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PERIODIC SOLUTIONS OF THE EQUATION $x''(t) + g(x(t)) = p(t)$

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1. INTRODUCTION, NOTATION AND MAIN RESULTS

Let R_N denote the N-dimensional Euclidean space with the usual norm. Let I be a compact nonempty interval in R_1. The following notation will be used:

$C^k(I)$ will denote the space of the functions which are k-times continuously differentiable on I (at the end-points of the interval I we mean of course the one-sided derivatives). For the sake of simplicity we use the notation $C^0(I) = C(I)$.

$L_1(I)$ will denote the space of all measurable functions u such that $|u|$ is integrable in the sense of Lebesgue, with the usual norm

$$\|u\| = \int_I |u(t)| \, dt .$$

Definition. Let $p \in L_1(I)$ and let g be a continuous real-valued function defined on the whole real line R_1. A function $x \in C^1(I)$ is said to be *a solution in the sense of Carathéodory* of the equation

$$x''(t) + g(x(t)) = p(t) \tag{1.1}$$

on the interval I if for any $a, t \in I$ it holds

$$x(t) = x(a) + (t - a) x'(a) + \int_a^t (t - s)(p(s) - g(x(s))) \, ds \tag{1.2}$$

or equivalently

$$x(t) = x(a) + (t - a) x'(a) + \int_a^t \left(\int_a^\tau (p(s) - g(x(s))) \, ds \right) d\tau . \tag{1.2'}$$

It is easy to see that x is a solution of (1.1) in the sense of Carathéodory if and only if x' is absolutely continuous and satisfies the equation (1.1) for almost all $t \in I$. Analogously, if $p \in C(I)$ then for the solution x of (1.1) in the sense of Carathéodory we immediately obtain $x \in C^2(I)$ and in this case the function x satisfies the equation (1.1) at any point of the interval I.

For our convenience put $J = \langle 0, 1 \rangle$.

This paper contains the following results:

Theorem 1. *Let a, b, c, d be real numbers and let g be a continuous real-valued function defined on R_1 and such that*

$$\lim_{|\xi| \to \infty} \frac{g(\xi)}{\xi} = +\infty . \tag{1.3}$$

Then the boundary value problem

$$\begin{gathered} x''(t) + g(x(t)) = p(t), \quad t \in J, \\ ax(0) + bx'(0) = 0, \quad cx(1) + dx'(1) = 0 \end{gathered} \tag{1.4}$$

has for each $p \in L_1(J)$ infinite number of distinct solutions in the sense of Carathéodory.

Corollary 1. *Under the assumptions of Theorem 1 the boundary value problem (1.4) has for each $p \in C(J)$ infinite number of distinct solutions.*

Theorem 2. *Let the function g satisfy the assumptions of Theorem 1. Then for any right hand side $p \in L_1(J)$ the periodic problem*

$$\begin{gathered} x''(t) + g(x(t)) = p(t), \\ x(0) = x(1), \quad x'(0) = x'(1) \end{gathered} \tag{1.5}$$

has at least one solution in the sense of Carathéodory.

Corollary 2. *Under the assumptions of Theorem 2 the periodic problem (1.5) has at least one solution for each $p \in C(J)$.*

At first the authors believed the result obtained in Theorem 1 to be new. After the preliminary communication (see [6]) was published ŠTEFAN SCHWABIK informed us that the same result has been obtained by H. EHRMANN (see [4]), who used also essentially the same method of proof. Since the proof of Theorem 2 requires the same auxiliary lemmas we present here also Theorem 1. By the authors' best knowledge the assertion of Theorem 2 has not been published until now under such general assumptions. Several authors considered the case of a special right hand side (see e.g. [2]) or introduced some additional assumptions (see e.g. [5], [7], [8]). For instance, in [7] it is proved that the equation (1.1) with $g(\xi) = 2\xi^3$ has a periodic solution with the period 1 provided the function p satisfies the following conditions:

p is even on R_1,

p', p'' are continuous on R_1,

p has the period 1.

$\int_J p(t)\,dt = 0$.

Note that the proofs of Corollaries can be obtained immediately from Theorems by applying the remark following Definition and thus they are omitted. The proofs of Theorems are based on the shooting method. Moreover, in the proof of Theorem 2 we use a certain fixed point theorem (see Lemma 9) the proof of which follows from the properties of Brouwer's topological degree of mapping (see e.g. [1]).

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2. AUXILIARY LEMMAS

Let g be a function satisfying the assumptions of Theorem 1. For $h \in (0, 1\rangle$ and $\xi \in R_1$ define

$$g_h(\xi) = \frac{1}{2h}\int_{\xi-h}^{\xi+h} g(\eta)\,d\eta\,. \tag{2.1}$$

Obviously the functions g_h satisfy on R_1 locally the Lipschitz condition. Moreover, it is easy to see that there exist nonnegative functions γ_1, γ_2 on R_1 and positive numbers ξ_0, m such that

$$\gamma_1(\xi) \leqq \frac{g_h(\xi)}{\xi} \leqq \gamma_2(\xi)$$

for all $|\xi| > \xi_0$ and $h \in (0, 1\rangle$,

$$\lim_{|\xi|\to\infty} \gamma_1(\xi) = +\infty\,,$$

$$\gamma_1(\xi) \geqq 1 \quad \text{for} \quad |\xi| \geqq \xi_0$$

and

$$|g_h(\xi)| \leqq m$$

for $|\xi| \leqq \xi_0$ and $h \in (0, 1\rangle$.

Denote by $M = M(\gamma_1, \gamma_2, \xi_0, m)$ the set of all functions f defined on R_1 and satisfying the following conditions:

(i) f is locally lipschitzian on R_1;
(ii) $\gamma_1(\xi) \leqq f(\xi)/\xi \leqq \gamma_2(\xi)$ for $|\xi| > \xi_0$;
(iii) $|f(\xi)| \leqq m$ for $|\xi| \leqq \xi_0$.

