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Časopis pro pěstování matematiky, Vol. 115 (1990), No. 1, 92--99

Persistent URL: `http://dml.cz/dmlcz/108722`

OSCILLATION BEHAVIOUR OF SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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(Received January 5, 1988)

Summary. In the present paper we study oscillatory behaviour of solutions of the neutral delay differential equation

$$\frac{d}{dt}[x(t) - \sum_{i=1}^{n} p_i(t)\, x(t - a_i)] + q_0(t)\, x(t) + \sum_{j=1}^{m} q_j(t)\, x(t - b_j) = 0\,, \quad t \geqq t_0\,.$$

We generalize the results of [3] for the equation

$$\frac{d}{dt}[x(t) - px(t - \tau)] + Q(t)\, x(t - \sigma) = 0\,, \quad t \geqq t_0\,,$$

where p, τ and σ are positive constants, $Q \in C([t_0, \infty), \mathbf{R}^+)$.

Keywords. Neutral delay differential equation; oscillatory solution; nonoscillatory solution.

AMS classification. 34K15, 34C10.

1. INTRODUCTION

This paper deals with the oscillatory behaviour of solutions of linear neutral delay differential equations in the form

$$\text{(1)} \qquad \begin{aligned} &\frac{d}{dt}[x(t) - \sum_{i=1}^{n} p_i(t)\, x(t - a_i)] + q_0(t)\, x(t) + \\ &+ \sum_{j=1}^{m} q_j(t)\, x(t - b_j) = 0\,, \quad t \geqq t_0\,, \end{aligned}$$

where ($\mathbf{R}^+ = [0, \infty)$)

(i) $p_i, q_j \in C'([t_0, \infty), \mathbf{R}^+)$ ($i = 1, 2, \ldots, n$; $j = 0, 1, \ldots, m$);

(ii) $\lim_{t\to\infty} \sum_{i=1}^{n} p_i(t) =: p$, $\lim_{t\to\infty} p_{i_0}(t) > 0$ exist, where $i_0 \in \{1, 2, \ldots, n\}$;

(iii) a_i, b_j are positive constants ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$).

Let $\varphi \in C([t_0 - \alpha, t_0], \mathbf{R})$, where $\alpha := \max_{i,j} \{a_i, b_j\}$. By a solution of (1) with the initial function φ at t_0 we mean a function $x \in C([t_0 - \alpha, \infty), \mathbf{R})$ such that $x(t) =$

$= \varphi(t)$ for $t_0 - \alpha \leqq t \leqq t_0$, $x(t) - \sum_{i=1}^{n} p_i(t)\, x(t - a_i)$ is continuously differentiable for $t \geqq t_0$ and $x(t)$ satisfies equation (1) for $t \geqq t_0$.

By the method of steps (see e.g. [1]) it can be proved that for any continuous initial function φ there exists a unique solution of (1) for $t \geqq t_0$.

A solution x of (1) is called oscillatory if there exists a sequence $\{t_i\}$ in $[t_0, \infty)$ with $\lim_{i\to\infty} t_i = \infty$ and $x(t_i) = 0$ for every $i = 1, 2, \ldots$. A solution x of (1) is called nonoscillatory if it is eventually positive or negative.

The object of this paper is to generalize the results in [3] where the equation (1) is of the following special form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - px(t - \tau)\right] + Q(t)\, x(t - \sigma) = 0\,, \quad t \geqq t_0\,,$$

with p, τ and σ being positive constants and $Q \in C([t_0, \infty), \mathbf{R}^+)$.

2. RESULTS

Lemma 1. *Let $a_i > 0$ be positive constants, $p_i \in C([t_0, \infty), \mathbf{R}^+)$ $(i = 1, 2, \ldots, n)$, $a = \max_i a_i$, $g\colon [t_0 - a, \infty) \to \mathbf{R}$ and let $\lim_{t\to\infty} \sum_{i=1}^{n} p_i(t) = p$, $\lim_{t\to\infty} p_{i_0}(t) = \beta > 0$ exist for some $i_0 \in \{1, 2, \ldots, n\}$. Set*

$$f(t) := g(t) - \sum_{i=1}^{n} p_i(t)\, g(t - a_i) \quad \text{for} \quad t \geqq t_0\,.$$

