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ANOTHER PROOF OF BORŮVKA'S CRITERION ON GLOBAL EQUIVALENCE OF THE SECOND ORDER ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Summary. By using a visible geometrical approach the author proves the criterion on global equivalence of the second order ordinary linear homogeneous differential equations in the real domain, originally derived by O. Borůvka in 1967 by an analytic method.

Keywords: Linear differential equations of the second order, Transformations, Global equivalence.

AMS Classification: 34A30, 34C10, 34C20.

I. INTRODUCTION

The most general pointwise global transformation that converts solutions of any ordinary linear homogeneous differential equation of the second order

(1) $$y'' + a_1(x)\, y' + b_1(x)\, y = 0$$

with real continuous coefficients defined on an open interval I of reals into solutions of an equation

(2) $$z'' + a_2(t)\, z' + b_2(t)\, z = 0\,,$$

$a_2, b_2\colon J \to \mathbb{R}$, is of the form

$$z(t) = f(t)\,.\, y(h(t))\,,$$

where

$f \in C^2(J)$, $f(t) \neq 0$ on J,

h is a C^2-diffeomorphism of J onto I, i.e.,

$h \in C^2(J)$, $h'(t) \neq 0$ on J, $h(J) = I$, and

z is a solution of (2) whenever y is a solution of (1), cf. [2] and [4].

If $a_1 = 0$, then $a_2 = 0$ if and only if $f(t) = c|h'(t)|^{-1/2}$ on J, where $c \neq 0$ is a constant, see e.g. [1]. Since $f \in C^2(J)$, we have $h \in C^3(J)$. Thus, in accordance with O. Borůvka [1], we shall consider the most general pointwise global transformations in the form

$$z(t) = c|h'(t)|^{-1/2}\, y(h(t))\,, \tag{3}$$

$h \in C^3(J)$, $h'(t) \neq 0$ on J, $h(J) = I$, c a nonzero constant. Such a transformation globally transforms solutions y of the equation

$$y'' + p(x)\, y = 0\,, \quad p \in C^0(I)\,, \tag{p}$$

into solutions of an equation

$$z'' + q(t)\, z = 0\,, \quad q \in C^0(J)\,, \tag{q}$$

in the sense of the formula (3). The function h in it is called the transformator, and the equations (p) and (q) are said to be globally equivalent.

In accordance with O. Borůvka [1] an equation (p) is said to be of finite type m, m a positive integer, if m is the maximal number of zeros of every nontrivial solution of (p) on the interval I. In this case, if there are two linearly independent solutions of (p) with $m - 1$ zeros, then (p) is of the finite type m and general, otherwise (p) is of the finite type m and special.

If an equation (p) is not of a finite type, then it is either one-side oscillatory or both-side oscillatory.

Two equations (p) and (q) are said to be of the same character, see again [1], if and only if they are

of the same finite type m, $m \geqq 1$, and either both are general, or both are special, or they are both one-side oscillatory, or they are both both-side oscillatory.

O. Borůvka proved in 1967 in [1]:

Borůvka's Criterion. *Equations (p) and (q) are globally equivalent if and only if they are of the same character.*

The aim of this paper is to give another proof of the criterion based on a geometric interpretation of global transformations first introduced in 1971 in [3].

II. NOTATION AND SOME BASIC FACTS

Denote by $\mathscr{T}[h]$ the transformation (3) with the transformator h, and write shortly

$$(q) = \mathscr{T}[h]\,(p)\,.$$

Let y_1 and y_2 be two linearly independent solutions of the equation (p), and let z_1 and z_2 be obtained by (3) where y_1 and y_2 stand instead of y. Denote by $\mathbf{y}$ and $\mathbf{z}$ the vector functions

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We will also write

$$\mathbf{z} = \mathscr{T}[h]\,\mathbf{y}\,.$$

It can be shown that the coordinates of $\mathbf{z}$ are again linearly independent solutions of (q). Moreover, since $\mathbf{y}$ is formed by two linearly independent solutions of (p) and h is a C^3-diffeomorphism of J onto I, the vector function

$$A\mathcal{T}[h]\,\mathbf{y}$$

is formed by linearly independent solutions of the equation (q) for any nonsingular 2 by 2 constant matrix A.

Denote by $W[\mathbf{y}](x)$ the Wronskian of the pair y_1, y_2 at the point $x \in I$.

