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TO THE INVERSION OF GÅRDING THEOREM

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Summary. The subject under consideration is the inversion of the known Gårding theorem concerning algebraic conditions (sufficient and) necessary for the so called ellipticity of some linear differential operators with very mild requirements (local integrability) on coefficients.

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The purpose of this paper is to extend the known inversion of Gårding theorem for partial differential equations, concerning the relation between the corresponding differential quadratic form and the quadratic form of coefficients, as given e.g. in Nečas, Chap. 3, Sect. 4.3, Théorème 4.7. Instead of the topological continuity of the coefficients including the boundary, we suppose only their local integrability. On the other hand, we restrict ourselves to the pure second order equations for the sake of simplicity, even if the method seems to work also in the general case.

The proof of our result (Theorem 13) is based, as usual, on the use of some form of Fourier transform. But the form used below is considerably different and essentially more complicated since we must work with the continuity in the absolute mean almost everywhere of our locally integrable coefficients (i.e. with their Lebesgue points) instead of the above mentioned topological continuity including the boundary. The auxiliary results are collected in a series of lemams.

1. We denote by $\mathbf{R}$ the real number field.

2. For arbitrary $d \in \{1, 2, \ldots\}$, $\mathbf{R}^d$ is the d-dimensional coordinate space in which we use the following abbreviations:

$$\langle \eta, \xi \rangle = \sum_{r=1}^{d} \eta_r \xi_r \text{ for } \xi, \quad \eta \in \mathbf{R}^d,$$

$$\mathrm{K}_h(\xi) = \{\eta : \eta \in \mathbf{R}^d, \max_{r \in \{1,2,\ldots,d\}} |\eta_r - \xi_r| \leqq h\} \quad \text{for} \quad h > 0 \quad \text{and} \quad \xi \in \mathbf{R}^d .$$

Further, μ is the usual Lebesgue measure in $\mathbf{R}^d$ and $\int_{\mathbf{R}^d}$ the corresponding Lebesgue integral.

3. $C_0^\infty(\mathbf{R}^d)$ is the set of all infinitely differentiable complex functions φ on $\mathbf{R}^d$ with compact support. The support of $\varphi \in C_0^\infty(\mathbf{R}^d)$ is denoted $\operatorname{supp}(\varphi)$ and the partial derivative with respect to the r-th variable $\partial\varphi/\partial\cdot_r$ or $\partial/\partial\cdot_r\, \varphi$ for $r \in \{1, 2, \ldots, d\}$.

4. In the sequel, Ω will be an arbitrary fixed open subset of $\mathbf{R}^d$.

5. Lemma. *Let f be a complex function on Ω. If the function f is locally integrable in Ω, then there exists a measurable set $N \subseteq \Omega$ such that*

(a) $\mu(N) = 0$,

(b) $\displaystyle\frac{1}{h^d}\int_{K_h(\xi)} (f - f(\xi)) \to 0 \ (h \to 0_+)$

for every $\xi \in \Omega \setminus N$.

Proof. Saks, Chap. IV, Theorem 6.3.

6. Let us define:

$$\psi_{\varrho\eta h\xi} = \frac{1}{h^{d/2}}\, e^{i\varrho\langle\eta, (\cdot/h) - \xi\rangle}\, \psi\left(\frac{\cdot}{h} - \xi\right)$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$.

7. Lemma. $\psi_{\varrho\eta h\xi} \in C_0^\infty(\mathbf{R}^d)$ *for every* $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ *and* $\xi \in \mathbf{R}^d$. *Moreover,*

$$\frac{\partial}{\partial\cdot_r}\psi_{\varrho\eta h\xi} = \frac{1}{h}\left[i\varrho\eta_r\psi_{\varrho\eta h\xi} + \left(\frac{\partial\psi}{\partial\cdot_r}\right)_{\varrho\eta h\xi}\right]$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ *and* $r \in \{1, 2, \ldots, d\}$.

8. Lemma. $\int_{\mathbf{R}^d} |\psi_{\varrho\eta h\xi}|^2 = \int_{\mathbf{R}^d} |\psi|^2$ *for every* $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ *and* $\xi \in \mathbf{R}^d$.