Assuming $0 < p \leqq 1$, g bounded on $[t_0 - a, \infty)$ and $\lim_{t\to\infty} f(t) = \gamma$ we obtain the following statements:

(a) *$p = 1$ implies $\gamma = 0$,*

(b) *$p < 1$ implies the existence of $\lim_{t\to\infty} g(t)$.*

Proof. Let $\{t_i\}$ and $\{t_i'\}$ be sequences of points in $[t_0, \infty)$, $\lim_{i\to\infty} t_i = \lim_{t\to\infty} t_i' = \infty$, such that

$$A := \limsup_{t\to\infty} g(t) = \lim_{i\to\infty} g(t_i)\,,$$

$$B := \liminf_{t\to\infty} g(t) = \lim_{i\to\infty} g(t_i')\,.$$

For every $\varepsilon > 0$ it is possible to determine a positive integer N such that $p_{i_0}(t_i) > 0$, $p_{i_0}(t_i') > 0$, $g(t_i - a_j) \leqq A + \varepsilon$, $g(t_i' - a_j) \geqq B - \varepsilon$ for every $i \geqq N$ and $j = 1, 2, \ldots, n$. For this i we then have

$$g(t_i - a_{i_0}) = \frac{1}{p_{i_0}(t_i)} [g(t_i) - f(t_i) - \sum_{\substack{j=1 \\ j \neq i_0}}^{n} p_j(t_i)\, g(t_i - a_j)] \geqq$$

$$\geqq \frac{1}{p_{i_0}(t_0)} [g(t_i) - f(t_i) - (A + \varepsilon) \sum_{\substack{j=1 \\ j \neq i_0}}^{n} p_j(t_i)] ,$$

$$g(t_i' - a_{i_0}) = \frac{1}{p_{i_0}(t_i')} [g(t_i') - f(t_i') - \sum_{\substack{j=1 \\ j \neq i_0}}^{n} p_j(t_i')\, g(t_i' - a_j)] \leqq$$

$$\leqq \frac{1}{p_{i_0}(t_i')} [g(t_i') - f(t_i') - (B - \varepsilon) \sum_{\substack{j=1 \\ j \neq i_0}}^{n} p_j(t_i')] .$$

Taking limits as $i \to \infty$ we obtain the inequalities

$$A \geqq \frac{1}{\beta} [A - \gamma - (A + \varepsilon)(p - \beta)] ,$$

$$B \leqq \frac{1}{\beta} [B - \gamma - (B - \varepsilon)(p - \beta)] ,$$

which are satisfied for every $\varepsilon > 0$. Then

$$\gamma \geqq A(1 - p) ,$$
$$\gamma \leqq B(1 - p) .$$

If $p = 1$, then $\gamma = 0$ and (a) is proved. If $p < 1$,

$$A \leqq \frac{\gamma}{1 - p} \leqq B ,$$

which implies that $A = B$ and (b) is proved.

Theorem 1. *Suppose the conditions* (i)–(iii) *are satisfied with* $p < 1$ *and*

$$(2) \qquad \int_{t_0}^{t} \sum_{j=0}^{m} q_j(t)\, \mathrm{d}t = \infty .$$

Then every nonoscillatory solution of (1) *tends to zero as* $t \to \infty$.

Proof. We can assume without any loss of generality that equation (1) has a nonoscillatory solution x, $x(t) > 0$ for $t \geqq T (\geqq t_0)$. Setting

$$w(t) := x(t) - \sum_{i=1}^{n} p_i(t)\, x(t - a_i) \quad \text{for} \quad t \geqq t_0$$

we have

$$w'(t) = - [q_0(t)\, x(t) + \sum_{j=1}^{m} q_j(t)\, x(t - b_j)] \leqq 0 , \quad t \geqq T + \alpha ,$$

which implies that w is nonincreasing on the interval $[T + \alpha, \infty)$. In particular

$x(t) \geqq w(t)$, $x(t - b_j) \geqq w(t - b_j) \geqq w(t)$ for $t \geqq T + 2\alpha$, $j = 1, 2, \ldots, m$, and thus

$$(3) \qquad q_0(t)\, x(t) + \sum_{j=1}^{m} q_j(t)\, x(t - b_j) \geqq w(t) \sum_{j=0}^{m} q_j(t)\,, \quad t \geqq T + 2\alpha\,.$$