We shall consider the family of ordinary differential equations of the first order

$$v'(t) = h(t, v(t)), \tag{2.2$_{f,p}$}$$

where

$$v(t) = [x(t), y(t)],$$
$$h(t, v(t)) = [y(t), \ p(t) - f(x(t))],$$
$$f \in M, \quad p \in L_1(J).$$

In accordance with the definition of a solution in the sense of Carathéodory of the equation (1.1) on the interval $I \subset R_1$ we define that a continuous vector valued function $v_{f,p}(t) = [x_{f,p}(t), y_{f,p}(t)]$ is a solution of $(2.2)_{f,p}$ in the sense of Carathéodory on I if for arbitrary $a, t \in I$ it holds

$$x_{f,p}(t) = x_{f,p}(a) + \int_a^t y_{f,p}(s)\,\mathrm{d}s,$$
$$y_{f,p}(t) = y_{f,p}(a) + \int_a^t (p(s) - f(x_{f,p}(s)))\,\mathrm{d}s.$$

For $\xi \in R_1$ and $f \in M$ denote

$$G_f(\xi) = \int_0^\xi f(s)\,\mathrm{d}s. \tag{2.3}$$

Lemma 1. *Let $v_{f,p} = [x_{f,p}, y_{f,p}]$ be a solution of $(2.2)_{f,p}$ (in the sense of Carathéodory) on the interval $I \subset J$. Then for any $a, t \in I$ it holds*

$$y^2_{f,p}(t) + 2G_f(x_{f,p}(t)) = y^2_{f,p}(a) + 2G_f(x_{f,p}(a)) + + 2\int_a^t p(s)\, y_{f,p}(s)\,\mathrm{d}s. \tag{2.4}$$

(The proof is trivial.)

Lemma 2. *Let G_f be a function defined by (2.3). Then there exists a constant $k \geqq 0$ such that the inequality*

$$G_f(\xi) \geqq \frac{\xi^2}{2} - k \tag{2.5}$$

holds for each $\xi \in R_1$ and $f \in M$.

Moreover, for $\xi \in R_1$ it is

$$(2.6) \qquad |G_f(\xi)| \leqq m\xi_0 + |\xi| \int_{\xi_0}^{|\xi|} \gamma_2(s)\,\mathrm{d}s\,.$$

Proof. It follows from the definition of the set M that $f(s) \geqq s$ for $|s| \geqq \xi_0$ and $f \in M$. Hence for $|\xi| \geqq \xi_0$ we have

$$G_f(\xi) = \int_0^{\xi} f(s)\,\mathrm{d}s = \int_0^{\xi_0 \operatorname{sgn}\xi} f(s)\,\mathrm{d}s + \int_{\xi_0 \operatorname{sgn}\xi}^{\xi} f(s)\,\mathrm{d}s \geqq \frac{\xi^2}{2} - \frac{\xi_0^2}{2} + \int_0^{\xi_0 \operatorname{sgn}\xi} f(s)\,\mathrm{d}s$$

which implies (2.5) with

$$k = \frac{\xi_0^2}{2} + m\xi_0\,.$$

The proof of the other part of the assertion is analogous.

Under the assumption (i) in the definition of the set M there exists for an arbitrary initial value $u \in R_2$ a unique solution $v_{f,p,u}(t)$ (in the sense of Carathéodory) of the equation $(2.2)_{f \in M,p}$ satisfying $v_{f,p,u}(0) = u$ which is defined on the maximal interval $J_{f,p,u} \subset J$ with $0 \in J_{f,p,u}$ (see e.g. [3, Chapter II]). For $u \in R_2$ and $t \in J_{f,p,u}$ put

$$(2.7) \qquad V_{f,p}(t, u) = v_{f,p,u}(t)\,.$$

Lemma 3. *Let $p \in L_1(J)$ and $f \in M$. Then the function $V_{f,p}$ defined by the relation (2.7) is defined on $J \times R_2$ and is a continuous mapping from $J \times R_2$ into R_2.*

Proof. First let us show that the mapping $V_{f,p}$ is defined on $J \times R_2$ (i.e., $J_{f,p,u} = J$ for any $u \in R_2$, $f \in M$ and $p \in L_1(J)$). To this end it is sufficient to prove an appropriate apriori estimate for weak solutions of the equation $(2.2)_{f,p}$.

Let $v_{f,p} = [x_{f,p}, y_{f,p}]$ be a weak solution of $(2.2)_{f,p}$ on an interval $I \subset J$, $0 \in I$, which satisfies the initial condition $v_{f,p,u}(0) = u = [u_1, u_2]$. In virtue of Lemma 1 for any $t \in I$ it holds

$$(2.8) \qquad y_{f,p}^2(t) + 2G_f(x_{f,p}(t)) = u_2^2 + 2G_f(u_1) + 2\int_0^t p(s)\,y_{f,p}(s)\,\mathrm{d}s\,.$$

Let us denote

$$(2.9) \qquad z_{f,p}(t) = \sup\{|y_{f,p}(s)|;\ s \in \langle 0, t\rangle\}\,, \quad t \in I\,.$$

Then we obtain from (2.8) and from Lemma 2

$$y_{f,p}^2(t) \leqq (u_2^2 + 2G_f(u_1) + 2k) + 2z_{f,p}(t)\left(\int_0^t |p(s)|\,\mathrm{d}s\right)$$

for $t \in I$. This implies by an elementary calculation the estimate

$$(2.10) \qquad y_{f,p}^2(t) \leqq z_{f,p}^2(t) \leqq$$
$$\leqq (1-\varepsilon_0)^{-1}(u_2^2 + 2G_f(u_1) + 2k) + \varepsilon_0(1-\varepsilon_0)^{-1}\left(\int_0^t |p(s)|\,\mathrm{d}s\right)^2$$

for $t \in I$ and $\varepsilon_0 \in (0, 1)$.