9. Lemma.

$$\frac{1}{\varrho^2}\int_{\mathbf{R}^d}\frac{\partial}{\partial\cdot_i}\psi_{\varrho\eta h\xi}\frac{\partial}{\partial\cdot_j}\overline{\psi}_{\varrho\eta h\xi} \to \frac{1}{h^2}\eta_i\eta_j\int_{\mathbf{R}^d}|\psi|^2 \ (\varrho \to \infty)$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ *and* $i, j \in \{1, 2, \ldots, d\}$.

Proof. By Lemma 7, we can write

$$\frac{1}{\varrho^2}\int_{\mathbf{R}^d}\frac{\partial}{\partial\cdot_i}\psi_{\varrho\eta h\xi}\frac{\partial}{\partial\cdot_j}\overline{\psi}_{\varrho\eta h\xi} =$$

$$= \frac{1}{\varrho^2}\frac{1}{h^2}\int_{\mathbf{R}^d}\left[i\varrho\eta_i\psi_{\varrho\eta h\xi} + \left(\frac{\partial\psi}{\partial\cdot_i}\right)_{\varrho\eta h\xi}\right]\left[-i\varrho\eta_j\overline{\psi}_{\varrho\eta h\xi} + \left(\frac{\partial\overline{\psi}}{\partial\cdot_j}\right)_{\varrho\eta h\xi}\right]$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ and $i, j \in \{1, 2, \ldots, d\}$.

The statement follows immediately by making use of Lemmas 7 and 8 and of the Schwarz inequality.

10. Lemma.

$$\frac{1}{\varrho^2}\int_{\mathbf{R}^d}\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|^2 \to \frac{1}{h^2}\eta_r^2\int_{\mathbf{R}^d}|\psi|^2 \quad (\varrho\to\infty)$$

for every $\psi\in C_0^\infty(\mathbf{R}^d)$, $\eta\in\mathbf{R}^d$, $h>0$ *and* $\xi\in\mathbf{R}^d$.

Proof. Immediate consequence of Lemma 9.

11. Sublemma. *For every* $\psi\in C_0^\infty(\mathbf{R}^d)$, *there exists a* $\varrho_0>0$ *such that*

$$\left(\int_{\mathbf{R}^d}|\psi|^2\right)\left[\varrho\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|\right]^2 \leqq$$

$$\leqq 2[\max|\psi|]^2\left[\varrho^2\int_{\mathbf{R}^d}|\psi|^2 + \sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho\left(\int_{\mathbf{R}^d}|\psi|^2\right)^{1/2}\left(\sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}\right]$$

for every $\varrho\geqq\varrho_0$.

Proof. The case $\psi=0$ is evident.

Suppose now $\psi\neq 0$.

Then we get immediately that

$$\frac{\varrho^2\int_{\mathbf{R}^d}|\psi|^2 + \sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho\left(\int|\psi|^2\right)^{1/2}\left(\sum_{r=1}^{d}\int\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}}{\left[\varrho\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|\right]^2} \to \frac{\int_{\mathbf{R}^d}|\psi|^2}{[\max|\psi|]^2}\ (\varrho\to\infty).$$

This implies that the left hand side of the preceding relation is

$$\geqq\frac{1}{2}\frac{\int_{\mathbf{R}^d}|\psi|^2}{[\max|\psi|]^2}$$

for sufficiently large $\varrho>0$.

The just proved inequality immediately gives the desired result.

12. Lemma. *For every* $\psi\in C_0^\infty(\mathbf{R}^d)$, *there exists a* $\varrho_0>0$ *such that*

$$h^d\left(\int_{\mathbf{R}^d}|\psi|^2\right)\left[\sum_{r=1}^{d}\max\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|\right]^2 \leqq$$

$$\leqq 2d\,[\max|\psi|]^2\sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|^2$$

for every $\varrho>\varrho_0$, $\eta\in\mathbf{R}^d$, $\sum_{r=1}^{d}\eta_r^2=1$, $h>0$ *and* $\xi\in\mathbf{R}^d$.

Proof. It is clear that

(1) $$\max|\psi_{\varrho\eta h\xi}| = \frac{1}{h^{d/2}}\max|\psi|$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$.

