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ON GRAPHS WITH RESTRICTED LINK GRAPHS AND THE CHROMATIC NUMBER AT MOST 3

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Summary. The j-link graph of a vertex x in a graph G is the subgraph of G induced by the vertices at distance j from x in G. The paper deals with some problems concerning the estimation of the chromatic number of G in terms of the chromatic numbers of its link graphs. The questions of Szamkołowicz are answered and a certain class of graphs with the chromatic number at most 3 and j-link graphs of a special type is described.

Keywords: link graph, chromatic number of a graph.

AMS Classification: 05C10.

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *distance between two vertices* x and y in G is the number of edges in a shortest path connecting x and y in G. The *eccentricity* of a vertex x in G is the distance between x and a farthest vertex from x in G. The *radius* $r(G)$ is the minimum eccentricity of the vertices in G. Let j be a natural number and x a vertex of G. The *j-link* of x in G, denoted by $L_j(x)$, is the subgraph of G induced by the vertices at distance j from x. Szamkołowicz [1, 2] looked for estimates of the chromatic number of a graph in terms of the chromatic numbers of its j-link graphs. In this paper we continue his study.

Let $\chi(G)$ be the chromatic number of G.

Let $\bar{N}(x)$ be the set of natural numbers j for which there exists a j-link of x in G. Let $\mathscr{K}$ be a class of graphs. By $\mathscr{G}(\mathscr{K})$ de wenote the class of graphs whose every vertex x has a j-link belonging to $\mathscr{K}$ for every $j \in \bar{N}(x)$. Let $\mathscr{K}_1$ be a class of graphs whose components are complete graphs with one or two vertices. Let $\mathscr{K}_2$ be the subclass of $\mathscr{K}_1$ containing graphs whose components are complete graphs K_2. L. Szamkołowicz [1] posed

Conjecture 1. $\chi(G) \leqq 3$ for every $G \in \mathscr{G}(\mathscr{K}_1)$.

Let $\mathscr{G}'$ be the subclass of $\mathscr{G}(\mathscr{K}_1)$ such that for every $G \in \mathscr{G}'$ there exists a vertex x whose j-link graphs belong to $\mathscr{K}_2$ for $j \in \bar{N}(x)$. L. Szamkołowicz [2] proved

Theorem 1. $\chi(G) \leqq 3$ *for* $G \in \mathscr{G}'$.

The following question has been proposed in [2]: Is it possible to enlarge the class $\mathscr{G}(\mathscr{K}_1)$ in Conjecture 1 to the class $\mathscr{G}(\mathscr{K}_3)$, where $\mathscr{K}_3$ is the class containing graphs whose components are K_1, K_2 or $K_{1,t}$, $t \geqq 2$? It is known that $\mathscr{K}_1$ cannot be replaced by the class $\mathscr{K}_4$ containing graphs whose components are K_1, K_2, P_3 or P_4. A simple counterexample is that of the complement of a cycle with 7 vertices, where the 1-link is P_4 and the 2-link is K_2 for every its vertex, and the chromatic number of the graph equals 4. Moreover, the underlying graph shows that we cannot replace $\mathscr{K}_1$ by $\mathscr{K}_4$ even if we restrict our considerations to planar graphs from $\mathscr{G}(\mathscr{K}_4)$. Table 1 lists the j-link graphs of the graph given in Figure 1.

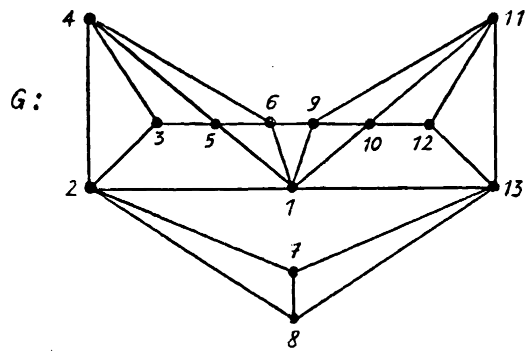

Fig. 1. $r(G) = 2$, $\chi(G) = 4$, and $L_j(x) \in \mathscr{K}_4$ for every vertex x of G and $j \in \bar{N}(x)$.

Table 1. The j-link graphs of vertices of the graph G shown in Figure 1, for $j = 1, 2, 3, 4$.

x	1	2, 13	3, 12	4, 11	5, 10	6, 9	7, 8
$L_1(x)$	$2K_1 \cup P_4$	$K_1 \cup 2K_2$	P_3	P_4	P_4	P_4	P_3
$L_2(x)$	$3K_2$	$K_1 \cup P_4$	$2K_2$	$2K_2$	$2K_1 \cup K_2$	$K_2 \cup P_3$	$K_1 \cup 2K_2$
$L_3(x)$		K_2	$K_1 \cup K_2$	P_3	$2K_2$	$K_1 \cup K_2$	P_4
$L_4(x)$			K_2	K_1			

Let us consider the graph H given in Figure 2.

