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ON KNESER PROBLEM FOR DIFFERENTIAL EQUATIONS OF THE 3RD ORDER

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Summary. In the paper sufficient conditions are found for the existence of a solution u of the third order nonlinear differential equation, satisfying $u(t) \geqq 0$, $u'(t) \leqq 0$, $u''(t) \geqq 0$ for $t \in \langle 0, \infty)$ and $\varphi(u(0), u'(0), u''(0)) = 0$, where φ is a continuous function.

Keywords: Kneser problem, a priori estimate, Carathéodory conditions, Arzelà-Ascoli theorem, Nagumo functions.

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1. INTRODUCTION

In this paper we consider the problem

(1) $$u''' = f(t, u, u', u''),$$

(2) $$u(t) \geqq 0, \quad u'(t) \leqq 0, \quad u''(t) \geqq 0 \quad \text{for} \quad t \in R_+,$$

(3) $$\varphi(u(0), u'(0), u''(0)) = 0.$$

Sufficient conditions are found for the existence of solutions of this problem.

We shall use the following notation:

$$R_+ = \langle 0, \infty), \quad R_- = (-\infty, 0\rangle, \quad D = R_+ \times R_- \times R_+, \quad J \subset R,$$

$C(J)$ is the set of all real continuous functions on J,

$AC^2(J)$ is the set of all real functions which are absolutely continuous with their second order derivatives on J,

$L(J)$ is the set of all real Lebesgue integrable on J functions,

a.e. = almost every,

$L_{loc}(J)$ is the set of all real functions which are Lebesgue integrable on each segment contained in J,

$\mathrm{Car}_{loc}(J \times I)$ is the set of all functions $f: J \times I \to R$ satisfying the local Carathéodory conditions on $J \times I$, i.e.

(i) for each $(x_1, x_2, x_3) \in I$, the mapping $t \mapsto f(t, x_1, x_2, x_3)$ is Lebesgue measurable on J,

(ii) for a.e. $t \in J$, the mapping $(x_1, x_2, x_3) \mapsto f(t, x_1, x_2, x_3)$ is continuous on I,

(iii) for each $\varrho > 0$ there exists $h_\varrho \in L_{\text{loc}}(J)$ such that

$$\sum_{i=1}^{3} |x_i| \leqq \varrho \Rightarrow |f(t, x_1, x_2, x_3)| \leqq h_\varrho(t) \quad \text{on} \quad I \times J .$$

A function $u \in AC^2(R_+)$ which fulfils (1) for a.e. $t \in R_+$ and satisfies (2), (3) for each $t \in R_+$ will be called a solution of the problem (1), (2), (3).

In what follows we shall assume

(4) $$f \in \text{Car}_{\text{loc}}(R_+ \times D), \; f(t, 0, 0, 0) = 0, \; f(t, x_1, x_2, 0) \leqq 0 \; \text{ on } \; R_+ \times D$$

$$(\text{which means for a.e. } t \in R_+ \text{ for every } x_1 \in R_+, \; x_2 \in R_-),$$

(5) $$\varphi \in C(D), \quad \varphi(0, 0, 0) < 0 .$$

Moreover, φ will satisfy exactly one of the following conditions:

$(\varphi 1)$ $\varphi(x_1, x_2, x_3) > 0$ for $x_1 > r$,

$(\varphi 2)$ $\varphi(x_1, x_2, x_3) > 0$ for $|x_2| > r$,

$(\varphi 3)$ $\varphi(x_1, x_2, x_3) > 0$ for $x_3 > r$,

$(\varphi 4)$ $\varphi(x_1, x_2, x_3) > 0$ for $x_1 + |x_2| > r$,

$(\varphi 5)$ $\varphi(x_1, x_2, x_3) > 0$ for $x_1 + x_3 > r$,

$(\varphi 6)$ $\varphi(x_1, x_2, x_3) > 0$ for $|x_2| + x_3 > r$,

$(\varphi 7)$ $\varphi(x_1, x_2, x_3) > 0$ for $x_1 + |x_2| + x_3 > r$,

where $r \in (0, \infty)$.