In [3] there was introduced the following

CONSTRUCTION

Consider our vector function $\mathbf{y}$ as a curve in the plane with coordinate functions y_1 and y_2 and the independent variable x as the parameter. Take the central projection of the curve onto the unit circle $\mathbf{S}_1$ in the plane and introduce a length parametrization with a suitable orientation into the projection. We obtain the curve $\mathbf{u}(s)$, $s \in I_u$, where

$$u_1(s) = \sin s\,, \quad u_2(s) = \cos s\,, \quad s \in I_u\,.$$

Lemma 1. *The differential equation of the second order whose solutions are u_1 and u_2, which is in fact*

$$u'' + u = 0 \quad \text{on} \quad I_u\,,$$

is globally equivalent to (p) on I.

Moreover, there was also showen the following geometrical interpretation of zeros of solutions.

Lemma 2. *To each straight line $\mathbf{d}$ passing through the origin in the plane,*

$$\mathbf{d} \equiv (d_1\zeta_1 + d_2\zeta_2 = 0)\,,$$

there exists one and only one (up to a constant nonzero factor) nontrivial solution $y_{\mathbf{d}}$ of (p) whose zeros are exactly the parameters of intersections of the line $\mathbf{d}$ with the curve $\mathbf{y}$.

Now we will give another proof of Borůvka's Criterion.

III. PROOF

Consider two equations (p) and (q), their pairs of linearly independent solutions $\mathbf{y}$ and $\mathbf{z}$, and the corresponding curves $\mathbf{u}$ and $\mathbf{v}$ on the unit circle given by Construction and defined on the intevals I_u and J_v, respectively. Due to Lemma 1, solutions y_1

and y_2 of (p) are globally transformed into $\sin s$ and $\cos s$, $s \in I_u$, in the sense of (3), i.e.,

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \frac{c_1}{|h_1'(x)|^{1/2}} \begin{pmatrix} \sin h_1(x) \\ \cos h_1(x) \end{pmatrix}$$

where h_1 is a C^3-diffeomorphism of I onto I_u.

Similarly

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \frac{c_2}{|h_2'(t)|^{1/2}} \begin{pmatrix} \sin h_2(t) \\ \cos h_2(t) \end{pmatrix}$$

where h_2 is a C^3-diffeomorphism of J onto J_v.

To each solution $k_1 y_1 + k_2 y_2$ of (p) we assign the solution $k_1 u_1 + k_2 u_2$ of the equation

$$(1_u) \qquad u'' + u = 0 \quad \text{on} \quad I_u$$

and conversely. Since $\boldsymbol{u}$ is the central projection of $\mathbf{y}$, Lemma 2 guarantees the same number of zeros of both $k_1 y_1 + k_2 y_2$ and $k_1 u_1 + k_2 u_2$, or that both equations are one-side oscillatory, or that they are both-side oscillatory at the same time, i.e., (p) and (1_u) are of the same character.

Similarly, the equations (q) and

$$(1_v) \qquad v'' + v = 0 \quad \text{on} \quad I_v$$

are globally equivalent and at the same time: either the number of zeros of each solution $r_1 z_1 + r_2 z_2$ (r_1, r_2 – constants) of (q) is the same as the number of zeros of the solution $r_1 v_1 + r_2 v_2$ of the equation (1_v), or both equations are one-side oscillatory, or they are both-side oscillatory, i.e., (q) and (1_v) are of the same character.

Now suppose that the equations (p) and (q) have the same character. In order to prove that (p) can be globally transformed into (q), it is sufficient to show that the equations (1_u) and (1_v), which are of the same character as well, are also globally equivalent. We know that

$$u_1(s) = \sin s, \quad u_2(s) = \cos s, \quad s \in I_u$$

and

$$v_1(\sigma) = \sin \sigma, \quad v_2(\sigma) = \cos \sigma, \quad \sigma \in I_v$$

are solutions of (1_u) and (1_v), respectively.

If the type of the equations (1_u) and (1_v) is finite and equal to m, then the length $\lambda(I_u)$ of the interval I_u satisfies

$$(m-1)\pi < \lambda(I_u) \leqq m\pi,$$

and, of course, also

$$(m-1)\pi < \lambda(I_v) \leqq m\pi.$$

If, moreover, (1_u) and (1_v) are general, then also

$$(4) \qquad (m-1)\pi < \lambda(I_u) < m\pi \quad \text{and} \quad (m-1)\pi < \lambda(I_v) < m\pi,$$

whereas if (1_u) and (1_v) are special then

$$\lambda(I_u) = \lambda(I_v) = m\pi\,, \tag{5}$$

see also Fig. 1 for $m = 1$.

