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ON CERTAIN PROPERTIES OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

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This paper deals with properties of the expression $\sum x_i(t)\, a_i$ depending on the non-homogenous terms $f_i(t)$, where $x_i(t)$ is a solution of the system of differential equations (1). Primarily, the derivative $\sum \dot{x}_i(t)\, a_i$ is investigated.

This paper was initiated by a generalization of Lemma 1 and 3 given in paper [1]. Lemma 3 in [1] is equivalent with the second part of Corollary 2. Lemma 1 is a particular case of Theorem 3 and remark 3 (if $\alpha_i = \alpha > 0$ and $p_{ij} = p_i \geqq 1$). Theorem 1, which is in fact a generalization of Lemma 3, is equivalent with the statement that each component $x_i(t)$ of a solution of the system (1) is also a solution of the difference-differential equation (2). Having proved this theorem, Theorem 3 can be proved in a simpler fashion.

We shall use the following notation: Let A be a constant matrix of complex numbers of the type $n \times n$, a, b vectors with n components. The expression (a, b) may have the meaning either of $\sum_{i=1}^{n} a_i b_i$ or of $\sum_{i=1}^{n} a_i \bar{b}_i$, where a_i, b_i are components of vectors a, b respectively, and $\bar{b}_i$ are complex-conjugates to b_i. In the first case A^* will denote the matrix symmetric to A and in the second case the matrix which is Hermitean symmetric to A.

Theorem 1. *Let the functions $f_i(t)$ be defined on $\langle 0, \infty)$ and integrable on every interval $\langle 0, T\rangle$. Put $\varphi(t) = (x(t), a)$, where $x(t)$ is a solution of the equation*

$$\dot{x} = Ax + f(t)\,. \tag{1}$$

Then to each $\gamma > 0$ there are n constants $0 < \alpha_1 < \ldots < \alpha_n < \gamma$ such that $\varphi(t)$ is a solution of the difference-differential equation

$$\dot{\varphi}(t) = \chi(t) + \psi(t) + \sum_{i=1}^{n} \mu_i\, \varphi(t + \alpha_i)\,, \tag{2}$$

where μ_i are certain constants, $\chi(t)$ is a linear combination of functions $f_i(t)$, $\psi(t)$

is a linear combination of functions $\int_0^{\alpha_l} e^{-\lambda_i \xi} \xi^k f_p(t+\xi)\,\mathrm{d}\xi$, λ_i are characteristic numbers of the matrix A and k is an integer with $0 \leqq k \leqq n_i$, where n_i is the multiplicity of λ_i.

Proof. Let the regular matrix T transform A into the Jordan's canonical form, and put $x = Ty$. We have $\varphi(t) = (x(t), a) = (y(t), b)$, where $b = T^*a$, $\dot{y} = By + f^+(t)$, B is the Jordan's canonical form of A and $f^+(t) = T^{-1} f(t)$. For each characteristic number λ_i there are r_i groups of differential equations

$$(3) \qquad \begin{aligned} \dot{y}^{(i)}_{\Theta,j} &= \lambda_i y^{(i)}_{\Theta,j} + y^{(i)}_{\Theta+1,j} + f^+_{\Theta,j} \quad &&\text{for} \quad \Theta = 1, 2, \ldots, q_{i,j} - 1\,, \\ \dot{y}^{(i)}_{\Theta,j} &= \lambda_i y^{(i)}_{\Theta,j} + f^+_{\Theta,j} &&\text{for} \quad \Theta = q_{i,j}\,, \quad j = 1, 2, \ldots, r_i\,. \end{aligned}$$

Let us denote $b^{(i)}_{\Theta,j}$ the coefficient at $y^{(i)}_{\Theta,j}$ in the expression (y, b). Now, we are going to transform this system of differential equations into a new one. If for a given i at least one coefficient $b^{(i)}_{\Theta,j}$ is different from zero, we put $w^{(i)}_1 = \sum_{\Theta,j} y^{(i)}_{\Theta,j} b^{(i)}_{\Theta,j}$ (or $\sum y^{(i)}_{\Theta,j} \bar{b}^{(i)}_{\Theta,j}$), where we are summing up those indices for which $b^{(i)}_{\Theta,j} \neq 0$. We derive easily the differenctial equation for $w^{(i)}_1$: $\dot{w}^{(i)}_1 = \sum_{\Theta,j} \dot{y}^{(i)}_{\Theta,j} b^{(i)}_{\Theta,j} = \lambda_i w^{(i)}_1 + \sum y^{(i)}_{\Theta+1,j} \delta^{(i)}_{\Theta+1,j} b^{(i)}_{\Theta,j} + \sum f^+_{\Theta,j} b^{(i)}_{\Theta,j}$, where $\delta^{(i)}_{\Theta+1,j} = 0$ for $\Theta \geqq q_{i,j}$ and $\delta^{(i)}_{\Theta+1,j} = 1$ for other Θ, j. Let us put $f^*_1 = \sum f^+_{\Theta,j} b^{(i)}_{\Theta,j}$ and further $w^{(i)}_2 = \sum y^{(i)}_{\Theta+1,j} \delta^{(i)}_{\Theta+1,j} b^{(i)}_{\Theta,j}$, where the summe is taken for such Θ, j (i is fixed) for which $\delta^{(i)}_{\Theta+1,j} b^{(i)}_{\Theta,j} \neq 0$. Then we get the equation $\dot{w}^{(i)}_1 = \lambda_i w^{(i)}_1 + w^{(i)}_2 + f^*_1$. Evidently, we can introduced new variables $w^{(i)}_\nu$, $\nu = 1, \ldots, s_i$, where $s_i \leqq \max_j q_{i,j}$ so that the following group of differential equations hold

$$(4) \qquad \begin{aligned} \dot{w}^{(i)}_\nu &= \lambda_i w^{(i)}_\nu + w^{(i)}_{\nu+1} + f^*_\nu \quad &&\text{for} \quad \nu = 1, 2, \ldots, s_i - 1\,, \\ \dot{w}^{(i)}_\nu &= \lambda_i w^{(i)}_\nu + f^*_\nu &&\text{for} \quad \nu = s_i\,. \end{aligned}$$