Remark. a) Clearly

$(\varphi 4) \Rightarrow (\varphi 1), (\varphi 2)$,

$(\varphi 5) \Rightarrow (\varphi 1), (\varphi 3)$,

$(\varphi 6) \Rightarrow (\varphi 2), (\varphi 3)$,

$(\varphi 7) \Rightarrow (\varphi 4), (\varphi 5), (\varphi 6)$.

b) In the special case $\varphi(x_1, x_2, x_3) = x_1 - r$ the condition (3) reduces to $u(0) = r$. In this case φ satisfies $(\varphi 1)$. Similarly for $\varphi(x_1, x_2, x_3) = |x_2| - r$ the condition (3) reduces to $u'(0) = -r$ and φ satisfies $(\varphi 2)$, and so on.

c) Similar problems for differential equations of n-th order and differential systems were solved in [1–10]. Here, for $n = 3$, stronger results are obtained.

2. THE MAIN RESULTS

From now on we shall assume that

(6) $a \in (0, \infty)$, $\alpha \in R_+$, $k_1, k_2 \in N$, $h_0, h_1, h_2 \in L_{\mathrm{loc}}(\langle a, \infty))$,

$\omega \in C(R_+)$ is a positive function and $\int_0^\infty \frac{ds}{\omega(s)} = +\infty$,

$$\Omega(x) = \int_0^x \frac{ds}{\omega(s)},$$

(7) $\delta(t, \cdot)$ is nondecreasing for any $t \in \langle 0, a \rangle$,

$\delta(\cdot, x) \in L(\langle 0, a \rangle)$ is nonnegative for any $x \in R_+$.

Theorem 1. *Let* (4), (5), (6), (φ1) *be fulfilled, let* $h \in L(\langle 0, a \rangle)$ *be a positive function and*

(8) $$\int_0^a \frac{t\, dt}{H(t)} = +\infty \quad \textit{where} \quad H(t) = \int_0^t h(\tau)\, d\tau.$$

Further, let

(9) $$-h(t)(1 + x_3)^2 \leqq f(t, x_1, x_2, x_3) \leqq 0$$

for any $(t, x_1, x_2, x_3) \in \langle 0, a \rangle \times \langle 0, r \rangle \times R_- \times R_+$,

(10) $$f(t, x_1, x_2, x_3) \leqq \left[h_0(t) + \sum_{i=1}^{2} h_i(t) |x_i|^{k_i} + \alpha x_3\right] \omega(x_3)$$

for any $(t, x_1, x_2, x_3) \in \langle a, \infty) \times \langle 0, r \rangle \times R_- \times R_+$.

Then the problem (1), (2), (3) *has at least one solution.*

Remark. Other existence theorems with the assumption (φ1) can be found in [11].

Theorem 2. *Let* (4), (5), (6), (7), (φ2) *be fulfilled and*

(11) $$\lim_{x \to \infty} \int_0^a t\delta(t, x)\, dt > r.$$

Let there exist $a_0 \in (0, \infty)$, $a_0 < a$ *and a positive function* $h \in L(\langle 0, a_0 \rangle)$ *such that*

(12) $$\int_0^{a_0} \frac{dt}{H(t)} = +\infty, \quad \textit{where} \quad H(t) = \int_0^t h(\tau)\, d\tau.$$

Further, let

(13) $$f(t, x_1, x_2, x_3) \leqq -\delta(t, x_1)$$

for any $(t, x_1, x_2, x_3) \in \langle 0, a \rangle \times R_+ \times \langle -r, 0 \rangle \times R_+$,

and let on the set $\langle 0, a_0 \rangle \times R_+ \times \langle -r, 0 \rangle \times R_+$ *the inequality* (9) *and on the set* $(a, \infty) \times R_+ \times \langle -r, 0 \rangle \times R_+$ *the inequality* (10) *be satisfied. Then the problem* (1), (2), (3) *has at least one solution.* (The theorem is proved in [12].)

Theorem 3. *Let* (4), (5), (6), (7), (φ3) *be fulfilled and*

$$(14)\qquad \lim_{x\to\infty}\int_0^a \delta(t,x)\,\mathrm{d}t > r\,.$$

Let us suppose that on the set $\langle 0, a\rangle \times R_+ \times R_- \times R_+$ *the inequality* (13) *and on the set* $(a, \infty) \times R_+ \times R_- \times R_+$ *the inequality* (10) *are satisfied.*
Then the problem (1), (2), (3) *has at least one solution.*