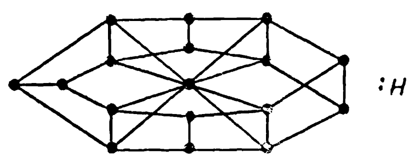

Fig. 2

It is easy to see that $L_j(x) \in \mathscr{K}_3$, $x \in V(H)$, $j \in \bar{N}(x)$. Moreover, if $K_{1,t}$ is a component of some $L_j(x)$ in H then $t = 2$. However, $\chi(H) = 4$. Since H contains a subgraph homeomorphic to the second graph of Kuratowski, i.e. $K_{3,3}$, it is not planar. In fact, H is the smallest graph with this property. Thus, in general, the class $\mathscr{K}_1$ cannot be replaced by $\mathscr{K}_3$ in Conjecture 1. Hence we have another question: is this possible for planar graphs? We present a partial solution of this problem. Let $\mathscr{G}''$ be a class of graphs which are planar, have radius not greater than 2, and the 1-link and 2-link graphs of their vertices belong to $\mathscr{K}_3$.

Theorem 2. *$\chi(G) \leqq 3$ for every $G \in \mathscr{G}''$.*

To prove the theorem we have to do some preliminary remarks. Let a planar embedding of a graph G be given. Let us assume that G has a subgraph presented in Figure 3. We will denote such a graph by $H(x; v, q; w, t)$.

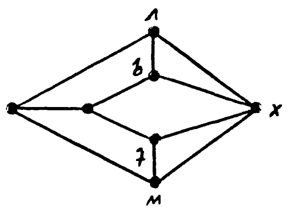

Fig. 3. $H(x; v, q; w, t)$.

Let $H_0 = H(x; w_0, t_0; w_{-1}, t_{-1})$, and assume that there is $H_1 \neq H_0$ in G such that $H_1 = H(x; w_1, t_1; w_0, t_0)$. Note that $w_1, t_1 \notin \{w_{-1}, t_{-1}\}$, or else G contains a subgraph homeomorphic to $K_{3,3}$. Further, if there is $H' \neq H_0$, $H' \neq H_1$ and $H' = H(x; w, t; w_0, t_0)$, then there is a subgraph in G homeomorphic to $K_{3,3}$ (it contains the vertices x, w_0, t_0 and w_{-1}, w_1, w). Therefore, there are at most two subgraphs H_0 and H_1 having a common edge $\{w_0, t_0\}$ in $L_1(x)$. We can assume, without loss of generality, that H_1 is contained in the interior face of the triangle induced by the vertices x, w_0, t_0. Similarly there is at most one subgraph $H_{-1} = H(x; w_{-1}, t_{-1}; w_{-2}, t_{-2})$ having an edge $\{w_{-1}, t_{-1}\}$ in common with H_0, and we can assume that H_{-1} is contained in the interior face of the triangle induced by the vertices x, w_{-1}, t_{-1}. Repeating this consideration for w_i, t_i, $i = \pm 1, \pm 2, \ldots$, we obtain a finite sequence of subgraphs $H_i = H(x; w_i, t_i; w_{i-1}, t_{i-1})$ such that H_i is contained in the interior face of the triangle induced by x, w_{i-1}, t_{i-1} for $i > 0$, and of the triangle induced by x, w_{i+1}, t_{i+1} for $i < -1$, and the edge $\{w_{-1}, t_{-1}\}$ is contained in the exterior face of the triangle induced by x, w_0, t_0.

Proof of Theorem 2. Let x be a vertex of eccentricity at most 2 in G. We define a special colouring of $L_1(x)$ and the vertex x in *Algorithm* A. This partial colouring leads to a proper colouring of all vertices of the graph G with at most 3 colours of the set $\{0, 1, 2\}$.