We can derive this group so that we consider $y^{(i)}_{\Theta,j}$, $\dot{y}^{(i)}_{\Theta,j}$, $f^+_{\Theta,j}$ as variables which obey the relations (3). Furthermore, if $w^{(i)}_\nu = \sum \eta_{\Theta,j} y^{(i)}_{\Theta,j}$, then we put $\dot{w}^{(i)}_\nu = \sum \eta_{\Theta,j} \dot{y}^{(i)}_{\Theta,j}$ We define $w^{(i)}_{\nu+1}$ as a linear combination of variables $y^{(i)}_{\Theta,j}$, which enter with nonzero coefficients in $\dot{w}^{(i)}_\nu - \lambda_i w^{(i)}_\nu$ expressed by $y^{(i)}_{\Theta,j}$, $f^+_{\Theta,j}$ (and which can be transformed by (3) so that there are no $\dot{y}^{(i)}_{\Theta,j}$). We define f^*_ν in a similar manner.

Put $m = \sum s_i$, $\omega(i) = \sum_{j=1}^{i-1} s_j$, $w_{\omega(i)+l} = w^{(i)}_l$ for $l = 1, \ldots, s_i$. Arranging the system (4) for w_l, we get a system of differential equations in Jordan's canonical form

$$(5) \qquad \dot{w} = Cw + f^*\,,$$

where w is a column vector with components w_l, $l = 1, \ldots, m$, $m \leqq n$. The characteristic numbers of the matrix C constitute a subset of characteristic numbers of matrix A, but to each characteristic number of matrix C there corresponds only one group (see (4)) of differential equations. Next, put

$$(6) \qquad c_{\omega(i)+1} = 1 \quad \text{for} \quad i = 1, 2, \ldots, \quad c_l = 0 \quad \text{for other } l\,.$$

Evidently, we can write $\varphi(t) = (w(t), c)$. Recalling that $c_{\omega(i)+1} \neq 0$ for c, the vector c does not belong to any invariant subspace of the matrix C^*. According to [2] pp. 149–150, (see (32) and the footnote), to each $\gamma > 0$ there are the numbers $0 < \alpha_1 < \ldots < \alpha_n < \gamma$ such that the vectors $e^{C^*\alpha_i}c$ are linearly independent (in (32) we can change the sign at ε and numbers $\alpha_1, \ldots, \alpha_n$ are dependent only on the matrix C and number γ).

Using the method of variation of constants we get

$$\varphi(t) = (w(t), c) = (e^{C(t-t_0)}w(t_0), c) + \left(\int_{t_0}^{t} e^{C(t-\tau)} f^*(\tau)\,\mathrm{d}\tau, c\right). \tag{7}$$

Putting $t = t_0 + \alpha_i$, we obtain after some arrangements

$$\varphi(t_0 + \alpha_i) = (w(t_0), e^{C^*\alpha_i}c) + \left(\int_0^{\alpha_i} e^{-C\xi} f^*(t_0 + \xi)\,\mathrm{d}\xi, e^{C^*\alpha_i}c\right).$$

Because the vectors $e^{C^*\alpha_i}c$ are linearly independent, the components of $w(t_0)$ are linear combinations of functions $\varphi(t_0 + \alpha_i)$ and

$$\int_0^{\alpha_i} e^{-\lambda_j\xi}\xi^k f_l(t_0 + \xi)\,\mathrm{d}\xi\,.$$

With regard to (4) we have $\dot{w}_l(t) = \lambda_i w_l(t) + w_{l+1}(t) + f_l^{**}(t)$ for $l \neq \omega(i + 1)$ and $\dot{w}_l = \lambda_i w_l + f_l^{**}$, for $l = \omega(i + 1)$. Then $\dot{w}_l(t)$ is a linear combination of $f_i(t)$ and of the former functions. With regard to (7) the statement of the theorem is true.

Without any further difficulty it can be verified that in fact the following remark was proved.

Remark 1. Let the matrix A be divided into blocks $A_{\mu\nu}$. Let $A_{\mu\nu} \equiv 0$ for $\mu \neq \nu$, and let $A_{\nu\nu}$ be square matrices. Let each matrix $A_{\nu\nu}$ have only one characteristic number λ_ν, $\lambda_\nu \neq \lambda_\mu$ for $\nu \neq \mu$. Denote $a_1^\nu, \ldots, a_l^\nu$ the components of the vector a which correspond to the matrix $A_{\nu\nu}$, and similarly for $x_1^\nu, \ldots, x_l^\nu$. Recalling the fact that in constructions mentioned formerly one component of vector w corresponds to all components $x_1^\nu, \ldots, x_l^\nu$, the statement of Theorem 1 is also true for $\sum_{i=1}^{l} x_i^\nu a_i^\nu$ or $\sum_{i=1}^{l} x_i^\nu \bar{a}_i^\nu$.