Theorem 4. *Let* (4), (5), (6), (8), (φ4) *be fulfilled. Let us suppose that on the set* $\langle 0, a\rangle \times \langle 0, r\rangle \times \langle -r, 0\rangle \times R_+$ *the inequality* (9) *and on the set* $\langle a, \infty) \times \langle 0, r\rangle \times \langle -r, 0\rangle \times R_+$ *the inequality* (10) *are satisfied.*
Then (1), (2), (3) *is solvable.*

Theorem 5. *Let* (4), (5), (6), (φ5) *be fulfilled. Let*

$$(15)\qquad f(t, x_1, x_2, x_3) \leqq 0 \quad \text{for any} \quad (t, x_1, x_2, x_3) \in \langle 0, a\rangle \times \langle 0, r\rangle \times R_- \times R_+\,.$$

and let (10) *be satisfied for any* $(t, x_1, x_2, x_3) \in \langle a, \infty) \times \langle 0, r\rangle \times R_- \times R_+$.
Then (1), (2), (3) *is solvable.*

Theorem 6. *Let* (4), (5), (6), (7), (14), (φ6) *be fulfilled. Let us suppose that* (13) *is satisfied on the set* $\langle 0, a\rangle \times R_+ \times \langle -r, 0\rangle \times R_+$ *and* (10) *is satisfied on the set* $(a, \infty) \times R_+ \times \langle -r, 0\rangle \times R_+$.
Then (1), (2), (3) *is solvable.*

Theorem 7. *Let* (4), (5), (6), (φ7) *be fulfilled, let* (15) *be satisfied on* $\langle 0, a\rangle \times \langle 0, r\rangle \times \langle -r, 0\rangle \times R_+$ *and* (10) *on* $\langle a, \infty) \times \langle 0, r\rangle \times \langle -r, 0\rangle \times R_+$.
Then (1), (2), (3) *is solvable.*

Remark. The assumption (13) in Theorems 2, 3, 6 is essential and cannot be omitted. For example, the problems

$$u''' = 0\,,\quad u(t) \geqq 0\,,\quad u'(t) \leqq 0\,,\quad u''(t) \geqq 0\,,\quad u'(0) = -r\,,$$

or

$$u''' = 0\,,\quad u(t) \geqq 0\,,\quad u'(t) \leqq 0\,,\quad u''(t) \geqq 0\,,\quad u''(0) = r\,,$$

or

$$u''' = 0\,,\quad u(t) \geqq 0\,,\quad u'(t) \leqq 0\,,\quad u''(t) \geqq 0\,,\quad u''(0) + |u'(0)| = r$$

have no solution although the function $f(t, x_1, x_2, x_3) = 0$ satisfies all assumptions of Theorem 2 or 3 or 6 except (13).

If the function f is nonpositive, i.e. satisfies

$$(4n)\qquad f \in \mathrm{Car}_{\mathrm{loc}}(R_+ \times D)\,,\ f(t, 0, 0, 0) = 0\,,\ f(t, x_1, x_2, x_3) \leqq 0 \ \text{on}\ R_+ \times D$$

instead of (4), we obtain the following corolaries.

Corollary 1. *Let* (4*n*), (5), ($\varphi 1$) *be fulfilled. Let there exist* $a \in (0, \infty)$ *and a positive function* $h \in L(\langle 0, a\rangle)$ *satisfying* (8) *such that* (9) *is fulfilled on* $\langle 0, a\rangle \times \langle 0, r\rangle \times R_- \times R_+$.
Then (1), (2), (3) *is solvable.*

Corollary 2. *Let* (4*n*), (5), (7), (11), (12), ($\varphi 2$) *be fulfilled. Let* (13) *be satisfied on* $\langle 0, a\rangle \times R_+ \times \langle -r, 0\rangle \times R_+$ *and* (9) *on* $\langle 0, a_0\rangle \times R_+ \times \langle -r, 0\rangle \times R_+$, *where* $a_0 \in (0, a)$.
Then (1), (2), (3) *is solvable.*

Corollary 3. *Let* (4*n*), (5), (7), (14), ($\varphi 3$) *be fulfilled and let on the set* $\langle 0, a\rangle \times R_+ \times R_- \times \langle 0, r\rangle$ *the inequality* (13) *be satisfied.*
Then (1), (2), (3) *is solvable.*

Corollary 4. *Let* (4*n*), (5), (8), ($\varphi 4$) *be fulfilled and on the set* $\langle 0, a\rangle \times \langle 0, r\rangle \times \langle -r, 0\rangle \times R_+$ *let the inequality* (9) *be satisfied.*
Then (1), (2), (3) *is solvable.*

