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WEAK HOMOMORPHISMS IN IMPLICATION ALGEBRA

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Summary. It is proven that the only surjective (semi-)weak homomorphisms of implication algebras are usual homomorphisms.

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Denote by $\mathrm{A} = (A, F)$ an algebra with a support A and a set F of (fundamental) operations. Given two algebtas $\mathrm{A} = (A, F)$, $\mathrm{B} = (B, G)$, a mapping $h: A \to B$ is called a *semi-weak homomorphism* (see [5]) if for each n-ary operation $f \in F$ there is an n-ary term g of B such that

$$(*) \qquad h(f(a_1, \ldots, a_n)) = g(h(a_1), \ldots, h(a_n)) \quad \text{for all} \quad a_i \in A .$$

If, in addition, for each n-ary operation $g \in G$ there exists an n-ary term f of A such that $(*)$ holds, h is called a *weak homomorphism.*

The concept of weak homomorphism was introduced by A. Goetz [4] and E. Marczewski [6] and intensively studied by some authors, see e.g. [2], [3], [4], [5], [7] and the references there. For some classes of algebras, the concept of (semi-)weak homomorphism coincides with the concept of homomorphism (or its dual). Especially, K. Głazek, J. Michalski, A. Goetz, T. Katriňák, T. Traczyk and M. Kolibiar gave a number of such classes among Boolean and Post algebras, p-algebras, lattices, integral domains, groups, semigroups and median algebras. The aim of this short note is to describe (semi-)weak homomorphisms in implication algebras (see [1]).

Definition. An algebra $\mathrm{A} = (A, \{\cdot\})$ with one binary operation is an *implication algebra* if it satisfies

$$(a \cdot b) \cdot a = a ,$$
$$(a \cdot b) \cdot b = (b \cdot a) \cdot a ,$$
$$a \cdot (b \cdot c) = b \cdot (a \cdot c)$$

for every elements a, b, c of A.

Lemma 1. *Let* $\mathrm{A} = (A, F)$ *and* $\mathrm{B} = (B, G)$ *be algebras. Any surjective (semi-)weak homomorphism* $h: A \to B$ *can be expressed in the form* $h = g \cdot i$, *where*

$g: (A, F) \to (B, F)$ is a (usual) homomorphism and $i: (B, F) \to (B, G)$ is a bijective (semi-)weak homomorphism; i is the identity map on B.

For the proof, see e.g. Lemma 2.2 and 2.4 in [5] or [10], p. 223.

Remark. Lemma 1 shows that investigations of (semi-)weak homomorphisms can be limited to usual homomorphisms and (semi-)weak homomorphisms of the form $i: (B, F) \to (B, G)$.

Lemma 2. *Every implication algebra* A *contains a constant* 1 *satisfying*

$$a \,.\, a = 1\,,$$

$$1 \,.\, a = a\,,$$

$$a \,.\, 1 = 1$$

for each $a \in A$.

For the proof, see e.g. Theorem 1 in [1].

Lemma 3. *The free implication algebra with two free generators x, y has exactly six elements, namely*

$$x,\ y,\ 1,\ x \,.\, y,\ y \,.\, x,\ (x \,.\, y) \,.\, y\,.$$

Lemma 3 can be easily proved by using the axioms from Definition and by Lemma 2. For full details, see Theorem 2 in [1].

Theorem. *The only surjective (semi-)weak homomorphisms of implication algebras are usual homomorphisms.*

Proof. By Lemma 1, it suffices to prove the assertion only for bijective (semi-)weak homomorphisms of A onto A. Let A be an implication algebra with at least two elements and h: A $\to$ A a bijective (semi-)weak homomorphism. Evidently, $h(1) = 1$. By Lemma 3, there are only six binary terms in A, i.e. there exist only six possibilities how to map the binary operation, namely

$$x \,.\, y \to x\,,$$

$$x \,.\, y \to y\,,$$

$$x \,.\, y \to 1\,,$$

$$x \,.\, y \to y \,.\, x\,,$$

$$x \,.\, y \to (x \,.\, y) \,.\, y\,,$$

$$x \,.\, y \to x \,.\, y\,.$$

(1) Try the case $x \,.\, y \to x$. Then $h(x \,.\, y) = h(x)$, i.e., for the choice $y = 1$ we obtain $h(x) = h(x \,.\, 1) = h(1) = 1$. Since h is bijective, this implies card A $= 1$, which is a contradiction.

(2) In the case $x \,.\, y \to y$, put $x = y$. We obtain (by Lemma 2) $h(1) = h(x \,.\, x) = h(x)$, also card A $= 1$, a contradiction.

(3) The case $x \,.\, y \to 1$ is clearly contradictory.

(4) Suppose $x \,.\, y \to y \,.\, x$. For the choice $y = 1$ we have

$$1 = h(1) = h(x \,.\, 1) = h(1) \,.\, h(x) = 1 \,.\, h(x) = h(x)\,,$$

which is also a contradiction.

(5) Suppose $x \,.\, y \to (x \,.\, y) \,.\, y$ and put $x = 1$. We obtain

$$h(y) = h(1 \,.\, y) = (h(1) \,.\, h(y)) \,.\, h(y) = (1 \,.\, h(y)) \,.\, h(y) = h(y) \,.\, h(y) = 1\,,$$

again card A = 1, a contradiction.

The last possibility $x \,.\, y \to x \,.\, y$ gives the usual homomorphism.

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Souhrn

SLABÉ HOMOMORFISMY IMPLIKATIVNÍCH ALGEBER

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Je dokázáno, že jedinými surjektivními (polo-)slabými homomorfismy na implikativních algebrách jsou homomorfismy.

Резюме

СЛАБЫЕ ГОМОМОРФИЗМЫ ИМПЛИКАТИВНЫХ АЛГЕБЕР

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Показано, что суръективные (полу-)слабые гомоморфизмы в импликативных алгебрах совпадают с гомоморфизмами.

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