Theorem 1 remains also true in the case of a complex variable. If the statement of Theorem 1 is weakened slightly a theorem is obtained which may be formulated in a more compact and for applications more suitable form.

Notation. Let H be a linear space of functions,

i) defined and locally integrable on $\langle 0, \infty)$,

ii) if $f(t) \in H$, then $f^{(h)}(t) = f(t + h) \in H$ for each $h \geqq 0$,

iii) if $f(t) \in H$, then $\int_0^\alpha e^{\lambda\xi}\xi^k f(t + \xi)\,\mathrm{d}\xi \in H$

for arbitrary $\alpha > 0$, λ and integer $k \geqq 0$.

Theorem 2. *Let $x(t)$ be a solution of equation* (1), *H an arbitrary linear space of functions which fullfil* i) *to* iii), $f_i(t) \in H$, $(x(t), a) \in H$. *Then also* $(\dot{x}(t), a) \in H$.

Using the Riemann integral in abstract spaces we can formulate the following remark.

Remark 2. Let $x(t)$ be a solution of equation (1) and let Q be a Banach's space of functions defined on $\langle 0, \infty)$ and locally integrable on every $\langle 0, T\rangle$ (or a Banach's space of classes of equivalent functions) which has the following properties:

i) if $f \in Q$ then $f^{(h)}(t) = f(t + h) \in Q$ for each $h \geqq 0$ (if f is an element of a class of equivalent functions from Q then the class of equivalent functions which contains $f^{(h)}(t)$ belongs into Q. If $f(t)$, $g(t)$ belong to the same class, then $f^{(h)}(t)$, $g^{(h)}(t)$ belong into the same class, too).

ii) The mapping of the interval $\langle 0, \infty)$ into the set Q defined by the relation $\vartheta(\xi) = f^{(\xi)}(t)$ has a bounded total variation [3] on each interval $\langle 0, T\rangle$. (In every class of equivalent functions there is at least one function for which $\vartheta(\xi)$ has a bounded total variation on each interval $\langle 0, T\rangle$.)

iii) $(x(t), a) \in Q$ (the class containing $(x(t), a)$ belongs into Q);

then $(\dot{x}(t), a) \in Q$ (Q contains the class of equivalent functions with $(\dot{x}(t), a)$).

By Theorem 3.3.2 in [3] there are integrals $\int_0^{\alpha_i} e^{-\lambda_j \xi} \xi^k f_i(t + \xi) \, d\xi$ which in according to definition 3.3.1 in [3] belong into Q. Instead of the assumption that $f^{(\xi)}(t)$ have a bounded total variation, we can assume that the mapping $\langle 0, \infty)$ into Q defined by the relation $\vartheta(\xi) = f^{(\xi)}(t) \in Q$ is strongly continuous.

A series of consequences may be derived from Theorem 2 and the remark 2. We shall mention only the most interesting ones.

Corollary 1. *Let the functions $(x(t), a)$ and $f_i(t)$ be bounded; then $(\dot{x}(t), a)$ is also bounded.*

Corollary 2. *Let the functions $f_i(t)$ and $(x(t), a)$ have limits as $t \to \infty$; then $(\dot{x}(t), a)$ has also a limit as $t \to \infty$. If the limits of $f_i(t)$ and $(x(t), a)$ are equal to zero, then the limit of $(\dot{x}(t), a)$ is also equal to zero.*

Corollary 3. *Let a nonnegative, nonincreasing function $\Theta(t)$ exist such that $|f_i(t)| \leqq C_i \, \Theta(t)$, $|(x(t), a)| \leqq C \, \Theta(t)$; then there is a constant D such that $|(\dot{x}(t), a)| \leqq D \, \Theta(t)$.*

Corollary 4. *Let the function $\Theta(t)$ equal either $t^n + \varepsilon$, $\varepsilon > 0$ or $e^{\kappa t}$, and let $|f_i(t)| \leqq C_i \, \Theta(t)$, $|(x(t), a)| \leqq C \, \Theta(t)$; then a constant D exists such that $|(\dot{x}(t), a)| \leqq D \, \Theta(t)$.*

Using the Hölder's inequality, the following corollary may easily be deduced:

Corollary 5. *Let the functions $f_i(t)$ and $(x(t), a)$ belong into L_p, $p \geqq 1$; then $(\dot{x}(t), a)$ also belongs into L_p.*

Corollary 6. *Let the functions $f_i(t)$ and $(x(t), a)$ be almost periodic; then $(\dot{x}(t), a)$ is also almost periodic. (Every continuous alsmost periodical function defined on $\langle 0, \infty)$ can be uniquely continuated on the whole interval $(-\infty, \infty)$.)*

We may also impose more complicated requirements on the space H.

Corollary 7. *Let constants $M_i \geqq 0$, $i = 0, 1, \ldots, n$ and nonnegative functions $g_i(t) \in L_p$, $p \geqq 1$, $i = 0, 1, \ldots, n$ exist such that*

$$|f_i(t)| \leqq M_i + g_i(t)\,, \quad i = 1, \ldots, n\,, \quad |(x(t), a)| \leqq M_0 + g_0(t)\,;$$

then a constant $N \geqq 0$ and a nonnegative function $h(t) \in L_p$ exist such that $|(\dot{x}(t), a)| \leqq N + h(t)$.