Corollary 5. *Let* (4*n*), (5), ($\varphi 5$) *be fulfilled. Then* (1), (2), (3) *is solvable.*

Corollary 6. *Let* (4*n*), (5), (7), (14), ($\varphi 6$) *be fulfilled and let* (13) *be satisfied on the set* $\langle 0, a\rangle \times R_+ \times \langle -r, 0\rangle \times \langle 0, r\rangle$. *Then* (1), (2), (3) *is solvable.*

Corollary 7. *Let* (4*n*), (5), ($\varphi 7$) *be fulfilled. Then* (1), (2), (3) *is solvable.*

3. PROOFS

To prove the above theorems we need some lemmas.

Lemma 1. *Let* (4), (5) *and* (φi) *be fulfilled, where* $i \in \{1, 2, 3, 4, 5, 6, 7\}$. *Suppose that*

$$|f(t, x_1, x_2, x_3)| \leqq f^*(t)$$

holds on the set $R_+ \times D$, *where* $f^* \in L_{loc}(R_+)$.
Then for any $c \in (0, \infty)$ *the boundary value problem*

$$u''' = f(t, u, u', u''),$$
$$\varphi(u(0), u'(0), u''(0)) = 0, \quad u(c) = u'(c) = 0$$

has at least one solution $u \in AC^2(\langle 0, c\rangle)$ *satisfying*

$$u(t) \geqq 0, \quad u'(t) \leqq 0, \quad u''(t) \geqq 0 \quad on \quad \langle 0, c\rangle.$$

Proof. Lemma 1 can be proved analogously to Lemma 3 in [12].

Lemma 2. *Let $c > 0$ and let $v \in C^2(\langle 0, c \rangle)$ be such that*

$$v(t) \geqq 0\,, \quad v'(t) \leqq 0\,, \quad v''(t) \geqq 0 \quad for \quad 0 \leqq t \leqq c\,.$$

Then the inequality

$$|v'(t)| \leqq v(0)/c + \sqrt{(2v(t)\, w(t))} \quad for \quad 0 \leqq t \leqq c$$

where $w(t) = \max\{|v''(s)| : t \leqq s \leqq c\}$ holds.

Proof. See Lemma 3 in [11].

Proof of Theorem 1. Without loss of generality we may assume that h_j ($j = 0, 1, 2$) are nonnegative functions.

First, suppose that there exists $f^* \in L_{loc}(R_+)$ such that

$$|f(t, x_1, x_2, x_3)| \leqq f^*(t) \quad \text{on} \quad R_+ \times D\,. \tag{16}$$

Then for any $p \in N$ the boundary value problem (1), (3)

$$u(a + p) = u'(a + p) = 0 \tag{17}$$

has at least one solution $u \in AC^2(\langle 0, a + p \rangle)$ satisfying

$$u(t) \geqq 0\,, \quad u'(t) \leqq 0\,, \quad u''(t) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq a + p\,. \tag{18}$$

(See Lemma 1.)

From (3), (φ1) and (18) it follows that

$$0 \leqq u(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a + p \tag{19}$$

and

$$u(0) = u(a) + a|u'(a)| + \int_0^a t\, u''(t)\, dt\,,$$

which implies

$$\int_0^a t\, u''(t)\, dt \leqq r\,. \tag{20}$$

By (9) we have

$$(1 + u''(t))' \geqq -h(t)\,(1 + u''(t))^2 \quad \text{for} \quad 0 \leqq t \leqq a\,. \tag{21}$$

Integrating the differential equation

$$z'(t) = -h(t)\, z^2(t)\,, \quad 0 \leqq t \leqq a\,, \tag{22}$$

we get $z(t) = (1/z(0) + H(t))^{-1}$ and by virtue of (8) there exist $\varepsilon \in (0, 1)$ and $a_0 \in (0, a)$ such that $\int_{a_0}^a t(z(t) - 1)\, dt > r$, where $z(0) = 1/\varepsilon$. Let us suppose that $1 + u''(t) \geqq z(t)$ for $a_0 \leqq t \leqq a$. Then $\int_{a_0}^a t\, u''(t)\, dt > r$ which contradicts (20). Thus it is necessary that there exist $t_0 \in (a_0, a)$ such that

$$1 + u''(t_0) < z(t_0)\,. \tag{23}$$

Now, from (21), (22), (23) by Chaplygin Lemma on differential inequalities (see [5]) we get $1 + u''(t) \leqq 1/\varepsilon$ for $0 \leqq t \leqq t_0$, and by (9)

$$(24) \qquad u''(t) \leqq r_1 \quad \text{for} \quad 0 \leqq t \leqq a\,, \quad \text{where} \quad r_1 = 1/\varepsilon - 1\,.$$