From (2.8), (2.10) and from Lemma 2 one can easily obtain also the estimate

$$(2.11) \qquad x_{f,p}^2(t) \leqq$$
$$\leqq (1 + (1-\varepsilon_0)^{-1})(u_2^2 + 2G_f(u_1) + 2k) + (1 + \varepsilon_0^{-1}(1-\varepsilon_0)^{-1})\left(\int_0^t |p(s)|\,\mathrm{d}s\right)^2.$$

It follows from (2.10), (2.11) and [3, Chapter II] that the mapping $V_{f,p}$ is defined on $J \times R_2$. Continuity of $V_{f,p}$ follows from the assumption (i) in the definition of the set M.

Now let us denote (for $f \in M$ and $p \in L_1(J)$) by $d_{f,p}$ a function defined on R_2 by

$$(2.12) \qquad d_{f,p}(u) = \inf\{|V_{f,p}(t, u)|;\ t \in J\}.$$

Then it holds

Lemma 4. $\lim_{|u|\to\infty} d_{f,p}(u) = +\infty$ *uniformly with respect to* $f \in M$ *and* p *from any boudend subset of* $L_1(J)$, *i.e., for any bounded set* $U \subset L_1(J)$ *and arbitrary* $K > 0$ *there exists* $\tau > 0$ *such that* $d_{f,p}(u) > K$ *for each* $|u| \geqq \tau$, $f \in M$ *and* $p \in U$.

Proof. Let $U = \{p \in L_1(J);\ \|p\| \leqq c\}$ and let $v_{f,p} = [x_{f,p}, y_{f,p}]$ be a weak solution of $(2.2)_{f,p}$ on the interval J with $v_{f,p}(0) = u = [u_1, u_2]$. Then from (2.8) we obtain

$$(2.13) \qquad y_{f,p}^2(t) + 2\,G_f(x_{f,p}(t)) \geqq u_2^2 + 2\,G_f(u_1) - 2\,z_{f,p}(t)\int_0^t |p(s)|\,\mathrm{d}s$$

for $t \in J$, where the function $z_{f,p}$ is defined by (2.9). Let $\varepsilon_0 \in (0, 1)$ be fixed. Then from (2.10), (2.13) we have for any $\varepsilon_1 > 0$ and $f \in M$

$$(2.14) \qquad y_{f,p}^2(t) + 2\,G_f(x_{f,p}(t)) \geqq (1 - \varepsilon_1(1-\varepsilon_0)^{-1})(u_2^2 + 2\,G_f(u_1)) - K_2,$$

where

$$(2.15) \qquad K_2 = 2\,\varepsilon_1(1-\varepsilon_0)^{-1}\,k + (\varepsilon_1\varepsilon_0^{-1}(1-\varepsilon_0)^{-1} + \varepsilon_1^{-1})\,c^2.$$

Let $\varepsilon_1 > 0$ be fixed and such that

$$K_1 = (1 - \varepsilon_1(1 - \varepsilon_0)^{-1}) > 0 .$$

Further let us denote by δ_f the function defined on R_2 by

$$\delta_f([x, y]) = y^2 + 2\,G_f(x)\,, \quad [x, y] \in R_2\,.$$

The relation (2.14) implies

$$\delta_f(V_{f,p}(t, u)) \geqq K_1\,\delta_f(u) - K_2 \geqq K_1 u_1^2 + K_1 u_2^2 - K_1 k - K_2 \tag{2.16}$$

for $u \in R_2$, $t \in J$, $f \in M$, $p \in U$.

Suppose that there exist $K > 0$, $U \subset L_1(J)$ a bounded set, $f_n \in M$, $p_n \in U$, $t_n \in J$, $u_n \in R_2$ such that $|u_n| \to \infty$ and

$$d_{f_n,p_n}(u_n) \leqq |V_{f_n,p_n}(t_n, u_n)| < K\,.$$

Since δ_f is continuous on R_2 we conclude that the sequence $\{\delta_{f_n}(V_{f_n,p_n}(t_n, u_n))\}$ is bounded which is a contradiction with (2.16).

Remark 1. Put

$$D_{f,p}(u) = \sup\,\{|V_{f,p}(t, u)|;\ t \in J\}$$

for $f \in M$, $p \in L_1(J)$ and $u \in R_2$. Then from (2.10), (2.11) and Lemma 2 the following inequality immediately follows:

$$|D_{f,p}(u)|^2 \leqq (1 + 2(1 - \varepsilon_0)^{-1}\left(u_2^2 + 2m\xi_0 + 2|u_1|\int_{\xi_0}^{|u_1|} \gamma_2(s)\,ds + 2k\right) +$$
$$+ (1 + 2\varepsilon_0^{-1}(1 - \varepsilon)^{-1})\left(\int_0^1 |p(s)|\,ds\right)^2 .$$

As usual, denote for an arbitrary nonzero complex number $z \in \mathbf{C}$

$$\operatorname{Arg} z = \left\{x \in R_1;\ e^{-ix} = \frac{z}{|z|}\right\}.$$

In the sequel we need the following.