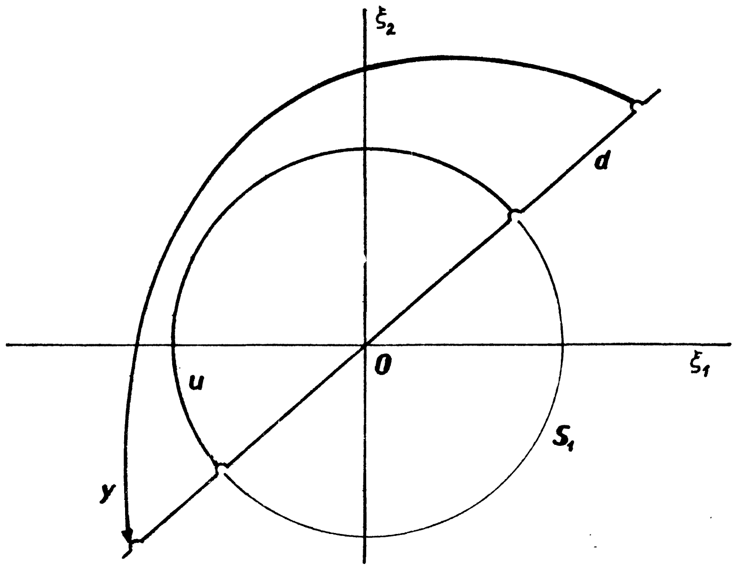

Fig. 1

First consider the relation (4). Let $I_u = (a_u, b_u)$ and $I_v = (a_v, b_v)$. Due to the relation (4), the vectors

$$\mathbf{u}(a_u) = \begin{pmatrix} \sin\, a_u \\ \cos\, a_u \end{pmatrix} \quad \text{and} \quad \mathbf{u}(b_u) = \begin{pmatrix} \sin\, b_u \\ \cos\, b_u \end{pmatrix}$$

are not parallel. The vectors $\mathbf{v}(a_v)$ and $\mathbf{v}(b_v)$ are not parallel, either. Hence there exists a centroaffine mapping of the plane with a nonsingular matrix A such that $\mathbf{v}(a_v) = A\,\mathbf{u}(a_u)$ and $\mathbf{v}(b_v) = A\,\mathbf{u}(b_u)$. However, in this case, the central projection and the length parametrization $\sigma \mapsto h(\sigma) = s$ (with a suitable initial value, i.e., $h(a_v) = a_u$) of the curve $A\,\mathbf{u}(s)$, $s \in I_u$, lying on the ellipse $A\mathbf{S}_1$ (described in Construction) gives exactly the curve $\mathbf{v}(\sigma)$, $\sigma \in I_v$, on the unit shere $\mathbf{S}_1$ with the length parametrization. Hence

$$\mathbf{u} = A\,\mathscr{T}[h]\,\mathbf{v}$$

or, the equations (1_u) and (1_v) are globally equivalent, see also Fig. 2 for type 1 and general.

Now, if (5) is satisfied, i.e., the equations (1_u) and (1_v) are of finite type m and special, then the transformator

$$s = h(\sigma) = \sigma + a_u - a_v$$

converters $\mathbf{u}(s)$, $s \in I_u$, into $\mathbf{v}(\sigma)$, $\sigma \in I_v$. Hence (1_u) and (1_v) are globally equivalent.

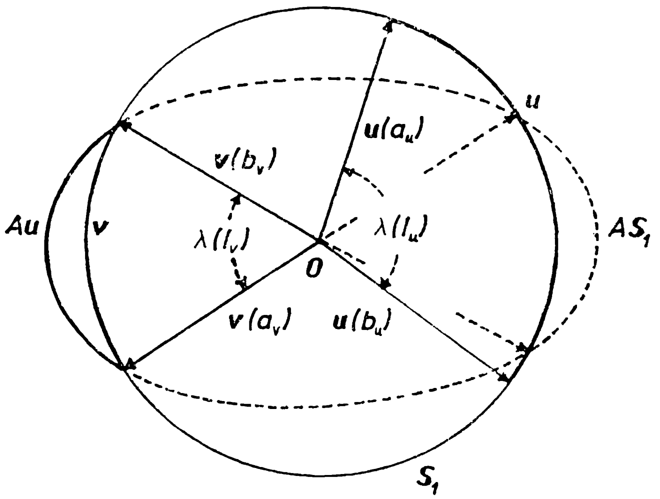

Fig. 2

In the case when both (1_u) and (1_v) are one-side oscillatory, Lemma 2 implies $I_u = (a_u, \infty)$ and $I_v = (a_v, \infty)$, for $\mathbf{u}$ and $\mathbf{v}$, otherwise (v_2, v_1) is taken instead of $\mathbf{v} = (v_1, v_2)$. Then again

$$s = h(\sigma) = \sigma + a_u - a_v$$

globally transforms (1_u) into (1_v), i.e., $(1_v) = \mathscr{T}[h]\,(1_u)$.