Corollary 8. *Let nonnegative, nonincreasing function $\Theta(t)$, nonnegative constants C_i, $i = 0, 1, \ldots, n$ and nonnegative functions $g_i(t) \in L_p$, $p \geqq 1$ exist such that*

$$|f_i(t)| \leqq C_i\, \Theta(t) + g_i(t)\,, \quad i = 1, \ldots, n\,, \quad |(x(t), a)| \leqq C_0\, \Theta(t) + g_0(t)\,;$$

then a constant $D \geqq 0$ and a function $h(t) \geqq 0$, $h(t) \in L_p$ exist such that $|(\dot{x}(t), a)| \leqq$ $\leqq D\, \Theta(t) + h(t)$.

Corollary 9. *Let the functions $f_i(t)$ and their derivatives be bounded $|f_i^{(n)}(t)| \leqq M_n$ for $n \geqq 0$, $M_0 \leqq M_1 \leqq \ldots$, $\liminf_{n\to\infty} n/\sqrt[n]{(M_n)} > 0$ and let $(x(t), a)$ be bounded; then the function $(x(t), a)$ may be expanded in a power series in a neighborhood of every point $t_0 > 0$. The radius of convergence is not less than $\varrho = \min(t_0, q)$, where q is a certain positive constant or $+\infty$.*

Proof. If $|f_i(t)| \leqq M$, $|(x(t), a)| \leqq M$, then $|(\dot{x}(t), a)| \leqq \delta M$, where δ is a positive number with $\delta \geqq 1$, which depends only on the matrix A and the vector a and is independent of functions $f_i(t)$ and the constant M. Let us choose M_{-1} so that $|(x(t), a)| \leqq \delta M_{-1}$. Using successively Theorem 2 we conclude that a nondecreasing integral-valued function $\nu(n)$ with $\nu(n) \leqq n$ exists such that $\left|\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}(x(t), a)\right| \leqq \delta^n M_{\nu(n)}$. For the coefficients of the power series for $(x(t), a)$ at t_0 we get $|a_n| \leqq \delta^n M_{\nu(n)}/n!$. The radius of convergence of this series is given by the formula

$$\liminf_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} \geqq \liminf_{n\to\infty} \frac{n}{\delta e \sqrt[n]{M_{\nu(n)}}} \geqq \liminf_{n\to\infty} \frac{n}{\delta e \sqrt[n]{(M_n)}}\,, \tag{8}$$

as M_i, $i = 0, 1, \ldots$ form a non-decreasing sequence and $\liminf (n/\delta e \sqrt[n]{[M_{-1}]}) = \infty$. But we have assumed that the last expression in (8) is positive and we may denote it by q.

We may expect that the knowledge of the derivative $\dot{\varphi}(t)$ could help us to establish more involved properties of $\varphi(t)$. This fact is exploited in the following theorems.

However, we shall not proceed directly in this way, but use the vector w for reaching this aim. Nevertheless, the idea mentioned above played an important role by the formulation of following results.

Theorem 3. *Let $h_{i,j}(t)$ be nonnegative functions defined on $\langle 0, \infty)$ and integrable on each $\langle 0, T\rangle$ such that*

$$\lim_{t\to\infty} h_{i,0}(t) = 0\,, \quad h_{i,j}(t) \in L_{p_{i,j}}\,, \quad p_{i,j} \geqq 1\,, \quad j \geqq 1\,, \quad |f_i(t)| \leqq \sum_{j=0}^{r_i} h_{i,j}(t)\,,$$

and let $g_i(t)$ be also nonnegative functions defined on $\langle 0, \infty)$ and integrable on each $\langle 0, T\rangle$ such that

$$\lim_{t\to\infty} g_0(t) = 0\,, \quad \int_0^\infty g_i^{\alpha_i}(t)\,\mathrm{d}t < \infty\,, \quad \alpha_i > 0\,, \quad i \geqq 1\,, \quad |(x(t), a)| \leqq \sum_{j=0}^{r_0} g_i(t)\,.$$

Then we have $\lim_{t\to\infty} (x(t), a) = 0$.

Proof. In the proof of Theorem 1 there was shown that the function $\varphi(t)$ can be written in the form $\varphi(t) = (w(t), c)$, where $w(t)$ is a solution of (5) and the vector c is defined by relations (6). Further, it was shown that each component $w_i(t)$ is expressible as a linear combination of functions $\varphi(t + \vartheta_i)$ and $\int_0^{\vartheta_i} e^{-\lambda_j\xi}\xi^k f_i(t + \xi)\,\mathrm{d}\xi$.

From this it follows that for each component $w_i(t)$ we have

$$|w_i(t)| \leqq \sum_{j=0}^{S_i} u_{i,j}(t)\,, \quad \lim_{t\to\infty} u_{i,0}(t) = 0\,, \quad \int_0^\infty |u_{i,j}(t)|^{\alpha_{i,j}}\,\mathrm{d}t < \infty\,, \quad \alpha_{i,j} > 0\,.$$