Lemma 5. *Let X be a topological space and let F be a continuous complex valued function defined on $J \times X$ satisfying $F(t, u) \neq 0$ for an arbitrary $[t, u] \in J \times X$. Then for any $u \in X$ there exists a continuous real valued function ψ_u on J such that*

$$\psi_u(t) \in \operatorname{Arg} F(t, u)\,, \quad t \in J\,.$$

Moreover, for an arbitrary $u \in X$ the value

$$\Psi(u) = \omega_u(1) - \omega_u(0)$$

is independent of the choice of a continuous function ω_u on J with $\omega_u(t) \in \operatorname{Arg} F(t, u)$, $t \in J$, and the mapping Ψ is continuous on X.

Proof. The assertions of Lemma except possibly the continuity of the function Ψ are well-known from the classical complex analysis. Now we shall prove the continuity of Ψ. Let $u_0 \in X$ be fixed and let $a = \min \{|F(t, u_0)|;\ t \in J\}$. Let $\varepsilon > 0$ be arbitrary and choose $\eta \in (0, a)$ such that

$$2 \arcsin \frac{\eta}{a} < \varepsilon .$$

Then there exists a neighborhood $U(u_0, \eta) \subset X$ of the point u_0 such that $|F(t, u) - F(t, u_0)| < \eta$ for each $t \in J$ and $u \in U(u_0, \eta)$ (this is true in virtue of the continuity of F and the compactness of J). For an arbitrary $u \in U(u_0, \eta)$ there exists a continuous function $\psi_u(t) \in \operatorname{Arg} F(t, u)$, $t \in J$ such that

$$|\psi_u(0) - \psi_{u_0}(0)| < \arcsin \frac{\eta}{a}, \tag{2.17}$$

where ψ_{u_0} is an arbitrary continuous function on J such that $\psi_{u_0}(t) \in \operatorname{Arg} F(t, u_0)$, $t \in J$. We shall prove that

$$|\psi_u(t) - \psi_{u_0}(t)| < \arcsin \frac{\eta}{a}$$

for each $t \in J$ and $u \in U(u_0, \eta)$. Suppose that there exist $t_1 \in J$ and $u_1 \in U(u_0, \eta)$ such that $|\psi_{u_1}(t_1) - \psi_{u_0}(t_1)| \geqq \arcsin (\eta/a)$. With respect to (2.17) there exists $t_0 \in (0, t_1)$ such that $|\psi_{u_1}(t_0) - \psi_{u_0}(t_0)| = \arcsin (\eta/a)$ and thus

$$\begin{aligned} &|F(t_0, u_1) - F(t_0, u_0)| = \\ &= \big||F(t_0, u_1)| - |F(t_0, u_0)| \exp (-i(\psi_{u_0}(t_0) - \psi_{u_1}(t_0)))\big| \geqq \\ &\geqq |F(t_0, u_0)|\ |\sin (\psi_{u_0}(t_0) - \psi_{u_1}(t_0)| \geqq \frac{\eta}{a} a = \eta \end{aligned}$$

which is a contradiction. So we have

$$|\Psi(u) - \Psi(u_0)| = |\psi_u(1) - \psi_u(0) + \psi_{u_0}(1) - \psi_{u_0}(0)| < 2 \arcsin \frac{\eta}{a} < \varepsilon$$

for an arbitrary $u \in U(u_0, \eta)$.

According to Lemma 4 there exists $r_0 > 0$ such that $|V_{f,p}(t, u)| \geqq 1$ for arbitrary $t \in J$, $|u| \geqq r_0, f \in M$ and $\|p\| \leqq \bar{q}$. Let us denote

$$X_{r_0} = \{u \in R_2;\ |u| \geqq r_0\} .$$

It follows from Lemma 5 that for any $f \in M$, $p \in L_1(J)$, $\|p\| \leqq \bar{q}$ and $u \in X_{r_0}$ there exists a continuous function $\varphi_{f,p,u}$ on the interval J such that

$$V_{f,p}(t, u) = |V_{f,p}(t, u)| \,[\cos \varphi_{f,p,u}(t),\ -\sin \varphi_{f,p,u}(t)] \tag{2.18}$$

and the real valued function

$$\Phi_{f,p}(u) = \varphi_{f,p,u}(1) - \varphi_{f,p,u}(0) \tag{2.19}$$

is continuous on X_{r_0}.

Lemma 6. *Let $p \in L_1(J)$. Then*

$$\lim_{|u| \to \infty} \Phi_{f,p}(u) = +\infty$$

uniformly with respect to $f \in M$.

Proof. Let $p_n \in C(J)$ be such a sequence that $\lim_{n \to \infty} \|p_n - p\| = 0$. Thus

$$\sup_{n=1,2,\ldots} \int_0^1 |p_n(s)| \,\mathrm{d}s = \bar{q} < +\infty .$$

Let $u \in X_{r_0}$ and let $\varphi_{f,p_n,u}$ be a continuous real valued function defined on J such that (2.18) holds. Then it is easy to see that $\varphi_{f,p_n,u}$ is continuously differentiable on J and satisfies the differential equation

$$\varphi'_{f,p_n,u}(t) = q_{f,p_n,u}(t, \varphi_{f,p_n,u}(t)) - r^{-1}_{f,p_n,u}(t)\, p_n(t) \cos \varphi_{f,p_n,u}(t) , \tag{2.20}$$

where $q_{f,p_n,u}(t, \varphi) = \sin^2 \varphi + r^{-1}_{f,p_n,u}(t) f(r_{f,p_n,u}(t) \cos \varphi) \cos \varphi$ and $r_{f,p_n,u}(t) = |V_{f,p_n}(t, u)|$ (hence $r_{f,p_n,u}(t) \geqq d_{f,p_n}(u)$), $(t \in J,\ \varphi \in R_1)$. Further, let us denote by $\varkappa_{f,p_n,u}$ the function defined by

$$\varkappa_{f,p_n,u}(\psi) = \inf \{s^{-1} f(s \cos \psi) \cos \psi;\ s \geqq d_{f,p_n}(u)\} .$$