If $w(t) \geqq 0$ for $t \geqq T + 2\alpha$, then (3) implies

$$w'(t) \leqq -w(t) \sum_{j=0}^{m} q_j(t)\,,$$

hence

$$(4) \qquad w(t) \leqq w(T + 2\alpha) \exp\left[-\int_{T+2\alpha}^{t} \sum_{j=0}^{m} q_j(s)\, ds\right], \quad t \geqq T + 2\alpha\,.$$

If $w(t) < 0$ for $t \geqq T_1 (\geqq T + 2\alpha)$, then

$$(5) \qquad x(t) < \sum_{i=1}^{n} p_i(t)\, x(t - a_i)\,, \quad t \geqq T_1\,.$$

Define $B := \max\{w(T + 2\alpha), 0\}$. From (4) and (5) we get

$$x(t) \leqq \sum_{i=1}^{n} p_i(t)\, x(t - a_i) + B \exp\left[-\int_{T+2\alpha}^{t} \sum_{j=0}^{m} q_j(s)\, ds\right]$$

for $t \geqq T_1$.

Now we prove that x is a bounded solution of (1). Let there exist a sequence $\{t_k\}$, $t_k \geqq T_1$, such that

$$(6) \qquad \lim_{k\to\infty} x(t_k) = \infty\,, \quad x(t_k) = \max_{T_1 \leqq t \leqq t_k} x(t)\,.$$

Then

$$x(t_k) \leqq \sum_{i=1}^{n} p_i(t_k)\, x(t_k - a_i) + B \exp\left[-\int_{T+2\alpha}^{t_k} \sum_{j=0}^{m} q_j(s)\, ds\right] \leqq$$

$$\leqq x(t_k) \sum_{i=1}^{n} p_i(t_k) + B \exp\left[-\int_{T+2\alpha}^{t_k} \sum_{j=0}^{m} q_j(s)\, ds\right].$$

Herefrom we obtain for k sufficiently large (so that $\sum_{i=1}^{n} p_i(t_k) < 1$)

$$x(t_k) \leqq \frac{B}{1 - \sum_{i=1}^{n} p_i(t_k)} \exp\left[-\int_{T+2\alpha}^{t_k} \sum_{j=0}^{m} q_j(s)\, ds\right].$$

It follows from (2) that

$$\lim_{k\to\infty} \frac{B}{1 - \sum_{i=1}^{n} p_i(t_k)} \exp\left[-\int_{T+2\alpha}^{t_k} \sum_{j=0}^{m} q_j(s)\, ds\right] =$$

$$= \frac{B}{1 - p} \lim_{k\to\infty} \exp\left[-\int_{T+2\alpha}^{t_k} \sum_{j=0}^{m} q_j(s)\, ds\right] = 0\,,$$

in contradiction to (6).

Applying Lemma 1 (b) with $f(t) = w(t)$, $g(t) = x(t)$, we conclude that $\lim_{t\to\infty} x(t) =: L$ exists. Consequently, w is bounded on $[t_0, \infty)$. If $L > 0$, then

$$x(t) > \frac{L}{2} \quad \text{for} \quad t \geqq T_2 \ (\geqq T)$$

and

$$w(t) - w(T_2 + \alpha) = - \int_{T_2+\alpha}^{t} [q_0(s)\,x(s) + \sum_{j=1}^{m} q_j(s)\,x(s - b_j)]\,ds \leqq$$

$$\leqq - \frac{L}{2} \int_{T_2+\alpha}^{t} \sum_{j=0}^{m} q_j(s)\,ds\,,$$

so that

$$w(t) \leqq w(T_2 + \alpha) - \frac{L}{2} \int_{T_2+\alpha}^{t} \sum_{j=0}^{m} q_j(s)\,ds \quad \text{for} \quad t \geqq T_2 + \alpha\,.$$

However, by virtue of (2) this leads to a contradiction with the boundedness of w. Theorem 1 is true completely proved.

Theorem 2. *Suppose the conditions* (i)–(iii) *and* (2) *are satisfied and* $\sum_{i=1}^{n} p_i(t) = 1$ *for* $t \geqq T(\geqq t_0)$.