The functions $u_{i,j}(t)$ may be chosen nonnegative and such that

$$|w_i(t)| = \sum_{j=0}^{S_i} u_{i,j}(t)\,. \tag{9}$$

Then we can write

$$w_i(t) = \varepsilon_i(t) \sum_{j=0}^{S_i} u_{i,j}(t)\,, \tag{10}$$

where $\varepsilon_i(t)$ assumes only values $+1$, -1. We prove first that

$$\liminf_{t\to\infty} |w_i(t)| = 0\,. \tag{11}$$

In the opposite case any $\bar{\varepsilon} > 0$, $t_0 > 0$ would exist such that $|w_i(t)| \geqq \bar{\varepsilon}$ for $t \geqq t_0$. Further, let us choose $t_1 \geqq t_0$ so that $u_0(t) < \bar{\varepsilon}/2$ for $t \geqq t_1$. According to (9) it follows that $\sum_{j=1}^{S_i} u_{i,j}(t) > \bar{\varepsilon}/2$ for $t \geqq t_1$. Denote A_j the set of those numbers t for which $t \geqq t_1$, $u_{i,j}(t) > \bar{\varepsilon}/2S_i$. We have $\sum_{j=1}^{S_i} A_j = \langle t_1, \infty)$ and consequently, there is at least one set A_j with infinite measure. Due to the fact that the functions $u_{i,j}$ are nonnegative we get $\int_0^\infty u_{i,j}^{\alpha_{i,j}}(t)\,dt = \infty$; but this is a contradiction with the assumptions. Hence, the relation (11) is proved. We shall now prove that $\lim_{t\to\infty} w_i(t) = 0$. Suppose conversely that $\limsup_{t\to\infty} |w_i(t)| > 0$; then there are numbers a, b, $0 < a < b < 1$ and an infinite number of intervals $\langle t_1^{(n)}, t_2^{(n)}\rangle$ such that

$$|w_i(t_1^{(n)})| = a, \ |w_i(t_2^{(n)})| = b, \ a \leqq |w_i(t)| \leqq b \quad \text{for} \quad t \in \langle t_1^{(n)}, t_2^{(n)}\rangle .$$

Assume now that there is a subsequence of such intervals that $|t_2^{(n)} - t_1^{(n)}| \geqq d$, where d is a positive constant. If we confine ourselves on t so large that $u_0(t) < a/2$ we get from (9) the inequality $\sum_{j=1}^{s_i} u_{i,j}(t) > a/2$. In the same way as before, we get $\int_0^\infty u_{i,j}^{\alpha_{i,j}}(t)\,dt = \infty$ for some j. From this it follows that the lengths of all mentioned intervals converge to zero, i.e.

$$(12) \qquad |t_2^{(n)} - t_1^{(n)}| \to 0 .$$

As the system (4) is in Jordan's canonical form, we can divide the components of a solution $w_i(t)$ into groups so that each group of components is a solution of one system of type (4). In each such system there is an equation

$$(13) \qquad \dot{w}_i(t) = \lambda\, w_i(t) + f_i^*(t)$$

where λ is a characteristic number of the matric C (see proof of Theorem 1). Using (10) the equation (13) can be expressed in the following form

$$(14) \qquad \dot{w}_i(t) = \lambda\, \varepsilon_i(t) \sum_{j=0}^{s_i} u_{i,j}(t) + \varepsilon_i^*(t) \sum_{j=0}^{r_i} v_{i,j}(t) .$$

The function $f_i^*(t)$ is a linear combination of functions $f_i(t)$. As $p_{i,j} \geqq 1$ a similar estimate for $f_i^*(t)$ as for $f_i(t)$ can be obtained. The functions $v_{i,j}$, ε^* are related to f_i^* in the same way as $u_{i,j}$, ε to $w_i(t)$.

Let us choose $\beta = \max(\alpha_{i,j}, \frac{1}{2})$, where $\alpha_{i,j} < 1$ for each fixed i. (The index i at β is omitted.) We have $0 < \beta < 1$. From (14) it follows that

$$(15) \qquad |b^\beta - a^\beta| = \left||w_i(t_2^{(n)})|^\beta - |w_i(t_1^{(n)})|^\beta\right| \leqq$$

$$\leqq |\lambda|\, \beta \int_{t_1(n)}^{t_2(n)} u_0(t) |w_i(t)|^{\beta-1}\,dt + |\lambda|\, \beta \sum_{j=1}^{s_i} \int_{t_1(n)}^{t_2(n)} u_{i,j} |w_i|^{\beta-1}\,dt +$$

$$+ \beta \int_{t_1(n)}^{t_2(n)} v_{i,0} |w_i|^{\beta-1}\,dt + \beta \sum_{j=1}^{r_i} \int_{t_1(n)}^{t_2(n)} v_{i,j} |w_i|^{\beta-1}\,dt .$$

As $|w_i(t)| \geqq a$ for $t \in \langle t_1^{(n)}, t_2^{(n)} \rangle$, $\beta < 1$ we can estimate the first and the third term on the right side of inequality (15) by

$$\leqq |\lambda| \, \beta a^{\beta-1} \int_{t_1(n)}^{t_2(n)} (u_0 + v_{i,0}) \, dt \, .$$

This expression tends to 0 as $n \to \infty$ because u_0, $v_{i,0}$ tend to 0 and (12) is fulfilled. In the second expression let us take first those terms for which $\alpha_{i,j} < 1$. From (9) it follows that $u_{i,j}(t) \leqq |w_i(t)|$. In the case $u_{i,j}(t) > 0$ we have $|w_i|^{\beta-1} \leqq |u_{i,j}|^{\beta-1} \leqq$ $\leqq |u_{i,j}|^{\alpha_{i,j}-1}$ (b is chosen so that $u_{i,j} \leqq b \leqq 1$). If $u_{i,j}(t) = 0$ then evidently $u_{i,j}(t) \, w_i^{\beta-1}(t) = u_{i,j}^{\alpha_{i,j}}(t) = 0$. For the above mentioned integrals we always get the estimate $\leqq |\lambda| \, \beta \sum \int_{t_1(n)}^{t_2(n)} u_{i,j}^{\alpha_{i,j}} \, dt$. Because $\int_0^\infty u_{i,j}^{\alpha_{i,j}} \, dt < \infty$ these terms tend to 0 as $n \to \infty$. The rest of the second expression, i.e. the integrals with $u_{i,j}$ for which $\alpha_{i,j} \geqq 1$, and also the terms of the last expression may be estimated by