An easy calculation shows that also the function $\varkappa_{f,p_n,u}$ is continuous. For $d_{f,p_n}(u) \,.\, |\cos \psi| \geqq \xi_0$ we have by the above

$$\varkappa_{f,p_n,u}(\psi) \geqq \gamma_1(d_{f,p_n}(u)\, |\cos \psi|) \cos^2 \psi .$$

If $d_{f,p_n}(u)\, |\cos \psi| \leqq \xi_0$ then for $s \in \langle d_{f,p_n}(u), \cos \psi|^{-1}\, \xi_0 \rangle$ it holds $|s^{-1} f(s \cos \psi) \,.\, \cos \psi| \leqq m\, d^{-1}_{f,p_n}(u)$, where $m > 0$ is a constant from the condition (iii) in the definition of the set M. Since for $s \geqq \xi_0 |\cos \psi|^{-1}$ it is $s^{-1} f(s \cos \psi) \cos \psi \geqq \cos^2 \psi \geqq 0$ we conclude that for $d_{f,p_n}(u)\, |\cos \psi| \leqq \xi_0$ it holds $\varkappa_{f,p_n,u}(\psi) \geqq -m\, d^{-1}_{f,p_n}(u)$.

Thus we obtain the following relations:

a) For $d_{f,p_n}(u)\,|\cos\psi| \leqq \xi_0$ we have

$$\sin^2\psi + \varkappa_{f,p_n,u}(\psi) \geqq 1 - \xi_0^2\, \mathrm{d}_{f,p_n}^{-2}(u) - m\, d_{f,p_n}^{-1}(u)\,;$$

b) For $d_{f,p_n}(u)\,|\cos\psi| \geqq \xi_0$ we have

$$\sin^2\psi + \varkappa_{f,p_n,u}(\psi) \geqq 1\,.$$

In other words, there exist $r_1 \geqq r_0$ and $c > 0$ such that

$$\sin^2\psi + \varkappa_{f,p_n,u}(\psi) - c \geqq c > 0$$

for each $|u| \geqq r_1$, $\psi \in R_1$, $f \in M$ and $n = 1, 2, \ldots$. Let us compare the differential equation (2.20) with the equation

$$\psi'_{f,p_n,u}(t) = \sin^2\psi_{f,p_n,u}(t) + \varkappa_{f,p_n,u}(\psi_{f,p_n,u}(t)) - c - d_{f,p_n}^{-1}(u)\,|p_n(t)| \tag{2.21}$$

with the initial condition $\psi_{f,p_n,u}(0) = \varphi_{f,p_n,u}(0)$. An elementary comparison theorem from the theory of differential equations yields

$$\varphi_{f,p_n,u}(1) - \varphi_{f,p_n,u}(0) \geqq \psi_{f,p_n,u}(1) - \psi_{f,p_n,u}(0)\,. \tag{2.22}$$

However, from (2.21) we have the simple relation

$$\int_{\psi_{f,p_n,u}(0)}^{\psi_{f,p_n,u}(1)} \frac{\mathrm{d}\psi}{\sin^2\psi + \varkappa_{f,p_n,u}(\psi) - c} =$$

$$= 1 - \int_0^1 \frac{d_{f,p_n}^{-1}(u)\,|p_n(t)|}{\sin^2\psi_{f,p_n,u}(t) + \varkappa_{f,p_n,u}(\psi_{f,p_n,u}(t)) - c}\,\mathrm{d}t$$

which implies

$$\int_{\psi_{f,p_n,u}(0)}^{\psi_{f,p_n,u}(1)} \frac{\mathrm{d}\psi}{\sin^2\psi + \varkappa_{f,p_n,u}(\psi) - c} \geqq 1 - c^{-1}\, d_{f,p_n}^{-1}(u) \left(\int_0^1 |p_n(t)|\,\mathrm{d}t\right) \geqq \tag{2.23}$$

$$\geqq 1 - c^{-1}\, d_{f,p_n}^{-1}(u)\,\bar{q}$$

for $|u| \geqq r_1, f \in M$ and $n = 1, 2, \ldots$. Since $\varkappa_{f,p_n,u}$ and sin are 2π – periodic functions, it follows from (2.23) that

$$\psi_{f,p_n,u}(1) - \psi_{f,p_n,u}(0) \geqq 2\pi(1 - c^{-1}\, d_{f,p_n}^{-1}(u)\,\bar{q})\,. \tag{2.24}$$

$$\cdot\left(\int_0^{2\pi} \frac{\mathrm{d}\psi}{\sin^2\psi + \varkappa_{f,p_n,u}(\psi) - c}\right)^{-1} - 2\pi$$

for arbitrary $f \in M$, $|u| \geqq r_1$ and $n = 1, 2, \ldots$. Hence we shall show that for any $\varepsilon > 0$ there exists $r_2 > r_1$ such that

$$\int_0^{2\pi} \frac{\mathrm{d}\psi}{\sin^2 \psi + \varkappa_{f,p_n,u}(\psi) - c} < \varepsilon \tag{2.25}$$

provided $|u| \geqq r_2$, $f \in M$ and $n = 1, 2, \ldots$. Let $\varepsilon > 0$ be arbitrary but fixed. There exist $\eta > 0$ and $\xi_1 > \xi_0$ such that

$$\int_{\{\psi \in (0,2\pi);\, |\cos\psi| < \tau\}} \frac{\mathrm{d}\psi}{\sin^2 \psi + \varkappa_{f,p_n,u}(\psi) - c} \leqq \int_{\{\psi \in (0,2\pi);\, |\cos\psi| < \tau\}} \frac{1}{c} \,\mathrm{d}\psi < \frac{\varepsilon}{2} \tag{2.26}$$