By Lemma 8, we have

(2) $$\int_{\mathbf{R}^d}|\psi_{\varrho\eta h\xi}|^2 = \int_{\mathbf{R}^d}|\psi|^2$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$,

(3) $$\int_{\mathbf{R}^d}\left|\left(\frac{\partial\psi}{\partial_{\bullet r}}\right)_{\varrho\eta h\xi}\right|^2 = \int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|^2$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ and $r \in \{1, 2, \ldots, d\}$.

On the other hand, we have by Lemma 7 that

(4) $$\frac{\partial}{\partial_{\bullet r}}\psi_{\varrho\eta h\xi} = \frac{1}{h}\left[\mathrm{i}\varrho\eta_r\psi_{\varrho\eta h\xi} + \left(\frac{\partial\psi}{\partial_{\bullet r}}\right)_{\varrho\eta h\xi}\right]$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ and $r \in \{1, 2, \ldots, d\}$.

It follows from (1) and (4) that

(5) $$\max\left|\frac{\partial}{\partial_{\bullet r}}\psi_{\varrho\eta h\xi}\right| \leqq \frac{1}{h^{d/2+1}}\left[\varrho|\eta_r|\max|\psi| + \max\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|\right]$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ and $r \in \{1, 2, \ldots, d\}$.

Now by (5), we obtain

(6) $$\sum_{r=1}^{d}\max\left|\frac{\partial}{\partial_{\bullet r}}\psi_{\varrho\eta h\xi}\right| \leqq$$
$$\leqq \frac{1}{h^{d/2+1}}\left[\varrho\sum_{r=1}^{d}|\eta_r|\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|\right] =$$
$$= \frac{d^{1/2}}{h^{d/2+1}}\left[\varrho\,(\sum_{r=1}^{d}|\eta_r|^2)^{1/2}\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|\right]$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$.

As an immediate consequence of (6) we have

(7) $$\left[\sum_{r=1}^{d}\max\left|\frac{\partial}{\partial_{\bullet r}}\psi_{\varrho\eta h\xi}\right|\right]^2 \leqq \frac{d}{h^{d+2}}\left[\varrho\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|\right]^2$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d}\eta_r^2 = 1$, $h > 0$ and $\xi \in \mathbf{R}^d$.

By (7), we can write

(8) $$h^d\left(\int_{\mathbf{R}^d}|\psi|^2\right)\left[\sum_{r=1}^{d}\max\left|\frac{\partial}{\partial_{\bullet r}}\psi_{\varrho\eta h\xi}\right|\right]^2 \leqq \frac{d}{h^2}\left(\int_{\mathbf{R}^d}|\psi|^2\right)\left[\varrho\max|\psi| + \sum_{r=1}^{d}\max\left|\frac{\partial\psi}{\partial_{\bullet r}}\right|\right]^2$$

for every $\psi \in C_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d}\eta_r^2 = 1$, $h > 0$ and $\xi \in \mathbf{R}^d$.

On the other hand, we obtain from (2), (3) and (4) under use of the Schwarz inequality that

$$(9)\qquad \int_{\mathbf{R}^d}\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|^2 = \frac{1}{h^2}\int_{\mathbf{R}^d}\left|\mathrm{i}\varrho\eta_r\psi_{\varrho\eta h\xi} + \left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|^2 \geqq$$

$$\geqq \frac{1}{h^2}\int_{\mathbf{R}^d}\left(|\mathrm{i}\varrho\eta_r\psi_{\varrho\eta h\xi}| - \left|\left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|\right)^2 =$$

$$= \frac{1}{h^2}\left[\varrho^2\eta_r^2\int_{\mathbf{R}^d}|\psi_{\varrho\eta h\xi}|^2 + \int_{\mathbf{R}^d}\left|\left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|^2 - 2\varrho|\eta_r|\int_{\mathbf{R}^d}|\psi_{\varrho\eta h\xi}|\left|\left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|\right] \geqq$$

$$\geqq \frac{1}{h^2}\left[\varrho^2\eta_r^2\int_{\mathbf{R}^d}|\psi_{\varrho\eta h\xi}|^2 + \int_{\mathbf{R}^d}\left|\left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|^2 - \right.$$

