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Andrzej Matraś

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ON HAVLÍČEK-TIETZE CONFIGURATION IN SOME NON-DESARGUESIAN PLANES

ANDRZEJ MATRAŚ, Olsztyn

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Summary. Examples of non-desarguesian planes in which Havlíček-Tietze configuration exists are studied, a necessary condition for existence of Havlíček-Tietze configurations in projective planes over regular nearfields of order p^2 is given, and a connection between the Havlíček-Tietze configurations and the circle-symmetries in Minkowski planes is established.

Keywords: Havlíček-Tietze configuration, non-desarguesian plane, projective plane over regular nearfield, circle-symmetry.

1. INTRODUCTION

By a Havlíček-Tietze (shortly H-T) configuration in the projective plane we shall mean a configuration of four triangles, no two of them having a common vertex, each two of them in six-fold homology, the centres of the homologies of any two triangles being the vertices of the other two while the axes are the corresponding opposite sides. The existence of an H-T configuration in the projective plane of order 4 was proved by Havlíček and Tietze [2].

This configuration was examined in detail in [4] and [5]. In particular, in [4] it was proved that in a desarguesian projective plane there exists an H-T configuration if and only if its coordinatizing field contains a root of the polynomial $x^2 + x + 1$ different from 1. H-T configurations exists also in some non-desarguesian planes, in particular in the translation projective planes of order 16 containing subplanes of order 4 (see [3]). The H-T configurations are closely connected with the Desargues Proposition, because every point of such a configuration is a vertex of some desarguesian configuration contained in this H-T configuration. In this paper we consider some examples of non-desarguesian planes in which there exists an H-T configuration although they do not contain subplanes of order 4. We obtain necessary conditions for the existence of H-T configurations in the projective planes over regular nearfields of order p^2. We establish a connection between the H-T configurations and the circle symmetries in some Minkowski planes over nearfields obtained by the extension of an affine plane [7].

This generalized Corollary 4,7 from [6].

2. H-T CONFIGURATIONS IN THE PROJECTIVE PLANES OVER SOME NEARFIELDS

We use nearfields (except the seven types in which the multiplication is defined by an additional relation with left distributivity) and the classical Hall's method of construction of projective planes over such nearfields [1].

Let us define the set of points as $P = K^2 \cup K \cup \{\infty\}$; the set of lines $=\{(x, y), (m); y = x \circ m + b;\ m, b \in K\} \cup \{x = c;\ c \in K\} \cup \{(m), (\infty);\ m \in K\}$, where K is a nearfield and ∞ is an element such that $\infty \notin K$.

By a result of Zassenhaus [1] all finite nearfields are known. In particular, we consider the nearfield $K_{p,2}$ which may be described as

$$K_{p,2} = \{\{\{\alpha + \beta b\};\ \alpha, \beta \in Z_p,\ b^2 = 2\};\ 0, 1, \oplus, \circ\}$$

where 0, 1 are the neutral elements of "$\oplus$" and "$\circ$" and

$$(\alpha + \beta b) \oplus (\alpha_1 + \beta_1 b) = (\alpha + \alpha_1) + (\beta + \beta_1)\, b\,,$$

$$a \circ b = \begin{cases} ab & \text{if} \quad b \in Q\,, \\ a^p b & \text{if} \quad b \notin Q \end{cases}$$

where $\alpha, \beta, \alpha_1, \beta_1 \in Z_p$ and Q is the group of all nonzero squares.

Theorem 1. *The projective Hall's plane over the finite regular nearfield $K_{p,2}$ in which there exists an element a satisfying the condition*

$$(*) \qquad a \neq 1\,, \quad a^3 = 1\,, \quad a + 1 \in Q$$

contains an H-T configuration.

Proof. Let a be an element of the nearfield satisfying the condition $(*)$ According to Theorem 1 of [4] it is sufficient to construct two triangles with no common vertices which are in six-fold homology. Let $X_1 = (0)$, $X_2 = (\infty)$, $X_3 = (0, 0)$, $A_1 = (1, 1)$, $A_2 = (a, a^2)$, $A_3 = (a^2, a)$ be points of the projective Hall's plane over the nearfield $K_{p,2}$ and let

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \\ A_3 & A_1 & A_2 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \\ A_2 & A_3 & A_1 \end{pmatrix}$$

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ A_3 & A_2 & A_1 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \\ A_1 & A_3 & A_2 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \\ A_2 & A_1 & A_3 \end{pmatrix}$$

be the pairs of ordered triples.