for $|u| \geqq r_1$, $f \in M$, $n = 1, 2, \ldots$, and

$$\eta^2 \gamma_1(\xi_1) - c \geqq \frac{4\pi}{\varepsilon} \tag{2.27}$$

Thus for $\psi \in (0, 2\pi)$ such that $|\cos \psi| \geqq \eta$, for $|u| \geqq r_1$ with $d_{f,p_n}(u) \geqq \eta^{-1}\xi_1$ and for $n = 1, 2, \ldots$ we have $\sin^2 \psi + \varkappa_{f,p_n,u}(\psi) - c \geqq \eta^2 \gamma_1(\xi_1) - c$ and hence

$$\int_{\{\psi \in (0,2\pi);\, |\cos\psi| \geqq \eta\}} \frac{\mathrm{d}\psi}{\sin^2 \psi + \varkappa_{f,p_n,u}(\psi) - c} \leqq \frac{2\pi}{\eta^2 \gamma_1(\xi_1) - c}. \tag{2.28}$$

Let $r_2 \geqq r_1$ be such that $d_{f,p_n}(u) \geqq \eta^{-1}\xi_1$ for $n = 1, 2, \ldots$, $f \in M$ and $|u| \geqq r_2$. The relations (2.25)–(2.28) imply

$$\int_0^{2\pi} \frac{\mathrm{d}\psi}{\sin^2 \psi + \varkappa_{f,p_n,u}(\psi) - c} < \varepsilon$$

for arbitrary $n = 1, 2, \ldots$, $f \in M$ and $|u| \geqq r_2$, which together with (2.24) and (2.22) implies:

For an arbitrary $K > 0$ there exists $r_2 > 0$ such that

$$\Phi_{f,p_n}(u) > K \quad \text{for} \quad f \in M\,, \quad n = 1, 2, \ldots, \quad \text{and} \quad |u| \geqq r_2\,.$$

Now it is easy to see that to complete the proof it is sufficient to show that $\lim_{n\to\infty} \Phi_{f,p_n}(u) = \Phi_{f,p}(u)$ for each $|u| \geqq r_0$ and $f \in M$. This fact follows from Lemma 5 for if $|u| \geqq r_0$, $f \in M$ then the mapping

$$[t, p] \mapsto V_{f,p}(t, u)$$

is continuous and non-vanishing on $J \times L_1(J)$ and thus (with respect to the assertion of Lemma 5) the mapping

$$p \mapsto \Phi_{f,p}(u)$$

is continuous on $L_1(J)$.

Lemma 7. *Let $\varrho > r_0$. Denote*

$$K(\varrho) = \sup \{D_{f,p}(u);\ f \in M,\ p \in L_1(J),\ \|p\|_{L_1(J)} < \varrho,\ u \in R_2,\ r_0 \leqq |u| \leqq \varrho\}$$

and $\bar{\gamma}_2(\varrho) = \max \{\gamma_2(s);\ |s| \leqq K(\varrho)\}$. *Then*

$$\Phi_{f,p}(u) \leqq 1 + \bar{\gamma}_2(\varrho) + m + \int_0^1 |p(s)| \,\mathrm{d}s$$

provided $f \in M,\ r_0 \leqq |u| \leqq \varrho,\ \int_0^1 |p(s)| \,\mathrm{d}s < \varrho$.

Proof. Let $p_n \in C(J)$ be such that $\lim_{n\to\infty} \|p_n - p\| = 0$. Let the notation introduced in (2.20) be observed. Then

$$q_{f,p_n,u}(t, \varphi) - r^{-1}_{f,p_n,u}(t)\, p_n(t) \cos \varphi \leqq$$
$$\leqq 1 + \gamma_2(d_{f,p_n}(u)\, |\cos \varphi|) + \frac{m}{d_{f,p_n}(u)} + \frac{|p_n(t)|}{d_{f,p_n}(u)} \leqq$$
$$\leqq 1 + \bar{\gamma}_2(\varrho) + m + |p_n(t)|\,.$$

Comparing the differential equation (2.20) with the initial problem

$$\zeta'_{f,p_n,u}(t) = 1 + \bar{\gamma}_2(\varrho) + m + |p_n(t)|$$
$$\zeta_{f,p_n,u}(0) = \varphi_{f,pn,u}(0)$$

we conclude

$$1 + \bar{\gamma}_2(\varrho) + m + \int_0^1 |p_n(t)| \,\mathrm{d}t = \zeta_{f,p_n,u}(1) - \zeta_{f,p_n,u}(0) \geqq$$
$$\geqq \varphi_{f,p_n,u}(1) - \varphi_{f,p_n,u}(0) = \Phi_{f,p_n}(u)\,.$$

Letting n tend to infinity we obtain our assertion.

3. BOUNDARY VALUE PROBLEM

Let $p \in L_1(J)$ be fixed. Let a, b, c, d be real numbers and consider the boundary value problem

$$(3.1)_f \qquad x''(t) + f(x(t)) = p(t)\,,$$
$$ax(0) + bx'(0) = 0\,, \quad cx(1) + dx'(1) = 0\,,$$

where $f \in M$.