$$\left. - 2\varrho|\eta_r|\left(\int_{\mathbf{R}^d}|\psi_{\varrho\eta h\xi}|^2\right)^{1/2}\left(\int_{\mathbf{R}^d}\left|\left(\frac{\partial\psi}{\partial_{\cdot r}}\right)_{\varrho\eta h\xi}\right|^2\right)^{1/2}\right] =$$

$$= \frac{1}{h^2}\left[\varrho^2\eta_r^2\int_{\mathbf{R}^d}|\psi|^2 + \int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho|\eta_r|\left(\int_{\mathbf{R}^d}|\psi|^2\right)^{1/2}\left(\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}\right]$$

for every $\psi \in \mathbf{C}_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$, $\xi \in \mathbf{R}^d$ and $r \in \{1, 2, \ldots, d\}$.

Let us recall that clearly

$$(10)\qquad \sum_{r=1}^{d}|\eta_r|\left(\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2} \leqq \left(\sum_{r=1}^{d}\eta_r^2\right)^{1/2}\left(\sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}$$

for every $\psi \in \mathbf{C}_0^\infty(\mathbf{R}^d)$ and $\eta \in \mathbf{R}^d$.

It follows from (9) and (10) that

$$(11)\qquad \sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|^2 \geqq$$

$$\geqq \frac{1}{h^2}\left[\varrho^2\sum_{r=1}^{d}\eta_r^2\int_{\mathbf{R}^d}|\psi|^2 + \sum_{r=1}^{d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho\sum_{r=1}^{d}|\eta_r|\left(\int_{\mathbf{R}^d}|\psi|^2\right)^{1/2}\left(\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}\right] \geqq$$

$$\geqq \frac{1}{h^2}\left[\varrho^2\sum_{r=1}^{d}\eta_r^2\int_{\mathbf{R}^d}|\psi|^2 + \sum_{r=1}^{d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho\left(\sum_{r=1}^{d}\eta_r^2\right)^{1/2}\left(\int_{\mathbf{R}^d}|\psi|^2\right)^{1/2}\left(\sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}\right]$$

for every $\psi \in \mathbf{C}_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$.

As a special case of (11) we have

$$(12)\qquad \sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial}{\partial_{\cdot r}}\psi_{\varrho\eta h\xi}\right|^2 \geqq$$

$$\geqq \frac{1}{h^2}\left[\varrho^2\int_{\mathbf{R}^d}|\psi|^2 + \sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2 - 2\varrho\left(\int_{\mathbf{R}^d}|\psi|^2\right)^{1/2}\left(\sum_{r=1}^{d}\int_{\mathbf{R}^d}\left|\frac{\partial\psi}{\partial_{\cdot r}}\right|^2\right)^{1/2}\right]$$

for every $\psi \in \mathbf{C}_0^\infty(\mathbf{R}^d)$, $\varrho > 0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d}\eta_r^2 = 1$, $h > 0$ and $\xi \in \mathbf{R}^d$.

Using Sublemma 11, we get from (8) and (12) that for every $\psi \in C_0^\infty(\mathbf{R}^d)$, there exists a $\varrho_0 > 0$ such that

$$h^d \int_{\mathbf{R}^d} |\psi|^2 \left[\sum_{r=1}^{d} \max \left| \frac{\partial}{\partial_{\cdot r}} \psi_{\varrho\eta h\xi} \right| \right]^2 \leqq$$

$$\leqq \frac{d}{h^2} \left(\int_{\mathbf{R}^d} |\psi|^2 \right) \left[\max |\psi| + \sum_{r=1}^{d} \max \left| \frac{\partial \psi}{\partial_{\cdot r}} \right| \right]^2 \leqq$$

$$\leqq \frac{2d}{h^2} [\max |\psi|]^2 \left[\varrho^2 \int_{\mathbf{R}^d} |\psi|^2 + \sum_{r=1}^{d} \int_{\mathbf{R}^d} \left| \frac{\partial \psi}{\partial_{\cdot r}} \right|^2 - 2\varrho \left(\int_{\mathbf{R}^d} |\psi|^2 \right)^{1/2} \left(\sum_{r=1}^{d} \int_{\mathbf{R}^d} \left| \frac{\partial \psi}{\partial_{\cdot r}} \right|^2 \right)^{1/2} \right] \leqq$$

$$\leqq 2d [\max |\psi|]^2 \left[\sum_{r=1}^{d} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial_{\cdot r}} \psi_{\varrho\eta h\xi} \right|^2 \right]$$

for every $\varrho > \varrho_0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $h > 0$ and $\xi \in \mathbf{R}^d$, and this is in fact the desired result.