Algorithm A

colour $(x) := 0$;
FOR every isolated vertex y of $L_1(x)$ **DO** colour $(y) := 2$;
FOR every star S in $L_1(x)$ **DO**
  colour $(s) := 2$; {for the vertex s of degree $t > 1$ in S}
  colour $(p) := 1$; {for every vertex p of degree 1 in S}
WHILE there is K_2 in $L_1(x)$ which has not been coloured **DO**
{Let w_0, t_0 be the vertices of K_2}.
  colour $(t_0) := 1$;
  coluur $(w_0) := 2$;
  Find the sequence $H_i = H(x; w_i, t_i; w_{i-1}, t_{i-1})$ such that $H_0 = H(x; w_0, t_0; w_{-1}, t_{-1})$, $i = 0, \pm 1, \pm 2, \dots$.
  {The sequence may be empty.}
  FOR every w_i and t_i of the sequence **DO**
    colour $(t_i) := 1$;
    colour $(w_i) := 2$;

If the eccentricity of x equals 2 we have to colour the vertices of the components of $L_2(x)$. Evidently, we can colour with 0 and 1 such components whose vertices are adjacent to isolated vertices in $L_1(x)$. Let us proceed to the colouring of the other components. Let F be a component of $L_2(x)$ which has not been coloured yet. One of the two underlying cases has to appear.

Case 1. There is a vertex y of F adjacent to a vertex w of a star S in $L_1(x)$. Let z be a neighbour of y in F.

Case 1.1. Let w be the center of F. Note that, if z has a neighbour belonging to S, then all other vertices of $L_1(x)$ adjacent to z or y are isolated in $L_1(x)$. We colour F as follows. If z is adjacent to w then if y is the center of F then colour $(y) := 1$, else colour $(y) := 0$. In both cases colour $(z) :=$ 1-colour (y) and we colour the remaining vertices of F with 0. Assume that z is not adjacent to w. If z is the center of F then colour $(z) := 0$ and colour $(u) := 1$ for the other vertices of F, else colour $(y) := 1$ and colour $(u) := 0$ for $u \in V(F) - \{y\}$. Assume that no neighbour of y in F is adjacent to a vertex of S. Let v be a vertex of $L_1(x)$ adjacent to the vertex z. If v is adjacent to y then one can see that every vertex different from w of $L_1(x)$ and adjacent to y or z is isolated in $L_1(x)$. In the opposite case C_3 or P_4 is contained in $L_2(p)$ for some vertex p of S, or P_4 is contained in the 2-link of a neighbour of v. Therefore we can properly colour the center of F with 1 and other vertices of F with 0. Assume that y is not adjacent to any neighbour of its adjacent vertices. If there is a vertex q in $L_1(x)$ adjacent to y, $q \neq w$, then it is adjacent to v, or else P_4 is contained in $L_2(v)$ or in $L_2(w)$. Further, v is the unique vertex of $L_1(x)$ adjacent to z, and either w or w and q are the only vertices of $L_1(x)$ adjacent to y. Hence, if z is the center of F we can colour it with the colour of q and the other vertices of F with 0. If y is the center of F then the other vertices of F are adjacent to v, or else P_4 is contained in $L_2(v)$ or $L_2(w)$. Hence, we can colour y with 0 and the other vertices of F with the colour of q. Consider the case when w is the unique neighbour of y belonging to $L_1(x)$. It is easy to see that if y is the center of F, then one can colour it with 1 and the other vertices of F with 0. Assume that z is the center of F. If every vertex belonging to $L_1(x)$ and adjacent to z is isolated in $L_1(x)$ then we can colour z with 1 and the other vertices of F with 0. If there is q adjacent to v in $L_1(x)$ then there is no other vertex in $L_1(x)$ adjacent to z, or else $L_2(w)$ contains either C_3 or P_4. Hence we can colour z with the colour of q and the other vertices of F with 0.

Case 1.2. Let w be not the center of S, i.e., w is a vertex of degree 1 in S. Evidently, no vertex of F is adjacent to the center of any star in $L_1(x)$, or else we have Case 1.1. If z has a neighbour in S then all vertices $L_1(x)$ adjacent to y or z belong to S, or else P_4 is contained in $L_2(v)$ for some vertex v adjacent to z or y and belonging to $L_1(x)$ but not belonging to S, or C_3 is contained in the 2-link of the center of S. Thus, we can colour the center of F with 2 and the other vertices of F with 0. In the opposite case, i.e., if no neighbour of y in F has a neighbour in S, then every neighbour of any vertex adjacent to y in F is isolated in $L_1(x)$, or else either C_3 or P_4 is contained in the 2-link of some vertex of S. It is clear that we can colour all vertices adjacent to y in F with the colour 1 and the other vertices of F with 0.

Case 2. There is a vertex y of F adjacent to a vertex w of a certain K_2, called K below, in $L_1(x)$. Let z be a neighbour of y in F. Let t be the vertex of K, $t \neq w$.