We shall suppose that both vectors $\omega_1 = [a, b]$, $\omega_2 = [c, d]$ are nonzero (in the other cases the existence of $(3.1)_f$ a solution of in the sense of Carathéodory is obvious). Let $\psi_1, \psi_2 \in R_1$ satisfy

$$\omega_1 = |\omega_1| [\cos \psi_1, -\sin \psi_1], \quad \omega_2 = |\omega_2| [\cos \psi_2, -\sin \psi_2].$$

It is easy to see that the set of all solutions (in the sense of Carathéodory) of the boundary value problem $(3.1)_f$ equals to the set of all solutions $x(t)$ of the initial problem

$$(3.2)_f \qquad x''(t) + f(x(t)) = p(t),$$
$$[x(0), x'(0)] = u = |u| [\cos \varphi_1, -\sin \varphi_1]$$

such that ($V_{f,p}(t, u)$ is defined by the relation (2.7))

$$V_{f,p}(1, u) = |V_{f,p}(1, u)| [\cos \varphi_2, -\sin \varphi_2],$$

where φ_1, φ_2 satisfy the relations

$$\varphi_1 = \psi_1 + (2n_1 + 1)\frac{\pi}{2}, \quad \varphi_2 = \psi_2 + (2n_2 + 1)\frac{\pi}{2}$$

for some integers n_1, n_2.

Let r_0 be a positive number defined above Lemma 6.

Lemma 8. *There exists a positive integer n_0 with the following property: For each $n \geqq n_0$ and $f \in M$ there exists $s_{n,f} \geqq r_0$ such that if $f \in M$ then $(3.1)_f$ has a solution $x_{n,f,p}$ (in the sense of Carathéodory) with*

$$[x_{n,f,p}(0), x'_{n,f,p}(0)] = s_{n,f}\left[\cos\left(\psi_1 + \frac{\pi}{2}\right), -\sin\left(\psi_1 + \frac{\pi}{2}\right)\right] = u_{s_n,f}$$

and it is $\Phi_{f,p}(u_{s_n,f}) = \psi_1 - \psi_2 + n\pi$, $n \geqq n_0$, $f \in M$.

Proof. Set $u_s = s[\cos(\psi_1 + \frac{1}{2}\pi), -\sin(\psi_1 + \frac{1}{2}\pi)]$. The function $s \mapsto \Phi_{f,p}(u_s)$ is defined and continuous on $\langle r_0, \infty)$ and it is

$$V_{f,p}(1, u_s) = |V_{f,p}(1, u_s)| \left[\cos\left(\psi_1 + \frac{\pi}{2} + \Phi_{f,p}(u_s)\right), -\sin\left(\psi_1 + \frac{\pi}{2} + \Phi_{f,p}(u_s)\right)\right].$$

This means (with respect to the note before Lemma) that any u_s for which $\Phi_{f,p}(u_s) = \psi_1 - \psi_2 + n\pi$ (n is an integer) defines a solution in the sense of Carathéodory of the boundary value problem $(2.29)_f$.

Let n_0 be such an integer that $\Phi_{f,p}(u_{r_0}) < \psi_1 - \psi_2 + n_0\pi$ for each $f \in M$ (this is possible according to Lemma 7). Since $\lim_{s \to \infty} \Phi_{f,p}(u_s) = \infty$ uniformly with respect to $f \in M$, for each $n \geqq n_0$ and $f \in M$ there exists $u_{s_n,f}$ such that

$$\Phi_{f,p}(u_{s_n,f}) = \psi_1 - \psi_2 + n\pi.$$

Proof of Theorem 1. Let n_0 be the positive integer the existence of which is guaranteed by Lemma 8. Let $n \geqq n_0$ be fixed. Then for each g_h defined by (2.1) there exists a solution $x_{n,g_h,p}$ of $(2.29)_{g_h}$ satisfying

$$[x_{n,g_h,p}(0),\ x'_{n,g_h,p}(0)] = u_{s_n,g_h}\,,$$

$$\Phi_{g_h,p}(u_{s_n,g_h}) = \psi_1 - \psi_2 + n\pi\,.$$

With respect to the assertion of Lemma 7 the set $\{u_{s_n,g_h}\}_{h\in(0,1)}$ is bounded. Now from Remark 1 it follows that $\{x_{n,g_h,p}\}_{h\in(0,1)}$ is a bounded set in the space $C^1(J)$. Thus there exists a sequence $h_k \searrow 0$ such that

$$x_k = x_{n,g_{h_k},p} \to x \quad \text{in} \quad C(J)$$

and

$$u_k = u_{s_n,g_{h_k}} \to u \quad \text{in} \quad R_2\,.$$

Moreover, the functions $s \mapsto g_{h_k}(x_k(s))$ are bounded independently of $k = 1, 2, \ldots$ and $\lim_{k\to\infty} g_{h_k}(x_k(s)) = g(x(s))$ for each $s \in J$. If in the relation

$$x_k(t) = [u_1]_k + t[u_2]_k + \int_0^t (t - s)\,(p(s) - g_{h_k}(x_k(s)))\,\mathrm{d}s$$

k tends to infinity we obtain (using Lebesgue's Dominated Convergence Theorem) that $x \in C^1(J)$ and this verifies (1.2) with $a = 0$ and $I = J$.

The proof of Theorem 1 is complete.

4. PERIODIC PROBLEM

If $K \subset R_N$ is a compact set denote by $G_\infty(K)$ the unbounded component of the et $R_N \setminus K$.

Lemma 9. *Let $F : R_N \to R_N$ be a continuous mapping defined on R_N. Let $\mathfrak{M}$ be a compact subset of R_N. Suppose*

a) $0 \notin \mathfrak{M}$,

b) $0 \notin G_\infty(\mathfrak{M})$,

c) *for each $u \in \mathfrak{M}$ it is*

$$F(u) = \frac{(F(u), u)}{(u, u)}\,u\,,$$

where $(.,.)$ denotes the usual inner product in R_N.

Then there exists at least one point $x_0 \in R_N$ such that $F(x_0) = x_0$.
(Note that from the assumption b) it immediately follows that $\mathfrak{M}$ is nonempty.)