13. Theorem. *Let a_{ij}, $i, j \in \{1, 2, \ldots, d\}$, be complex functions on Ω, and δ a real function on the system of open bounded sets $G \subseteq \Omega$ such that $\bar{G} \subseteq \Omega$. If*

(α) *the functions a_{ij}, $i, j \in \{1, 2, \ldots, d\}$, are locally integrable in Ω,*

(β) *for every open bounded set $G \subseteq \Omega$ such that $\bar{G} \subseteq \Omega$, there exists a constant $\lambda \in \mathbf{R}$ so that*

$$\operatorname{Re} \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial \varphi}{\partial_{\cdot i}} \frac{\partial \bar{\varphi}}{\partial_{\cdot j}} + \lambda \int_\Omega |\varphi|^2 \geqq \delta(G) \sum_{r=1}^{d} \int_\Omega \left| \frac{\partial \varphi}{\partial_{\cdot r}} \right|^2$$

for every $\varphi \in C_0^\infty(\mathbf{R}^d)$ such that $\operatorname{supp}(\varphi) \subseteq G$, then for every open bounded set $G \subseteq \Omega$ such that $\bar{G} \subseteq \Omega$, we have

$$\sum_{i,j=1}^{d} \operatorname{Re} a_{ij}(\xi)\, \eta_i \eta_j \geqq \delta(G) \sum_{r=1}^{d} \eta_r^2$$

for almost every $\xi \in G$ and every $\eta_1, \eta_2, \ldots, \eta_d \in \mathbf{R}$.

Proof. Let us first recall that we can apply Lemma 5 as seen from the assumption (α), and thus we can fix a measurable set $N \subseteq \Omega$ such that

(1) $\mu(N) = 0$,

(2) $\dfrac{1}{h^d} \displaystyle\int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)| \to 0 \quad (h \to 0_+)$

for every $\xi \in \Omega \setminus N$ and $i, j \in \{1, 2, \ldots, d\}$.

Let us fix a function $\chi \in \mathbf{R}^d \to \mathbf{R}$ such that

(3) $\chi \in C_0^\infty(\mathbf{R}^d)$,

(4) $\chi \neq 0$,

(5) $\operatorname{supp}(\chi) \subseteq \{\sigma : \sigma \in \mathbf{R}^d, \max_{r \in \{1,2,\ldots,d\}} |\sigma_r| \leqq 1\}$.

It is easy to see from (5) that

(6) $\operatorname{supp}(\chi_{\varrho\eta h\xi}) \subseteq K_h(\xi)$ for every $\varrho > 0$, $\eta \in \mathbf{R}^d$, $h > 0$ and $\xi \in \mathbf{R}^d$.

Moreover, let us fix a constant a constant $h_0(\xi)$, $\xi \in \Omega$, such that

(7) $h_0(\xi) > 0$ for every $\xi \in \Omega$,

(8) $K_h(\xi) \subseteq \Omega$ for every $\xi \in \Omega$ and $0 < h \leqq h_0(\xi)$.

It follows from (6), (7) and (8) that

(9) $\operatorname{supp}(\chi_{\varrho\eta h\xi}) \subseteq \Omega$ for every $\varrho > 0$, $\eta \in \mathbf{R}^d$, $0 < h \leqq h_0(\xi)$ and $\xi \in \Omega$.