Case 2.1. Let z have a neighbour in K. If not vertex of any component in $L_1(x)$ different from K is incident to y or z then one can colour F with at most 3 colours of the set $\{0, 1, 2\}$. Assume that there is a vertex v, $v \neq w$, in $L_1(x)$ which is adjacent to y. If z is adjacent to w then every neighbour of y in F is adjacent to the unique vertex w in $L_1(x)$, or else $L_2(t)$ contains C_3 or P_4, or $L_2(z)$ contains P_4. Therefore, we can colour y with 0, z with the colour of t, and the other vertices of F with the colour which has been assigned to the vertex of degree 1 in F. Assume now that z is adjacent to t, and no neighbour of y in F is adjacent to w. If v defined above is adjacent to z then every neighbour of y or z belonging to $L_1(x)$ but not belonging to K is isolated in $L_1(x)$, or else C_3 or P_4 is contained in the 2-link of t or w or other vertex of $L_1(x)$. Moreover, all the other vertices of F have isolated neighbours in $L_1(x)$. Hence, we can colour z with 0 because we can assume without loss of generality that the vertex w is coloured with 2, and the other vertices of F with 1. Assume that v is not adjacent to z. If z has a neighbour q belonging to $L_1(x)$, $q \neq t$, then q and v induce K_2, or else $L_2(v)$ contains P_4. Moreover, v and w are the unique neighbours of y, and t and q are the unique neighbours of z in $L_1(x)$. Since *Algorithm* A assigns the same colour to v and w, say 2, and the colour 1 to t and q, we can colour y with 1, z with 2 and the other vertices with 0. Now assume that t is the unique neighbour of z belonging to $L_1(x)$. If z is the center of F then we colour it with the colour of w, and the other vertices with 0. If y is the center of F then no neighbour of y in F is adjacent to w

or v, or else $L_2(z)$ contains P_4. If every vertex of F different from y is adjacent to t, then we can colour y with 0 and the other vertices of F with the colour of w. In the opposite case, there is a neighbour u of y in F adjacent to q of $L_1(x)$, $q \neq v, w, t$. Further, q is adjacent to v, or else $L_2(q)$ contains P_4. Evidently, every vertex of F has its neighbours in $\{w, v, t, q\}$ among the vertices of $L_1(x)$. Therefore, we can colour y with 0 and the other vertices of F with the colour of w, or v.

Case 2.2. Let no neighbour of y in F have a neighbour in K. Let us assume that v is the neighbour of z in $L_1(x)$. If v is not isolated in $L_1(x)$ then v is the unique neighbour of z belonging to $L_1(x)$, or else $L_2(w)$ contains C_3 or P_4. This implies that w is the unique neighbour of y in $L_1(x)$, or else $L_2(v)$ contains C_3 or P_4. Hence, we can colour y with the colour of t and the other vertices of F with 0. Assume that all neighbours of z belonging to $L_1(x)$ are isolated in $L_1(x)$. Then we can colour y with 0, z with 1, and the other vertices of F with 0 if z is the center of F, and with 1 in the opposite case.

Finally, every vertex isolated in $L_2(x)$ can be coloured with 0. This completes the proof.

We know no example of a planar graph G belonging to $\mathcal{G}(\mathcal{K}_3)$ whose radius is greater than 2 and whose chromatic number is greater than 3. Let $\mathcal{G}_P(\mathcal{K}_3)$ be the subclass of planar graphs of the class $\mathcal{G}(\mathcal{K}_3)$.

Conjecture 2. $\chi(G) \leqq 3$ for every $G \in \mathcal{G}_P(\mathcal{K}_3)$.

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Souhrn

O GRAFECH SE SPECIÁLNÍMI SPOJOVÝMI GRAFY S CHROMATICKÝM ČÍSLEM ROVNÝM NEJVÝŠE TŘEM

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j-spojový graf vrcholu x v grafu G je podgraf grafu G indukovaný vrcholy, které mají vzdálenost od x v G rovnu j. Článek se zabývá odhady chromatického čísla grafu G pomocí chromatických čísel jeho spojových grafů. Je dána odpověď na otázky L. Szamkołowicze a popsána jistá třída grafů s chromatickým číslem nejvýše 3 a spojové grafy speciálního typu.

Резюме

ГРАФЫ СО СПЕЦИАЛЬНЫМИ ЛИНКОВЫМИ ГРАФАМИ И ХРОМАТИЧЕСКИМ ЧИСЛОМ НЕПРЕВЫШАЮЩИМ 3

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По определению „j— линковый граф" вершины x графа G — это подграф в G индуцированный вершинами, расстояние которых до x равно j. В статье изучаются оценки хроматического числа графа G при помощи хроматических чисел его линковых графов, даётся ответ на вопросы Л. Шамколовича и описывается некоторый класс графов с линковыми графами специального типа и хроматическим числом непрвышающим 3.

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