Using Lemma 2 and taking into account (18), (19), (24) we obtain

$$(25) \qquad |u'(t)| \leqq r_2 \quad \text{for} \quad 0 \leqq t \leqq a + p\,, \quad \text{where} \quad r_2 = r/a + \sqrt{(2rr_1)}\,.$$

By (10) we have

$$(u''(t))' \leqq [h_0(t) + \sum_{i=1}^{2} h_i(t)\, |u^{(i-1)}(t)|^{k_i} + \alpha\, u''(t)]\, \omega(u''(t))$$

and integrating from a to t and using (19), (24), (25) we get

$$(26) \qquad u''(t) \leqq \varrho(t) \quad \text{for} \quad a \leqq t \leqq a + p\,,$$

where $\varrho(t) = \Omega^{-1}(\Omega(r_1) + \alpha r_2 + (r^{k_1} + r_2^{k_2} + 1) \int_a^t \sum_{i=0}^{2} h_i(\tau)\, d\tau)$. Now, if f does not satisfy (16), we put

$$\sigma(t) = \begin{cases} r + r_2 + r_1 & \text{for} \quad 0 \leqq t \leqq a \\ r + r_2 + \varrho(t) & \text{for} \quad a < t \leqq a + p\,, \end{cases}$$

$$\chi(t, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq \sigma(t) \\ 2 - s/\sigma(t) & \text{for} \quad \sigma(t) \leqq s \leqq 2\,\sigma(t) \\ 0 & \text{for} \quad 2\,\sigma(t) \leqq s\,, \end{cases}$$

$$\tilde{f}(t, x_1, x_2, x_3) = \chi(t, \sum_{i=1}^{3} |x_i|)\, f(t, x_1, x_2, x_3)\,.$$

Since $\tilde{f}$ satisfies (16) and all assumptions of Theorem 1, the boundary value problem

$$u''' = \tilde{f}(t, u, u', u'')\,, \quad (3), (17)$$

has at least one solution u_p satisfying (18), (19), (24), (25), (26) and so

$$(27) \qquad \sum_{i=1}^{3} |u_p^{(i-1)}(t)| \leqq \sigma(t) \quad \text{for} \quad 0 \leqq t \leqq a + p\,.$$

Thus u_p is also a solution of the problem (1), (3), (17) on $\langle 0, a + p \rangle$. Now, denote

$$f_p(t, x_1, x_2, x_3) = \begin{cases} f(t, x_1, x_2, x_3) & \text{for} \quad 0 \leqq t \leqq a + p \\ 0 & \text{for} \quad t > a + p\,. \end{cases}$$

Then $|f_p(t, x_1, x_2, x_3)| \leqq |f(t, x_1, x_2, x_3)|$ for any $p \in N$ and $\lim_{p \to \infty} f_p(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3)$ on $R_+ \times D$. Since

$$\sup \{ \sum_{i=1}^{3} |u_p^{(i-1)}(t)| : p \in N \} \leqq \sigma(t) \quad \text{for} \quad t \in R_+\,,$$

we can prove by the Arzelà-Ascoli theorem that the sequence $\{u_p\}_{p=1}^{\infty}$ contains a subsequence $\{u_{p_j}\}_{j=1}^{\infty}$ which is locally uniformly converging together with $\{u'_{p_j}\}_{j=1}^{\infty}$ and $\{u''_{p_j}\}_{j=1}^{\infty}$ on R_+, and $u(t) = \lim_{j \to \infty} u_{p_j}(t)$ is a solution of (1), (2), (3) on R_+.

Proof of Theorem 3. The first part of this proof is similar to that of Theorem 1 and $\boldsymbol{u}$ denotes again a solution of (1), (3), (17) satisfying (18).