Now we get from (α), (6) and (9) that

$$
\begin{aligned}
(10)\qquad & \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} (a_{ij} - a_{ij}(\xi)) \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h\xi} = \\
&= \operatorname{Re} \sum_{i,j=1}^{d} \int_{K_h(\xi)} (a_{ij} - a_{ij}(\xi)) \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h\xi} \leqq \\
&\leqq \sum_{i,j=1}^{d} \int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)| \left|\frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi}\right| \left|\frac{\partial}{\partial_{\bullet j}} \chi_{\varrho\eta h\xi}\right| \leqq \\
&= \sum_{i,j=1}^{d} \int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)| \max \left|\frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi}\right| \max \left|\frac{\partial}{\partial_{\bullet j}} \chi_{\varrho\eta h\xi}\right| \leqq \\
&\leqq \left[\max_{i,j\in\{1,2,\ldots,d\}} \int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)|\right] \left[\sum_{i,j=1}^{d} \max \left|\frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi}\right| \max \left|\frac{\partial}{\partial_{\bullet j}} \chi_{\varrho\eta h\xi}\right|\right] = \\
&= \left[\max_{i,j\in\{1,2,\ldots,d\}} \int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)|\right] \left[\sum_{r=1}^{d} \max \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h\xi}\right|\right]^2
\end{aligned}
$$

for every $\varrho > 0$, $\eta \in \mathbf{R}^d$, $0 < h \leqq h_0(\xi)$ and $\xi \in \Omega$.

By Lemma 12, we see from (3) and (4) that we can fix a $\varrho_0 > 0$ such that

$$
\begin{aligned}
(11)\qquad & \left[\sum_{r=1}^{d} \max \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h\xi}\right|\right]^2 \leqq \\
&\leqq \frac{2d}{h^d} \frac{(\max|\chi|)^2}{\int_{\mathbf{R}^d} |\chi|^2} \sum_{r=1}^{d} \int_{\mathbf{R}^d} \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h\xi}\right|^2
\end{aligned}
$$

for every $\varrho \geqq \varrho_0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $h > 0$ and $\xi \in \mathbf{R}^d$.

It follows from (10) and (11) that

$$
\begin{aligned}
(12)\qquad & \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} (a_{ij} - a_{ij}(\xi)) \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h\xi} \leqq \\
&\leqq \frac{2d}{h^d} \frac{(\max|\chi|)^2}{\int_{\mathbf{R}^d} |\chi|^2} \left[\max_{i,j\in\{1,2,\ldots,d\}} \int_{K_h(\xi)} |a_{ij} - a_{ij}(\xi)|\right] \left[\sum_{r=1}^{d} \int_{\mathbf{R}^d} \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h\xi}\right|^2\right]
\end{aligned}
$$

for every

$$\varrho \geqq \varrho_0\,,\quad \eta \in \mathbf{R}^d\,,\quad \sum_{r=1}^{d} \eta_r^2 = 1\,,\quad 0 < h \leqq h_0(\xi)\quad \text{and}\quad \xi \in \Omega\,.$$

Let us now consider a fixed but arbitrary set G such that

(13) $G \subseteq \Omega$,

(14) $\bar{G} \subseteq \Omega$,

(15) the set G is open.

It follows from (2), (4), (7) and (15) that we can fix a constant $h(\xi, \varepsilon)$, $\xi \in G \setminus N$, $\varepsilon > 0$, such that

(16) $0 < h(\xi, \varepsilon) \leqq h_0(\xi)$ for every $\xi \in G \setminus N$ and $\varepsilon > 0$,

(17) $K_{h(\xi,\varepsilon)} \subseteq G$ for every $\xi \in G \setminus N$ and $\varepsilon > 0$,

$$\text{(18)}\qquad \max_{i,j \in \{1,2,\ldots,d\}} \left[\frac{1}{h(\xi, \varepsilon)^d} \int_{K h(\xi,\varepsilon)} |a_{ij} - a_{ij}(\xi)|\right] \leqq \varepsilon \frac{\int_{\mathbf{R}^d} |\chi|^2}{2d(\max |\chi|)^2}$$

for every $\xi \in G \setminus N$ and $\varepsilon > 0$.