$$\leqq |\lambda| \, \beta \int_{t_1(n)}^{t_2(n)} (\sum u_{i,j} + \sum v_{i,j}) \, dt \leqq |\lambda| \, \beta a^{\beta-1} \Bigg[\sum \left(\int_{t_1(n)}^{t_2(n)} u_{i,j}^{\alpha_{i,j}} \, dt \right)^{1/\alpha_{i,j}} . \, (t_2^{(n)} - t_1^{(n)})^{1-1/\alpha_{i,j}} +$$
$$+ \sum \left(\int_{t_1(n)}^{t_2(n)} v_{i,j}^{p_{i,j}} \, dt \right)^{1/p_{i,j}} . \, (t_2^{(n)} - t_1^{(n)})^{1-1/p_{i,j}} \Bigg] .$$

The last inequality follows from Hölder's inequality. According to (12) these expressions also tend to 0 as $n \to \infty$. Thus, we have proved the convergence of the right side of inequality (15) to 0 as $n \to \infty$. However, the left hand side is constant. This contradiction shows that $\lim_{t \to \infty} w_i(t) = 0$ must be true. The proof can now be easily finished by induction. Each component $w_i(t)$ is given by the equation $\dot{w}_i = \lambda w_i +$ $+ w_{i+1} + f_i^*$. We have $\lim_{t \to \infty} w_{i+1}(t) = 0$ in accordance with the induction assumption. Thus, we can put $\hat{f}_i(t) = f_i^*(t) + w_{i+1}(t)$ and so get the equation already treated; hence, $\lim_{t \to \infty} w_i(t) = 0$. Summarizing, we have proved that

$$\lim_{t \to \infty} w(t) = 0 \, . \tag{16}$$

From (6) it follows that $\lim_{t \to \infty} \varphi(t) = \lim_{t \to \infty} (x(t), a) = 0$.

Remark 3. If the matrix A has the same type as in the remark 1 and if the assumptions of Theorem 3 are fulfilled, then

$$\lim_{t \to \infty} \sum x_k(t) \, a_k = 0 \quad (\text{or } \lim_{t \to \infty} \sum x_k \bar{a}_k = 0) \, ,$$

where the summation is extended over indices corresponding to $A_{\nu\nu}$.

Theorem 4. *Let nonnegative functions $g_i(t)$, $h_{i,j}(t)$ defined on $\langle 0, \infty)$ and integrable on each $\langle 0, T \rangle$ exist such that $h_{i,0}(t)$ is constant, $\int_0^\infty h_{i,j}^{p_{i,j}}(t) \, dt < \infty$, $p_{i,j} \geqq 1$,*

$j \geqq 1$, $|f_i(t)| \leqq \sum_{j=0}^{r_i} h_{i,j}(t)$; *let* $g_0(t)$ *be constant and* $\int_0^\infty g_i^{\alpha_i}(t)\,\mathrm{d}t < \infty$, $\alpha_i > 0$, $i \geqq 1$, $|(x(t), a)| \leqq \sum_{i=0}^{r_0} g_i(t)$. *Then the function* $(x(t), a)$ *is bounded on* $\langle 0, \infty)$.

Proof. We shall proceed as before and use again the function $w(t)$ for which the estimates as for $(x(t), a)$ are true. Consequently, (9) is true too. As before consider the equation $\dot{w}_i = \lambda w_i + f_i^*(t)$. First we are going to prove that $\liminf_{t\to\infty} |w_i(t)| < \infty$. If we had $\lim_{t\to\infty} |w_i(t)| = \infty$ it would be $\lim_{t\to\infty} \sum_{j=1}^{S_i} u_{i,j}(t) = \infty$ i.e. $\sum_{j=1}^{S_i} u_{i,j}(t) \geqq S_i$ for t sufficiently large. Thus we get as before $u_{i,j}(t) \geqq 1$ on the set A_j such that $\mu(A_j) = \infty$. This would imply that $\int_0^\infty u_{i,j}^{\alpha_{i,j}}\,dt = \infty$. This proves that $\liminf_{t\to\infty} |w_i(t)| < \infty$; further we shall prove that also $\limsup_{t\to\infty} |w_i(t)| < \infty$. Let us assume that $\limsup_{t\to\infty} |w_i(t)| = \infty$. Then we can choose a sequence of intervals $\langle t_1^{(n)}, t_2^{(n)}\rangle$ and a number a such that $\max(u_{i,0}, 1) < a = |w_i(t_1^{(n)})|$, $|w_i(t_2^{(n)})| = n$, $a \leqq |w_i(t)| \leqq n$ for $t \in \langle t_1^{(n)}, t_2^{(n)}\rangle$, $u_{i,0} = \sup u_{i,0}(t)$, $v_{i,0} = \sup v_{i,0}(t)$. Next, put $\beta = \min(\alpha_{i,j}, \frac{1}{2})$, where the minimum is taken on the set of all $\alpha_{i,j}$ with i fixed. (We do not mark the index i at β.) We have $0 < \beta < 1$. In the same way as in (15) we obtain

$$(17)\quad |n^\beta - a^\beta| \leqq \left||w_i(t_2^{(n)})|^\beta - |w_i(t_1^{(n)})|^\beta\right| \leqq \beta(|\lambda|\, u_{i,0} + v_{i,0}) \int_{t_1^{(n)}}^{t_2^{(n)}} |w_i|^{\beta-1}\,dt + \\ + \beta|\lambda| \sum_{j=1}^{S_i} \int_{t_1^{(n)}}^{t_2^{(n)}} u_{i,j}|w_i|^{\beta-1}\,\mathrm{d}t + \beta \sum_{j=1}^{r_i} \int_{t_1^{(n)}}^{t_2^{(n)}} v_{i,j}|w_i|^{\beta-1}\,\mathrm{d}t\,.$$