Then every solution of (1) *oscillates.*

Proof. On the contrary, without any loss of generality let us assume that equation (1) has a nonoscillatory solution x, $x(t) > 0$ for $t \geqq T_1$ ($\geqq T$). Putting

$$w(t) := x(t) - \sum_{i=1}^{n} p_i(t)\,x(t - a_i)\,, \quad t \geqq t_0$$

we have

$$w'(t) = -[q_0(t)\,x(t) + \sum_{j=1}^{m} q_j(t)\,x(t - b_j)] \leqq 0 \quad \text{for} \quad t \geqq T_1 + \alpha\,.$$

Consequently, w is nonincreasing on $[T_1 + \alpha, \infty)$.

Assumption (2) then implies $w(t) \not\equiv 0$ in a neighbourhood of ∞. Let $w(t) < 0$ for $t \geqq T_2$ ($\geqq T_1$). Hence

$$w(t) \leqq w(T_2) \ (<0) \quad \text{for} \quad t \geqq T_2. \tag{7}$$

To arrive at a contradiction we assume x to be not bounded. Then there exists a sequence $\{t_j\}$ such that $t_j \in [T_2 + \alpha, \infty)$, $\lim_{j\to\infty} t_j = \infty$, $\lim_{j\to\infty} x(t_j) = \infty$, $x(t_j) = \max_{T_2 \leqq t \leqq t_j} x(t)$. From (7) we obtain

$$x(t_j) \leqq w(T_2) + \sum_{i=1}^{n} p_i(t_j)\,x(t_j - a_i) \leqq w(T_2) +$$

$$+ x(t_j) \sum_{i=1}^{n} p_i(t_j) = w(T_2) + x(t_j)\,.$$

Therefore $w(T_2) \geqq 0$ contrary to $w(T_2) < 0$. Thus x is a bounded function and hence w is also a bounded function. From Lemma 1 (a) we obtain $\lim_{t\to\infty} w(t) = 0$ in contradiction to $w'(t) \leqq 0$, $w(t) < 0$ for $t \geqq T_2 + \alpha$. From this contradiction we conclude that $w(t) > 0$ for $t \geqq T_1 + \alpha$. Consequently,

$$x(t) > \sum_{i=1}^{n} p_i(t)\, x(t - a_i) \quad \text{for} \quad t \geqq T_1 + \alpha\,. \tag{8}$$

If $\liminf_{t\to\infty} x(t) = 0$ then there exists a sequence $\{t_j\}$, $t_j \in [T_1 + \alpha, \infty)$, such that

$$\lim_{j\to\infty} x(t_j) = 0\,, \quad x(t_j) = \min_{T_1 \leqq t \leqq t_j} x(t)$$

and (8) then yields

$$x(t_j) > \sum_{i=1}^{n} p_i(t_j)\, x(t_j - a_i) \geqq x(t_j) \sum_{i=1}^{n} p_i(t_j) = x(t_j)$$

and

$$x(t_j) > x(t_j)\,, \quad j = 1, 2, \ldots\,.$$

Hence there exists a positive constant $\beta > 0$ such that

$$x(t) \geqq \beta \quad \text{for} \quad t \geqq T_3 \; (\geqq T_1 + \alpha)\,. \tag{9}$$

Integrating (1) from $T_3 + \alpha$ to t we get

$$w(t) - w(T_3 + \alpha) + \int_{T_3+\alpha}^{t} [q_0(s)\, x(s) + \sum_{j=1}^{m} q_j(s)\, x(s - b_j)]\, \mathrm{d}s = 0$$

and by (9) we conclude

$$w(T_3 + \alpha) > \int_{T_3+\alpha}^{t} [q_0(s)\, x(s) + \sum_{j=1}^{m} q_j(s)\, x(s - b_j)]\, \mathrm{d}s \geqq$$

$$\geqq \beta \int_{T_3+\alpha}^{t} \sum_{j=0}^{m} q_j(s)\, \mathrm{d}s\,.$$

Hence

$$w(T_3 + \alpha) > \beta \int_{T_3+\alpha}^{t} \sum_{j=0}^{m} q_j(s)\, \mathrm{d}s \quad \text{for} \quad t \geqq T_3 + \alpha\,,$$

which, as $t \to \infty$, is contrary to assumption (2).

Remark 1. In the following Example 1 we shall demonstrate that the assumption $\sum_{i=1}^{n} p_i(t) = 1$ for $t \geqq T$ in Theorem 2 cannot be replaced by a weaker assumption $p = 1$.