If (1_u) and (1_v) are both-side oscillatory, they coincide and, of course, they are globally equivalent.

Now, it remains to show the converse, i.e., if the equation (p) can be globally transformed into (q), then (p) and (q) are of the same character. We have seen at the beginning of the proof that the equations (p) and (1_u) are globally equivalent and, due to Lemmas 1 and 2 that both the equations are of the same character. Similarly for the equations (q) and (1_v).

Hence it is sufficient to show that if (1_u) can be globally transformed into (1_v), then these equations are of the same character.

For this purpose, in accordance with the definition of global transformations, suppose

$$(6) \qquad \begin{pmatrix} \sin \sigma \\ \cos \sigma \end{pmatrix} = A|h'(\sigma)|^{-1/2} \begin{pmatrix} \sin h(\sigma) \\ \cos h(\sigma) \end{pmatrix}, \quad \sigma \in I_v$$

h being a C^3-diffeomorphism of I_v onto I_u, A being a nonsingular 2 by 2 matrix.

Define the following bijection of the set of nontrivial solutions of (1_u) onto the set of nontrivial solutions of (1_v):

To each nontrivial solution of (1_u)

$$k_1 \sin s + k_2 \cos s\,, \quad s \in I_u\,, \quad (k_1^2 + k_2^2 \neq 0)\,,$$

we have the corresponding solution

$$(k_1, k_2)\, A^{-1} |h'(\sigma)|^{1/2} \begin{pmatrix} \sin(\sigma) \\ \cos(\sigma) \end{pmatrix}, \quad \sigma \in I_v\,,$$

of the equation (1_v).

Due to the relation (6), and bijectivity of the correspondence, to each nontrivial solution u of (1_u) and its zero s_0 there correspond exactly one nontrivial solution v of (1_v) and its zero $\sigma_0 = h^{-1}(s_0)$. The definition of the character is based only on the cardinality and ordering (up to orientation) of the set of zeros of solutions, which is, of course, preserved by each C^3-diffeomorphism h. Hence the equations (1_u) and (1_v) have the same character, which completes the proof.

Literature

[1] *O. Borůvka:* Lineare Differentialtransformationen 2. Ordnung. VEB Berlin 1967. Linear Differential Transformations of the Second Order. The English Univ. Press, London 1971.

[2] *M. Čadek:* Form of general pointwise transformations of linear differential equations. Czechoslovak Math. J. *35* (*110*) (1985), 617—624.

[3] *F. Neuman:* Geometrical approach to linear differential equations of the n-th order. Rend. Mat. *5* (1972), 579—602 (Abstract: Some results on geometrical approach to linear differential equations of the n-th order. Comment. Math. Univ. Carolin. *12* (1971), 307—315).

[4] *P. Stäckel:* Über Transformationen von Differentialgleichungen. J. Reine Angew. Math. *111* (1893), 290—302.

Souhrn

JINÝ DŮKAZ BORŮVKOVA KRITERIA O GLOBÁLNÍ EKVIVALENCI OBYČEJNÝCH LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU

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Užitím názorného geometrického přístupu je dokázáno kriterium globální ekvivalence lineárních obyčejných homogenních diferenciálních rovnic 2. řádu v reálném oboru, které bylo poprvé odvozeno v roce 1967 O. Borůvkou analytickou metodou.

Резюме

ДРУГОЕ ДОКАЗАТЕЛЬСТВО КРИТЕРИЯ О. БОРУВКИ О ГЛОБАЛЬНОЙ ЭКВИВАЛЕНТНОСТИ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА

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С помощью наглядного геометрического подхода доказывается критерий глобальной эквивалентности линейных обыкновенных однородных дифференциальных уравнений второго порядка в вещественной области, который впервые был доказан О. Борувкой в 1967 году на основе аналитического метода.

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