Using the inequality $|w_i|^{\beta-1} \leqq a^{\beta-1}$ the first integral can be estimated as

$$(18)\qquad \beta(|\lambda|\, u_{i,0} + v_{i,0}) \int_{t_1^{(n)}}^{t_2^{(n)}} |w_i|^{\beta-1}\,dt \leqq \beta(|\lambda|\, u_{i,0} + v_{i,0})\, a^{\beta-1}|t_2^{(n)} - t_1^{(n)}|\,.$$

As in Theorem 3 it can be proved that $|t_2^{(n)} - t_1^{(n)}| \to 0$. We choose from the second expression first such terms for which $\alpha_{i,j} < 1$ and from (9) we get again $u_{i,j}(t) \leqq \leqq |w_i(t)|$. Estimating these integrals however, we must proceed more carefully. We devide the interval $\langle t_1^{(n)}, t_2^{(n)}\rangle$ into two parts: A_n, where $u_{i,j}(t) \leqq 1$ and B_n where $u_{i,j}(t) > 1$. We have

$$(19)\qquad \beta|\lambda| \int_{A_n} u_{i,j}|w_i|^{\beta-1}\,\mathrm{d}t \leqq \beta|\lambda|\, a^{\beta-1}|t_2^{(n)} - t_1^{(n)}|\,.$$

As for the set B_n, we proceed as in the proof of the preceding theorem.

$$(20)\qquad \beta|\lambda| \int_{B_n} u_{i,j}|w_i|^{\beta-1}\,\mathrm{d}t \leqq \beta|\lambda| \int_{t_1^{(n)}}^{t_2^{(n)}} u_{i,j}^{\alpha_{i,j}}\,\mathrm{d}t\,.$$

The integrals from the second expression, for which $\alpha_{i,j} \geqq 1$, and from the third expression can be estimated as

$$(21) \qquad \beta \int_{t_1(n)}^{t_2(n)} v_{i,j} |w_i|^{\beta-1} \, \mathrm{d}t \leqq \beta a^{\beta-1} \int_{t_1(n)}^{t_2(n)} v_{i,j} \, \mathrm{d}t \leqq$$

$$\leqq \beta a^{\beta-1} \left(\int_{t_1(n)}^{t_2(n)} v_{i,j}^{p_{i,j}} \, \mathrm{d}t \right)^{1/p_{i,j}} . \left| t_2^{(n)} - t_1^{(n)} \right|^{1-1/p_{i,j}} .$$

From (18) to (21) and from the assumptions of the theorem it follows that the right-hand side of the inequality (17) is bounded while the left-hand side diverges as $n \to \infty$. This contradiction shows that $\limsup_{t \to \infty} |w_i(t)| < \infty$; hence, $w_i(t)$ is bounded. It can be also proved as in Theorem 3 that $w(t)$ and $\varphi(t) = (x(t), a) = (w(t), c)$ are bounded.

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Výtah

O JISTÝCH VLASTNOSTECH SOUSTAV LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC

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Nechť $x(t)$ jest řešení soustavy lineárních diferenciálních rovnic (1) a $(x(t), a)$ jest skalární součin vektorů $x(t)$, a.

Věta 1. *Nechť funkce $f_i(t)$ jsou definované a lokálně integrovatelné na $\langle 0, \infty)$. K libovolnému $\gamma > 0$ existuje n konstant $0 < \alpha_1 < \ldots < \alpha_n < \gamma$ tak, že $\varphi(t) = (x(t), a)$ je řešení diferenčně-diferenciální rovnice (2), kde μ_i jsou jisté konstanty, $\chi(t)$ je lineární kombinace funkcí $f_i(t)$, $\psi(t)$ je lineární kombinace funkcí $\int_0^{\alpha_i} e^{-\lambda_i \xi} \xi^k \, . \, f_i(t + \xi) \, \mathrm{d}\xi$, λ_i jsou charakteristická čísla matice A, k je celé číslo $0 \leqq k \leqq n_i$, n_i je násobnost charakteristického čísla λ_i.*

Tuto větu lze formulovat také v poněkud slabší, ale přehlednější formě. Nechť H je lineární prostor funkcí:

1. definovaných a lokálně integrovatelných na $\langle 0, \infty)$,
2. jestliže $f(t) \in H$, pak $f^{(h)}(t) = f(t + h) \in H$ pro každé $h \geqq 0$,
3. jestliže $f(t) \in H$, pak $\int_0^\alpha e^{\lambda\xi} \xi^k f_i(t + \xi) \, d\xi \in H$ pro libovolné λ, kladné α a celé číslo $k \geqq 0$.

Věta 2. *Nechť $x(t)$ jest řešení diferenciální rovnice* (1), *H je lineární prostor funkcí splňující* 1) ... 3), *$f_i(t) \in H$ a $(x(t), a) \in H$, pak také $(\dot{x}(t), a) \in H$.*

Z těchto vět je odvozena řada důsledků. Na základě metody použité v důkazu věty 1 lze dokázat další věty.