Example 1. Let

$$p(t) := \frac{(t-1)(1+\ln^2(t-1))\exp\left[\operatorname{arctg}\dfrac{1}{\ln t}\right]}{t(1+\ln^2 t)\exp\left[\operatorname{arctg}\dfrac{1}{\ln(t-1)}\right]} \cdot$$

$$\cdot\left(1 - \frac{1+\ln^2 t}{\ln t \ln(\ln t)} \frac{\exp\left[\operatorname{arctg}\dfrac{1}{\ln t}\right] - 1}{\exp\left[\operatorname{arctg}\dfrac{1}{\ln t}\right]}\right),$$

$$Q(t) := \frac{1}{t \ln t \ln(\ln t)} + p'(t)\frac{\exp\left[\operatorname{arctg}\dfrac{1}{\ln(t-1)}\right] - 1}{\exp\left[\operatorname{arctg}\dfrac{1}{\ln t}\right] - 1}$$

for $t \geqq 3$. Since $p'(t) = O(1/(t \ln t(\ln(\ln t))^2))$ for $t \to \infty$, we have $Q(t) > 0$ for $t \geqq T(\geqq 3)$ and $\int_T^\infty Q(s)\,ds = \infty$. The equation

$$[x(t) - p(t)\,x(t-1)]' + Q(t)\,x(t) = 0\,, \quad t \geqq T$$

has a nonoscillatory solution $x(t) = \exp[\operatorname{arctg}(1/\ln t)] - 1$, $\lim_{t\to\infty} x(t) = 0$.

Theorem 3. *Suppose the conditions* (i)–(iii) *are satisfied with* $p < 1$, $q_0(t) = 0$ *for* $t \geqq t_0$ *and*

$$\liminf_{t\to\infty} \int_{t-b}^{t} \sum_{j=1}^{m} q_j(s)\,ds > \frac{1}{e}, \tag{10}$$

where $b := \min_j b_j$.

Then every solution of (1) *oscillates.*

Proof. On the contrary without loss of generality let us assume that (1) has a nonoscillatory solution x, $x(t) > 0$ for $t \geqq T$ ($\geqq t_0$). Theorem 1 then implies $\lim_{t\to\infty} x(t) = 0$. Setting

$$w(t) := x(t) - \sum_{i=1}^{n} p_i(t)\,x(t-a_i)\,, \quad t \geqq t_0\,,$$

we conclude that w is a nonincreasing function on $[T + \alpha, \infty)$. Since $w(t) \leqq x(t)$ for $t \geqq T + \alpha$ we get

$$w'(t) + w(t-b)\sum_{j=1}^{m} q_j(t) \leqq w'(t) + \sum_{j=1}^{m} q_j(t)\,w(t-b_j) \leqq$$

$$\leqq w'(t) + \sum_{j=1}^{m} q_j(t)\,x(t-b_j) = 0\,,$$

thus

$$w'(t) + w(t-b)\sum_{j=1}^{m} q_j(t) \leqq 0 \quad \text{for} \quad t \geqq T + \alpha\,. \tag{11}$$

Condition (10) implies (see Theorem 2 [2]) that the equation

$$x'(t) + x(t-b)\sum_{j=1}^{m} q_j(t) \leqq 0\,, \quad t \geqq T + \alpha\,,$$

cannot have an eventually positive solution, in contradiction to $x(t) > 0$ for $t \geqq T$.

Remark 2. If $p_i(t) = 0$ for $t \geqq t_0$, $i = 1, 2, \ldots, n$, then Theorem 3 follows from Corollary 2 [4].

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Souhrn

OSCILAČNÍ VLASTNOSTI ŘEŠENÍ NEUTRÁLNÍCH DIFERENCIÁLNÍCH ROVNIC SE ZPOŽDĚNÝM ARGUMENTEM

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V práci jsou uvedeny podmínky, které jsou postačující k tomu, aby všechna řešení rovnice (1) byla oscilatorická, a dále postačující podmínky k tomu, aby všechna neoscilatorická řešení konvergovala k nule pro $t \to \infty$.

Резюме

КОЛЕБАТЕЛЬНЫЕ СВОЙСТВА РЕШЕНИЙ НЕЙТРАЛЬНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

Svatoslav Staněk

В статье приводятся достаточные условия для колебания всех решений уравнений (1) и далее достаточные условия для того, чтобы все неколеблющиеся решения стремились к нулью для $t \to \infty$.

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