Now, let us choose $c_0 \in (r, \infty)$ and a function δ_0 satisfying (7) and (14) such that $\delta(t, x) \geqq \delta_0(t, x)$ on $\langle 0, a\rangle \times R_+$ and $\delta_0(t, x) = \delta_0(t, c_0)$ on $\langle 0, a\rangle \times \langle c_0, \infty)$. From ($\varphi$3), (13) and (18) it follows that

(28) $$u''' \leqq -\delta(t, u) \leqq -\delta_0(t, u)\,,$$

(29) $$0 \leqq u''(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a\,.$$

According to (14) there exist $r_0 \in \langle r, \infty)$ and $a_0 \in (0, a)$ such that $\int_0^{a_0} \delta_0(t, r_0)\,\mathrm{d}t > r$. Integrating (28) we obtain by (29) $\int_0^{a_0} \delta_0(t, u(a_0))\,\mathrm{d}t \leqq r$. Therefore $u(a_0) < r_0$ and by (18) we get

(30) $$0 \leqq u(t) \leqq r_0 \quad \text{for} \quad a_0 \leqq t \leqq a + p\,, \quad p \in N\,.$$

The equality $u(a_0) = u(a) + |u'(a)|\,(a - a_0) + \int_{a_0}^a (t - a_0)\,u''(t)\,\mathrm{d}t$ yields

(31) $$|u'(a)| \leqq r_0/(a - a_0)\,.$$

From the equality $u(0) = u(a) + |u'(a)|\,a + \int_0^a t\,u''(t)\,\mathrm{d}t$ we get by (29), (30) and (31) $u(0) \leqq r_1$, where $r_1 = r_0 + ar_0/(a - a_0) + a^2 r$, thus

(32) $$0 \leqq u(t) \leqq r_1 \quad \text{for} \quad 0 \leqq t \leqq a + p\,.$$

Now, using Lemma 2, we obtain by (18)

(33) $$|u'(t)| \leqq r_2 \quad \text{for} \quad 0 \leqq t \leqq a + p\,, \quad \text{where} \quad r_2 = r_1/a + \sqrt{(2r_1 r)}\,.$$

Similarly as in the proof of Theorem 1 we obtain from (10)

(34) $$u''(t) \leqq \varrho(t) \quad \text{for} \quad a \leqq t \leqq a + p\,,$$

where

$$\varrho(t) = \Omega^{-1}\Big(\Omega(r) + \alpha r_2 + (r_1^{k_1} + r_2^{k_2} + 1) \int_a^t \sum_{i=0}^{2} h_i(\tau)\,\mathrm{d}\tau\Big)\,.$$

Now, if f does not satisfy (16), we put

$$\sigma(t) = \begin{cases} r_1 + r_2 + r & \text{for} \quad 0 \leqq t \leqq a \\ r_1 + r_2 + \varrho(t) & \text{for} \quad a < t \leqq a + p\,, \end{cases} \qquad c_1 = \max\{c_0, r_1\}\,,$$

$$\sigma_1(s) = \begin{cases} s & \text{for} \quad 0 \leqq s \leqq c_1\,, \\ c_1 & \text{for} \quad s > c_1\,, \end{cases} \qquad \sigma_2(s) = \begin{cases} s & \text{for} \quad -r_2 \leqq s \leqq 0 \\ -r_2 & \text{for} \quad s < -r_2\,, \end{cases}$$

$$\sigma_3(t, s) = \begin{cases} s & \text{for} \quad 0 \leqq s \leqq \varrho(t) \\ \varrho(t) & \text{for} \quad \varrho(t) < s\,, \end{cases}$$

$$\chi(t, s) = \begin{cases} 1 & \text{for} \quad 0 \leqq s \leqq \sigma(t) \\ 2 - s/\sigma(t) & \text{for} \quad \sigma(t) < s \leqq 2\,\sigma(t)\,, \\ 0 & \text{for} \quad 2\,\sigma(t) < s \end{cases}$$

$$\tilde{f}(t, x_1, x_2, x_3) = \begin{cases} f(t, \sigma_1(x_1), \sigma_2(x_2), \sigma_3(t, x_3)) & \text{for} \quad 0 \leqq t \leqq a \\ \chi(t, \sum_{i=1}^{3} |x_i|)\, f(t, x_1, x_2, x_3) & \text{for} \quad a < t \leqq a + p\,. \end{cases}$$