Věta 3. *Nechť existují nezáporné funkce $h_{i,j}(t)$, $g_i(t)$ definované a lokálně integrovatelné na $\langle 0, \infty)$ tak, že $\lim_{t\to\infty} h_{i0}(t) = 0$, $h_{ij}(t) \in L_{p_{ij}}$, $p_{ij} \geqq 1$, $j \geqq 1$, $\lim_{t\to\infty} g_0(t) = 0$, $\int_0^\infty g_i^{\alpha_i}(t) \, dt < \infty$, $\alpha_i > 0$, $i \geqq 1$, $|f_i(t)| \leqq \sum_{j=0}^{r_i} h_{ij}(t)$, $|(x(t), a)| \leqq \sum_{j=0}^{r_0} g_i(t)$, pak platí $\lim_{t\to\infty} (x(t), a) = 0$.*

Věta 4. *Nechť existují nezáporné funkce $h_{ij}(t)$, $g_i(t)$ definované a lokálně integrovatelné na $\langle 0, \infty)$ tak, že $h_{i0}(t)$, $g_0(t)$ jsou konstantní, $h_{ij}(t) \in L_{p_{ij}}$, $p_{ij} \geqq 1$, $j \geqq 1$, $\int_0^\infty g_i^{\alpha_i}(t) \, dt < \infty$, $\alpha_i > 0$, $i \geqq 1$, $|f_i(t)| \leqq \sum_{j=0}^{r_i} h_{ij}(t)$, $|(x(t), a)| \leqq \sum_{j=0}^{r_0} g_i(t)$, pak funkce $(x(t), a)$ je omezená na $\langle 0, \infty)$.*

Резюме

ОБ ОПРЕДЕЛЕННЫХ СВОЙСТВАХ СИСТЕМ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Пусть $x(t)$ – решение системы линейных дифференциальных уравнений (1) и $(x(t), a)$ – скалярное произведение векторов $x(t)$, a.

Теорема 1. *Пусть функции $f_i(t)$ определены на $\langle 0, \infty)$ и суммируемы на каждом $\langle 0, T\rangle$. К любому $\gamma > 0$ существует n постоянных $0 < \alpha_1 < \ldots < \alpha_n < \gamma$ так, что $\varphi(t) = (x(t), a)$ есть решение разностно-дифференциального уравнениея* (2), *где μ_i – какие-то постоянные, $\chi(t)$ – линейная комбинация функций $f_i(t)$, $\psi(t)$ – линейная комбинация функций $\int_0^{\alpha_i} e^{-\lambda_i \xi} \xi^k f_l(t + \xi) \, d\xi$, λ_i – характеристические числа матрицы A, k – целое число $0 \leqq k \leqq n_i$, n_i – кратность характеристического числа k_i.*

Эту теорему можно сформулировать также в немного ослабленной, но более наглядной форме.

Пусть H — линейное пространство функций:

1. определенных на $\langle 0, \infty)$ и суммируемых на каждом $\langle 0, T\rangle$,
2. если $f(t) \in H$, то $f^{(h)}(t) = f(t + h) \in H$ для любого $h \geqq 0$,
3. если $f(t) \in H$, то $\int_0^\alpha e^{\lambda\xi}\xi^k f_l(t + \xi)\,\mathrm{d}\xi \in H$ для любого λ, положительного α и целого $k \geqq 0$.

Теорема 2. *Пусть* $x(t)$ *— решение дифференциального уравнения* (1), H *— линейное пространство функций, выполняющее* 1) ... 3), $f_i(t) \in H$, *и* $(x(t), a) \in H$, *тогда* $(\dot{x}(t), a) \in H$.

Из этих теорем можно вывести ряд следствий. На основе метода, использованного в доказательстве теоремы 1, можно доказать следующие теоремы.

Теорема 3. *Пусть существуют неотрицательные функции* $h_{ij}(t)$, $g_i(t)$, *определенные на* $\langle 0, \infty)$, *суммируемые на каждом* $\langle 0, T\rangle$ *а такие, что* $\lim\limits_{t\to\infty} h_{i0}(t) = 0$, $\lim\limits_{t\to\infty} g_0(t) = 0$, $h_{ij}(t) \in L_{p_{ij}}$, $p_{ij} \geqq 1$, $\int_0^\infty g_i^{\alpha_i}(t)\,\mathrm{d}t < \infty$, $\alpha_i > 0$, $i \geqq 1$, $|f_i(t)| \leqq \sum\limits_{j=0}^{r_1} h_{ij}(t)$, $|(x(t), a)| \leqq \sum\limits_{j=0}^{r_0} g_j(t)$; *тогда* $\lim\limits_{t\to\infty} (x(t), a) = 0$.

Теорема 4. *Пусть существуют неотрицательные функции* $h_{ij}(t)$, $g_i(t)$, *определенные на* $\langle 0, \infty)$, *суммируемые на каждом* $\langle 0, T\rangle$ *так, что* $h_{i0}(t), g_0(t)$ *— постоянные,* $h_{ij}(t) \in L_{p_{ij}}$, $p_{ij} \geqq 1$, $j \geqq 1$, $\int_0^\infty g_i^{\alpha_i}(t)\,\mathrm{d}t < \infty$, $\alpha_i > 0$, $i \geqq 1$, $|f_i(t)| \leqq \sum\limits_{j=0}^{r_i} h_{ij}(t)$, $|(x(t), a)| \leqq \sum\limits_{j=0}^{r_0} g_j(t)$, *тогда функция* $(x(t), a)$ *ограничена на* $\langle 0, \infty)$.