Proof. Let us denote

$$\mathfrak{M}_1 = \left\{u \in \mathfrak{M}\,;\ F(u) = \frac{(F(u), u)}{(u, u)}\,u\,,\ \ (F(u), u) \geqq (u, u)\right\},$$

$$\mathfrak{M}_2 = \left\{u \in \mathfrak{M}\,;\ F(u) = \frac{(F(u), u)}{(u, u)}\,u\,,\ \ (F(u), u) \leqq (u, u)\right\}.$$

If $\mathfrak{M}_1 \cap \mathfrak{M}_2 \neq \emptyset$ then F has clearly a fixed point in $\mathfrak{M}$. Hence let us suppose that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$.

First let us show that $0 \notin G_\infty(\mathfrak{M}_j)$ at least for one $j = 1, 2$. Indeed, let U be such a component of the set $R_N \setminus \mathfrak{M}$ which contains 0. Since $\partial U \subset \mathfrak{M}$, $\partial U = H_1 \cup H_2$, where $H_j = \partial U \cap \mathfrak{M}_j$, $j = 1, 2$. If $0 \in G_\infty(\mathfrak{M}_j)$ $(j = 1, 2)$ then $0 \in G_\infty(H_j)$ $(j = 1, 2)$ and hence there exist continuous functions v_j on J with values in R_N such that $v_j(0) = 0$, $v_j(t) \notin H_j$ for $t \in J$ and $|v_j(1)| > \sup\{|u|;\ u \in \mathfrak{M}\}$ $(j = 1, 2)$ and, moreover, $v_1(1) = v_2(1) = v$. Let us define a function h from $J \times \partial U$ into $R_N \setminus \{0\}$ by

$$h(t, u) = \begin{cases} u - v_1(t) & t \in J\,,\quad u \in H_1 \\ u - v_2(t) & t \in J\,,\quad u \in H_2\,. \end{cases}$$

Then h is a homotopy in $R_N \setminus \{0\}$ between the identity mapping E and the mapping $E - v$ and hence by the well-known theorem from the theory of Brouwer's topological degree (see e.g. [1]) it is $\mathrm{d}(E, U, 0) = \mathrm{d}(E - v, U, 0) = 0$ which is a contradiction with the assumption $0 \notin G_\infty(\mathfrak{M})$. Let us suppose for the sake fo brevity that $0 \notin G_\infty(\mathfrak{M}_1)$ (the other case may be proved in the same way). Let V be a component of the set $R_N \setminus \mathfrak{M}_1$ for which $0 \in V$. Let us define the mapping h_1 from $J \times \partial V$ into $R_N \setminus \{0\}$ by

$$h_1(t, u) = tu - (1 - t)(u - F(u))\,.$$

It is easy to see that h_1 is a homotopy in $R_N \setminus \{0\}$ between the mappings E and $F - E$ and hence $\mathrm{d}(F - E, V, 0) = \mathrm{d}(E, V, 0) = 1$. Since Brouwer's degree of the mapping $F - E$ is nonzero there exists $u_0 \in V$ such that $F(u_0) = u_0$.

Proof of Theorem 2. Let g satisfy locally the Lipschitz condition. Let us denote by F the mapping from R_2 into R_2 defined by $F(u) = V_{g,p}(1, u)$. The mapping F is continuous on R_2. Since $\lim\limits_{|u| \to \infty} \Phi_{g,p}(u) = +\infty$ there exists $\varrho > r_0$ such that

$$\inf\{\Phi_{g,p}(u);\ |u| = \varrho\} - \sup\{\Phi_{g,p}(u);\ |u| = r_0\} \geqq 2\pi\,.$$

Further let us set

$$\mathfrak{M} = \left\{u \in R_2;\ r_0 \leqq |u| \leqq \varrho \ \text{ and } \ F(u) = \frac{(F(u), u)}{(u, u)}\,u\right\}.$$

It is clear that $\mathfrak{M}$ is a compact subset of R_2 and the assumptions a) and c) of Lemma 9 are satisfied. Let us show that $0 \notin G_\infty(\mathfrak{M})$. Let v be an arbitrary continuous function from J into R_2 such that $v(0) = 0$ and $|v(1)| > \varrho$. Let us set $t_1 = \sup\{t \in J; |v(t)| = r_0\}$, $t_2 = \inf\{t \in J;\ t > t_1,\ |v(t)| = \varrho\}$. Then for $t \in \langle t_1, t_2 \rangle$ it is $r_0 \leqq |v(t)| \leqq \varrho$. The function $t \mapsto \Phi_{g,p}(v(t))$ is continuous on $\langle t_1, t_2 \rangle$, $\Phi_{g,p}(v(t_2)) - \Phi_{g,p}(v(t_1)) \geqq 2\pi$ and hence there exists $t_0 \in \langle t_1, t_2 \rangle$ and an integer n_0 such that $\Phi_{g,p}(v(t_0)) = 2\pi n_0$. If $v(t_0) = |v(t_0)| [\cos \varphi_0, -\sin \varphi_0]$; then $F(v(t_0)) = |F(v(t_0))| \cdot [\cos(\varphi_0 - \Phi_{g,p}(v(t_0))), -\sin(\varphi_0 - \Phi_{g,p}(v(t_0)))] = |F(v(t_0))|\, |v(t_0)|^{-1}\, v(t_0)$ and so $v(t_0) \in \mathfrak{M}$.

Thus every continuous curve connecting 0 and a point outside of $\mathfrak{M}$ with the norm at least ϱ intersects the set $\mathfrak{M}$. This fact implies $0 \notin G_\infty(\mathfrak{M})$.

According to the assertion of Lemma 9, the mapping F has at least one fixed point which proves Theorem 2 in the case of locally lipschitzian function g. If g is a continuous function we can use the same procedure as in the end of Section 3.

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