We obtain the points $B_1 = (a^2, a^2)$, $B_2 = (1, a)$, $B_3 = (a, 1)$, $C_1 = (a, a)$, $C_2 = (a^2, 1)$, $C_3 = (1, a^2)$ as centres and the lines

$$b_1\colon y = -x + a + 1 \quad b_2\colon y = -x \circ a - a \quad b_3\colon y = x \circ (a + 1) - 1$$

$$c_1\colon y = -x - a \qquad c_2\colon y = -x \circ a - 1 \quad c_3\colon y = x \circ (a + 1) + a + 1$$

as axes of the H-T configuration.

The triangles with vertices $X_1, X_2, X_3, A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$, together with the respective lines form the H-T configuration in the sense of [4].

The coordinates of all points and coefficients in the equations of the lines are squares in GF (p^2) while the multiplication is the same as in GF (p^2).

Corollary. *The projective Hall's plane over the nearfield $K_{5,2}$ contains an* H-T *configuration.*

Proof. Indeed, by Theorem 1 we can choose 2 + 2b or 2 + 3b (these elements generate the same H-T configuration) as an element a of the nearfield $K_{5,2}$ satisfying the conditions(*).

3. H-T CONFIGURATION AND CIRCLE SYMMETRIES IN SOME MINKOWSKI PLANES

Let K be a regular nearfield of order p^2 where p is such that in K there exists an element satisfying (*). The Minkowski plane $M_n(K) = (M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ over K is defined (see [7]) as $M = \bar{K} \times \bar{K}$ where $\bar{K} = K \cup \{\infty\}$ and ∞ is an element such that $\infty \notin K$,

$$\begin{aligned} \mathscr{L}_1 &= \{\{(k, x_2);\ \ x_2 \in \bar{K}\};\ k \in \bar{K}\}, \\ \mathscr{L}_2 &= \{\{(x_1, k);\ \ x_1 \in \bar{K}\};\ k \in \bar{K}\}, \\ \mathscr{C} &= \{(x_1, x_2);\ \ x_2 = \Phi(x_1)\}\ \Phi \in PGL(K) \end{aligned}$$

where $PGL(K)$ is the set of all permutations $K \cup \{\infty\}$ of the following forms

$$\Phi(x) = \begin{cases} x \circ a + b & \text{if } x \in K \\ \infty & \text{if } x = \infty, \end{cases}$$

$$\Phi(x) = \begin{cases} (x + b) \circ a + b & \text{if } x \in K \\ \infty & \text{if } x = -b \\ c & \text{if } x = \infty, \end{cases}$$

$$a, b, c \in K, \quad a \neq 0.$$

Let $(M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ be an arbitrary Minkowski plane. The residual plane M_p with respect to a point $p \in M$ is the set $M \setminus (\mathscr{L}_1 \cup \mathscr{L}_2)$ where $\mathscr{L}_1, \mathscr{L}_2$ are the lines through p, provided with all non-empty subsets $K \cap M_p$ for $K \in \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{C}$. We refer to the bundle of circles as to the set of circles with two points.

The existence of an H-T configuration in a projective plane implies some connections between the Minkowski inversion and the six-fold homologies of triangles. Some properties of a circle-symmetry and the points of an H-T configuration are described in the following lemma:

Lemma. *Let $M_n(K) = (M, \mathscr{L}_1, \mathscr{L}_2, \mathscr{C})$ be a Minkowski plane over a finite regular nearfield K of order p^2, containing an element satisfying the property (*). In $M_n(K)$ there exist points A_i, B_i, C_i $(i = 1, 2, 3)$ and circles α, β, γ (affine Minkowski*

„hyperbolas"), l_1, l_2, l_3 (affine straight lines) such that the following assertions hold:

1) *The triples α, β, γ and l_1, l_2, l_3 belong to two of circles in $M_n(K)$ such that $A_i \in \alpha$, $B_i \in \beta$, $C_i \in \gamma$; $A_i, B_i, C_i \in l_i$ $(i = 1, 2, 3)$.*

2) *The system of points A_i, B_i, C_i and affine lines l_1, l_2, l_3 of the affine plane M_p (where $p \in l_1, l_2, l_3$) can be completed to an H-T configuration in the associated projective plane.*

3) *The inversions with respect to α, β, γ, l_1, l_2, l_3 are the permutations of the points A_i, B_i, C_i (for example $\sigma_\alpha(B_i) = C_i$, $i = 1, 2, 3$, $\sigma_{l_1}(A_2) = A_3$, $\sigma_{l_1}(B_2) = B_3$, $\sigma_{l_1}(C_2) = C_3$, where $\sigma\varphi$ is the inversion with respect to the circle φ).*

4) *Each inversion with respect to these 6 circles interchanges two elements of a triple and fixes another triple (but not pointwise).*

Proof. By Corollary, in the projective extension of the affine plane M_p there exist points X_i, A_i, B_i, C_i $(i = 1, 2, 3)$ with coordinates as in Theorem 1. They form an H-T configuration. Let x^{-1} denote the inverse of x in the field $GF(p^2)$ and let $\alpha, \beta, \gamma, l_1, l_2, l_3$ be the circles $y = x^{-1}$, $y = x^{-1}a^2$, $y = x^{-1} \circ a$, $y = x$, $y = x \circ a$, $y = x \circ a^2$, respectively (the symbol $\circ$ denotes the multiplication in the nearfield defined above).

Circle inversions for a point $(x, y) \neq \infty$ $(x, y \in Q)$ can be obtained in the following way:

$$\sigma_\alpha(x, y) = (y^{-1}, x^{-1}), \quad \sigma_\beta(x, y) = (a \circ y^{-1}, x^{-1} \circ a),$$
$$\sigma_\gamma(x, y) = (a^2 \circ y^{-1}, x^{-1} \circ a^2)$$

and

$$\sigma_{l_1}(x, y) = (y, x), \quad \sigma_{l_2}(x, y) = (y \circ a^2, x \circ a^2), \quad \sigma_{l_3}(x, y) = (y \circ a, x \circ a^2).$$

By definition of the multiplication in a nearfield $(a, a^2 \in Q)$ and by the equation of an inversion it is easy to verify the properties 2, 3 and 4. Our Lemma immediately implies

Theorem 2. *In the affine Minkowski plane over a finite regular nearfield $K_{p,2}$ with an element satisfying the assumption (*) there exist concentric Minkowski circles such that each pair of them is inversely conjugated with respect to the remaining one.*

Remark. In our analysis we use the terminology of [7]. Instead of circles the expression "chain" or "cycle" can be applied and instead of lines — "generators". In [6] the term Minkowski circle is reserved for some special hyperbolas.

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Souhrn

HAVLÍČKOVA-TIETZOVA KONFIGURACE V JISTÝCH NE-DESARGUESOVSKÝCH ROVINÁCH

ANDRZEJ MATRAŚ

Jsou zkoumány příklady ne-desarguesovských rovin, v nichž existuje Havlíčkova-Tietzova konfigurace. Je podána dostatečná podmínka existence H-T konfigurace v projektivní rovině nad regulárním skorotělesem řádu p^2, a je ukázána souvislost H-T konfigurací s kruhovými symetriemi v Minkowského rovinách.

Резюме

КОНФИГУРАЦИЯ ГАВЛИЧЕКА-ТИЦЕ В НЕКОТОРЫХ НЕДЕЗАРГОВЫХ ПЛОСКОСТЯХ

ANDRZEJ MATRAŚ

Рассматриваются примеры недезарговых плоскостей, в которых существует конфигурация Гавличека-Тице. Дано достаточное условие для существования такой конфигурации в проективной плоскости над почтителом порядка p^2 и показана их связь с симметриями относительно окружностей в плоскостях Минковского.

Author's address: Agricultural and Technical Academy in Olsztyn, 10-740 Olsztyn.