Clearly $\tilde{f}$ satisfies (4), (10) and (16). Further,

$$(35) \qquad \tilde{f}(t, x_1, x_2, x_3) = f(t, x_1, x_2, x_3) \quad \text{for} \quad t > a\,, \quad \sum_{i=1}^{3} |x_i| \leqq \sigma(t)\,,$$

and for

$$(t, x_1, x_2, x_3) \in \langle 0, a \rangle \times \langle 0, c_1 \rangle \times \langle -r_2, 0 \rangle \times \langle 0, \varrho(t) \rangle$$

we have

$$(36) \qquad \tilde{f}(t, x_1, x_2, x_3) \leqq -\delta(t, \sigma_1(x_1)) \leqq -\delta_0(t, x_1) \quad \text{on} \quad \langle 0, a \rangle \times D\,.$$

Therefore the boundary value problem

$$u''' = \tilde{f}(t, u, u', u'')\,, \quad (3),\ (17)$$

has at least one solution $u_p \in AC^2(\langle 0, a + p \rangle)$ satisfying (18), (28)–(34) and so u_p is also a solution of (1), (3), (17) on $\langle 0, a + p \rangle$. The last part of this proof is the same as in the proof of Theorem 1.

Proof of Theorem 4. The difference between the assumptions of Theorems 1 and 4 is only in the boundedness of x_2. So we can prove Theorem 4 in the same way as Theorem 1 because the boundedness of u', where u is a solution of (1), (3), (17), follows from (φ4).

Proof of Theorem 5. Similarly as in the proof of Theorem 1 we can obtain a solution u of (1), (3), (17) satisfying (18). From (φ5), (15) and (18) it follows that

$$(37) \quad 0 \leqq u(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a + p\,, \quad 0 \leqq u''(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a\,.$$

Using Lemma 2 and taking into account (18), (37) $|u'(t)| \leqq r_1$ for $0 \leqq t \leqq a + p$, where $r_1 = r/a + 2r$. Now we can proceed as in the proof of Theorem 1.

Proof of Theorem 6. Similarly as in the proof of Theorem 1 we can obtain a solution u of (1), (3), (17) satisfying (18). From (φ6), (13) and (18) it follows that

$$0 \geqq u'(t) \geqq -r \quad \text{for} \quad 0 \leqq t \leqq a + p, \quad 0 \leqq u''(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a\,.$$

Analogously as in the proof of Theorem 3 we choose $c_0 \in (r, \infty)$ and a function δ_0 and get the estimate (32). Now we can proceed as in the proof of Theorem 1.

Proof of Theorem 7. Similarly as in the proof of Theorem 1 we obtain a solution u of (1), (3), (17) satisfying (18). From (φ7), (15) and (18) it follows that

$$0 \leqq u(t) \leqq r\,, \quad -r \leqq u'(t) \leqq 0 \quad \text{for} \quad 0 \leqq t \leqq a + p\,,$$
$$0 \leqq u''(t) \leqq r \quad \text{for} \quad 0 \leqq t \leqq a\,.$$

As in the proof of Theorem 1 we obtain from (10) the estimate $0 \leqq u''(t) \leqq \varrho(t)$ for $a \leqq t \leqq a + p$, where

$$\varrho(t) = \Omega^{-1}(\Omega(r) + \alpha r + (r^{k_1} + r^{k_2} + 1) \int_a^t \sum_{i=0}^{2} h_i(\tau)\, d\tau)\,.$$

The rest of the proof is analogous to that of Theorem 1.

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Souhrn

KNESEROVA ÚLOHA PRO DIFERENCIÁLNÍ ROVNICE 3. ŘÁDU

IRENA RACHŮNKOVÁ

V práci jsou nalezeny postačující podmínky pro existenci řešení u nelineární diferenciální rovnice 3. řádu, splňujícího podmínky $u(t) \geqq 0$, $u'(t) \leqq 0$, $u''(t) \geqq 0$ pro $t \in \langle 0, \infty)$ a $\varphi(u(0), u'(0), u''(0)) = 0$, kde φ je spojitá funkce.

Резюме

ЗАДАЧА КНЕЗЕРА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 3-ГО ПОРЯДКА

IRENA RACHŮNKOVÁ

В работе приведены достаточные условия для существования решения u нелинейного дифференциального уравнения третьего порядка, удовлетворяющего условиям $u(t) \geqq 0$, $u'(t) \leqq 0$, $u''(t) \geqq 0$ для $t \in \langle 0, \infty)$ и $\varphi(u(0), u'(0), u''(0)) = 0$, где φ — непрерывная функция.

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