As an immediate consequence of (6) and (17) we have

$$\text{(19)}\qquad \operatorname{supp}(\chi_{\varrho\eta h(\xi,\varepsilon)\xi}) \subseteq G \quad \text{for every}\quad \varrho > 0\,,\quad \eta \in \mathbf{R}^d\,,\quad \xi \in G \setminus N \quad \text{and}\quad \varepsilon > 0.$$

Now, in view of (13), (14) and (19), we see from (β) that we can fix a $\lambda \in \mathbf{R}$ such that

$$\text{(20)}\qquad \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h(\xi,\varepsilon)\xi} + \lambda \int_{\Omega} |\chi_{\varrho\eta h(\xi,\varepsilon)\xi}|^2 \geqq$$
$$\geqq \delta(G) \sum_{r=1}^{d} \int_{\mathbf{R}^d} \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi}\right|^2$$

for every $\varrho > 0$, $\eta \in \mathbf{R}^d$, $\xi \in G \setminus N$ and $\varepsilon > 0$.

On the other hand, we get from (4), (12), (16) and (18) that

$$\text{(21)}\qquad \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} (a_{ij} - a_{ij}(\xi)) \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h(\xi,\varepsilon)\xi} \leqq$$
$$\leqq \varepsilon \sum_{r=1}^{d} \int_{\mathbf{R}^d} \left|\frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi}\right|^2$$

for every $\varrho \geqq \varrho_0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $\xi \in G \setminus N$ and $\varepsilon > 0$.

$$\text{(22)}\qquad \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(\xi) \int_{\mathbf{R}^d} \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h(\xi,\varepsilon)\xi} + \lambda \int_{\mathbf{R}^d} |\chi_{\varrho\eta h(\xi,\varepsilon)\xi}|^2 =$$
$$= \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h(\xi,\varepsilon)\xi} + \lambda \int_{\Omega} |\chi_{\varrho\eta h(\xi,\varepsilon)\xi}|^2 -$$

$$- \operatorname{Re} \sum_{i,j=1}^{d} \int_{\Omega} (a_{ij} - a_{ij}(\xi)) \frac{\partial}{\partial_{\bullet i}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \frac{\partial}{\partial_{\bullet j}} \bar{\chi}_{\varrho\eta h(\xi,\varepsilon)\xi} \geqq$$

$$\geqq (\delta(G) - \varepsilon) \sum_{r=1}^{d} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial_{\bullet r}} \chi_{\varrho\eta h(\xi,\varepsilon)\xi} \right|^2$$

for every $\varrho > \varrho_0$, $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $\xi \in G \setminus N$ and $\varepsilon > 0$.

In view of (3), we can apply Lemmas 9 and 10 and we get from (22), dividing this inequality by ϱ^2 and letting $\varrho \to \infty$,

$$(23) \qquad \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(\xi) \eta_i \eta_j \frac{1}{h(\xi, \varepsilon)^2} \int_{\mathbf{R}^d} |\chi|^2 \geqq (\delta(G) - \varepsilon) \frac{1}{h(\xi, \varepsilon)^2} \int_{\mathbf{R}^d} |\chi|^2$$

for every $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $\xi \in G \setminus N$ and $\varepsilon > 0$.

Since $\int_{\mathbf{R}^d} |\chi|^2 > 0$ by (4) and $h(\xi, \varepsilon) > 0$ by (16), (23) implies

$$(24) \quad \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(\xi) \eta_i \eta_j \geqq \delta(G) - \varepsilon$$

for every $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$, $\xi \in G \setminus N$ and $\varepsilon > 0$.

It is clear that (24) is equivalent with

$$(25) \quad \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(\xi) \eta_i \eta_j \geqq \delta(G)$$

for every $\eta \in \mathbf{R}^d$, $\sum_{r=1}^{d} \eta_r^2 = 1$ and $\xi \in G \setminus N$.

Let us finally write (25) in the form

$$(26) \quad \operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(\xi) \eta_i \eta_j \geqq \delta(G) \sum_{r=1}^{d} \eta_r^2$$

for every $\eta \in \mathbf{R}^d$ and $\xi \in G \setminus N$.

Now the desired statement follows from (1) and (26).

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Souhrn

K OBRÁCENÍ GÅRDINGOVY VĚTY

Miroslav Sova

Obrácení Gårdingovy věty je dokázáno pro koeficienty pouze lokálně integrovatelné.

Резюме

ОБ ОБРАЩЕНИИ ТЕОРЕМЫ ГОРДИНГА

MIROSLAV SOVA

Доказано обращение теоремы Гординга для коэффициентов, которые интегрируемы лишь локально.

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