!\,(\alpha_j-1)!\,\alpha_{j+1}!\dots\alpha_n!} =$$

$$= \sum_{|\alpha|=m}\frac{1}{p_\alpha}\frac{(m-1)!\,(\alpha_1+\alpha_2+\dots+\alpha_n)}{\alpha_1!\,\alpha_2!\dots\alpha_n!} =$$

$$= \sum_{|\alpha|=m}\begin{bmatrix} m \\ \alpha \end{bmatrix}\frac{1}{p_\alpha}. \quad \square$$

4.2 **Corollary.** *Suppose that*

$$\sum_{j=1}^{n}\frac{1}{p_{\beta[j]}} > m$$

for all β satisfying $|\beta| = m - 1$. Then

$$\frac{1}{n^m} \sum_{|\alpha|=m} \begin{bmatrix} m \\ \alpha \end{bmatrix} \frac{1}{p_\alpha} > \frac{m}{n}$$

and if the numbers q_β are defined by

$$\frac{1}{q_\beta} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} - \frac{1}{n}$$

then for all γ satisfying $|\gamma| = m - 2$ we have

$$\sum_{i=1}^{n} \frac{1}{q_{\gamma[i]}} > m - 1 .$$

Proof. By Lemma 4.1 and (13)

$$\frac{1}{n^m} \sum_{|\alpha|=m} \begin{bmatrix} m \\ \alpha \end{bmatrix} \frac{1}{p_\alpha} = \frac{1}{n_m} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} > \frac{m}{n^m} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} = \frac{m}{n} .$$

Also

$$\sum_{i=1}^{n} \frac{1}{q_{\gamma[i]}} = \frac{1}{n} \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \frac{1}{q_{\gamma[i][j]}} - 1 \right] > \frac{1}{n} \sum_{i=1}^{n} (m-1) = m - 1 . \quad \square$$

4.3 **Theorem.** *Let $p_\alpha \geqq 1$ for all α with $|\alpha| = m$. Suppose that for every β with $|\beta| = m - 1$ we have*

$$\sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} > m . \tag{14}$$

Then there exists a constant K such that the inequality

$$\|u\|_q \leqq K \sum_{|\alpha|=m} \|D^\alpha u\|_{p_\alpha} \tag{15}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$, where

$$\frac{1}{q} = \frac{1}{n^m} \sum_{|\alpha|=m} \begin{bmatrix} m \\ \alpha \end{bmatrix} \frac{1}{p_\alpha} - \frac{m}{n} . \tag{16}$$

Proof. We proceed by induction on m; Theorem 3.2 is the case $m = 1$. Suppose the case $m - 1$ has been established, and consider the case m. For each β satisfying $|\beta| = m - 1$ let q_β be defined by

$$\frac{1}{q_\beta} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} - \frac{1}{n} .$$

By Corollary 4.2, $\sum_{i=1}^{n} 1/q_{\gamma[i]} > m - 1$ for each γ satisfying $|\gamma| = m - 2$. Thus we may apply the induction hypothesis and obtain the inequality

$$\|u\|_q \leqq K \sum_{|\beta|=m-1} \|D^\beta u\|_{q_\beta}$$

where

$$\frac{1}{q} = \frac{1}{n^{m-1}} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} \frac{1}{q_\beta} - \frac{m-1}{n}.$$

But Theorem 3.2 implies that for each β

$$\|D^\beta u\|_{q_\beta} \leqq K \sum_{j=1}^{n} \|D_j D^\beta u\|_{p_{\beta[j]}} = \sum_{j=1}^{n} \|D^{\beta[j]} u\|_{p_{\beta[j]}}.$$

Thus

$$\|u\|_q \leqq K \sum_{|\beta|=m-1} \sum_{j=1}^{n} \|D^{\beta[j]} u\|_{p_{\beta[j]}} \leqq K \sum_{|\alpha|=m} \|D^\alpha u\|_{p_\alpha}.$$

To complete the induction observe that

$$\frac{1}{q} = \frac{1}{n^{m-1}} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} \left[\frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} - \frac{1}{n} \right] - \frac{m-1}{n} =$$

$$= \frac{1}{n^m} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} - \frac{1}{n^m} \sum_{|\beta|=m-1} \begin{bmatrix} m-1 \\ \beta \end{bmatrix} - \frac{m-1}{n} =$$

$$= \frac{1}{n^m} \sum_{|\alpha|=m} \begin{bmatrix} m \\ \alpha \end{bmatrix} \frac{1}{p_\alpha} - \frac{m}{n}$$

by Lemma 4.1 and (13). □

4.4 Remark. The number of distinct partial derivatives of order m for a smooth function of n variables (that is the cardinality of the set $\{\alpha: |\alpha| = m\}$ is given by a binomial coefficient:

$$N(m, n) = \begin{bmatrix} n+m-1 \\ m \end{bmatrix} = \frac{(n+m-1)!}{m!\,(n-1)!}.$$

To see this, observe that to each α there corresponds a finite *nondecreasing* sequence $\{i_1, i_2, \ldots, i_m\}$ of elements selected from $\{1, 2, \ldots, n\}$. (Thus $i_1 = \ldots = i_{\alpha_1} = 1$, $i_{\alpha_1+1} = \ldots = i_{\alpha_2} = 2, \ldots$). Such nondecreasing sequences are in one-to-one correspondence with *strictly increasing* sequences $\{i_1, i_2 + 1, i_3 + 2, ,\ldots, i_m + m - 1\}$ selected from $\{1, 2, \ldots, n + m - 1\}$. There are evidently $N(m, n)$ of these latter.

Conditions (14) of Theorem 4.3 place $N(m-1, n)$ restrictions on the $N(m, n)$ numbers p_α. These conditions guarantee that every derivative $D^\beta u$ of order $m - 1$ can be estimated in terms of the quantities $\{\|D^\alpha u\|_{p_\alpha}: |\alpha| = m\}$. By induction (based on Corollary 4.2) so can all lower order derivatives.

4.5 Remark. By repeated applications of Lemma 3.1, if conditions (14) are satisfied no m'th order anisotropic Sobolev inequality of type (15) is possible for values of q other than that specified by (16). However, the author has been unable to generalize Lemma 3.1 directly to the m'th order case, so it remains open whether any inequalities of type (15) are possible when conditions (14) are not satisfied.

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Souhrn

ANISOTROPNÍ SOBOLEVOVY NEROVNOSTI

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Anisotropií Sobolevova prostoru se zde rozumí fakt, že $D^i u \in L_{p_i}$ s obecně různými p_i. Je dokázána věta o vnoření ve tvaru Sobolevovy nerovnosti pro odhad smíšené L_q-normy funkce pomocí příslušných L_{p_i} norem prvních derivací, zahrnující známé výsledky (Krbec, Kružkov a Kolodij, Rákosník), a dále zobecněná i na prostory vyšších řádů.

Резюме

АНИЗОТРОПНЫЕ НЕРАВЕНСТВА СОБОЛЕВА

R. A. ADAMS

Анизотропия здесь обозначает, что $D^i u \in L_{p_i}$, где числа p_i не обязательно одинаковы. В статье приведено доказательство теоремы вложения в виде неравенства Соболева для оценки смешанной L_q-нормы функции при помощи соответствующих L_{p_i}-норм производных первого порядка. Эта теорема, которая содержит в себе некоторые известные результаты (Крбец, Кружков и Колодий, Ракосник), далее обобщена для пространств высшего порядка.

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