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Peter Bugata; Mirko Horňák; Stanislav Jendroľ

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ON GRAPHS WITH GIVEN NEIGHBOURHOODS

PETER BUGATA, MIRKO HORŇÁK, STANISLAV JENDROĽ, Košice

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Summary. A relationship between 1- and 2-realizability of graphs is established. Zykov problem of 1-realizability is solved for two classes of graphs. A concept of $\bar{k}$-realizability of graphs is introduced; some necessary and some sufficient conditions of $\bar{k}$-realizablity are presented

Keywords: k-neighbourhood, k-realizability of graphs, closed k-neighbourhood, $\bar{k}$-realizability. of graphs.

AMS Classification: 05C99.

1. INTRODUCTION

In this paper a graph will mean a finite non-oriented graph without loops and multiple edges: the edge set $E(G)$ of a graph G is a subset of $\mathscr{P}_2(V(G))$ – the set of all 2-element subsets of the vertex set $V(G)$ of G. The usual concepts (but not necessarily the notation) of the graph theory are taken from Harary [2]: $\deg_G(v)$ denotes the degree of a vertex v in a graph G, $d_G(v, w)$ the distance in G between its vertices v, w, $e_G(v)$ the eccentricity of v in G, $r(G)$ the radius, $d(G)$ the diameter, $Z(G)$ the centre and $\bar{G}$ the complement of G, $G\langle U\rangle$ the subgraph of G induced by a set $U \subseteq V(G)$, $G_1 \times G_2$ the Cartesian product, $G_1 \cup G_2$ the disjoint union, $G_1 + G_2$ the join (Zykov sum) and $G_1[G_2]$ the composition of graphs G_1, G_2. $\prod_{i=1}^{n} G_i$, $\bigcup_{i=1}^{n} G_i$ and $\sum_{i=1}^{n} G_i$ are natural generalizations of the above operations on graphs; for $G_i \cong G$, $i = 1, \ldots, n$, $\bigcup_{i=1}^{n} G_i$ is shortened to nG. A cycle, a path or a star with k vertices will be denoted by C_k, P_k or S_k, respectively.

We define the *periphery* of G by $P(G) = \{v \in V(G): \forall z \in Z(G)\ d_G(z, v) = r(G)\}$, the *k-neighbourhood* of a vertex v in G by $N_k(v, G) = G\langle\{w \in V(G): d_G(v, w) = k\}\rangle$ and the *closed k-neighbourhood* (or *$\bar{k}$-neighbourhood*) of v in G by $N_{\bar{k}}(v, G) = G\langle\{w \in V(G): d_G(v, w) \leqq k\}\rangle$. A graph H is said to be *k-realizable*, $k \in \bigcup_{m=1}^{\infty} \{m, \bar{m}\}$, if there exists a graph $G \neq K_0 = (\emptyset, \emptyset)$ (a *k-realization* of H) such that $N_k(v, G) \cong H$ for all $v \in V(G)$.

The problem of 1-realizability posed by Zykov [9] is algorithmically unsolvable (Bulitko [5]), nevertheless it has been an object of study of many authors — see e.g. Blass-Harary-Miller [2], Brown-Connelly [3, 4], Hell [7], Sedláček [9]. Bielak [1] defined the term k-realizability and showed that a) the problem of 2-realizability is non-trivial, by founding an infinite number of examples of graphs which are 2-realizable, as well as of those which are not, and b) for any $k \geqq 3$ and a non-empty graph H the composition $C_{2k}[H]$ is a k-realization of H.

In the first part of this paper we show that the problem of 2-realizability is more difficult than the problem of 1-realizability. We present some results concerning 1- and 2-realizability of graphs. In the second part we deal with $\bar{k}$-realizability and give some necessary and some sufficient conditions for the $\bar{k}$-realizability.

2. GRAPHS WITH GIVEN 1- OR 2-NEIGHBOURHOODS

2.1. Theorem. *A non-empty graph H is 1-realizable if and only if the graph $\bar{H}$ is 2-realizable by a graph with diameter 2.*

Proof. a) If G is a 1-realization of H, then no component of G is K_1, hence $\bar{G} + \bar{G}$ is a 2-realization of $\bar{H}$ (see [1]) and $d(\bar{G} + \bar{G}) = 2$.

b) If G is a 2-realization of $\bar{H}$ with $d(G) = 2$, then $\bar{G}$ is a 1-realization of H. Indeed, for $v \in V(\bar{G}) = V(G)$ we have $N_2(v, G) \cong \bar{H}$, $V(N_1(v, \bar{G})) = \{w \in V(\bar{G}): \{v, w\} \in E(\bar{G})\} = \{w \in V(G): \{v, w\} \notin E(G)\} = \{w \in V(G): d_G(v, w) = 2\} = V(N_2(v, G))$ and $E(N_1(v, \bar{G})) = \{\{x, y\} \in \mathscr{P}_2(V(\bar{G})): d_{\bar{G}}(x, y) = d_{\bar{G}}(v, x) = d_{\bar{G}}(v, y) = 1\} = \{\{x, y\} \in \mathscr{P}_2(V(G)): d_G(x, y) = d_G(v, x) = d_G(v, y) = 2\} = E(\overline{N_2(v. G)})$.

2.2. Lemma. *If G is a 2-realization of a non-empty graph H with $d(H) \leqq 2$, then there exists a 2-realization K of H such that $d(K) = \min\{d(G), 2\}$.*

Proof. For $d(G) > 2$ define K by $V(K) = V(G)$ and $E(K) = \{\{v, w\} \in \mathscr{P}_2(V(G)): d_G(v, w) \neq 2\}$. Then evidently $d(K) = 2$ and $V(N_2(x, K)) = V(N_2(x, G))$, $E(N_2(x, K)) \supseteq E(N_2(x, G))$ for each $x \in V(K)$. The assumption $\{v, w\} \in E(N_2(x, K)) - E(N_2(x, G))$ would lead to $3 \leqq d_G(v, w) \leqq d_{N_2(x,G)}(v, w)$ in contradiction with $N_2(x, G) \cong H$ and $d(H) \leqq 2$; hence, $N_2(x, K) = N_2(x, G) \cong H$.

2.3. Theorem. *A disconnected graph H is 1-realizable if and only if $\bar{H}$ is 2-realizable.*

Proof. a) With respect to Theorem 2.1 a disconnected (obviously non-empty) 1-realizable graph H the graph $\bar{H}$ is 2-realizable.

b) Since the disconnectedness of H implies $d(\bar{H}) \leqq 2$ and $\bar{H} \neq K_0$, any 2-realization of $\bar{H}$ has diameter at least 2 and by Lemma 2.2 there exists a 2-realization G of $\bar{H}$ with $d(G) = 2$, and according to Theorem 2.1 H is 1-realizable.

The following lemma is in fact (in a more special form) proved in [7], but it is not formulated there as a special statement.

2.4. Lemma. (i) *If H is a 1-realizable graph, then for every positive integer n the graph $H \cup nK_1$ is 1-realizable, too.*

(ii) *If H is a graph and there exists a positive integer n such that the graph $H \cup nK_1$ is 1-realizable, then the graph H is 1-realizable, too.*

Proof. (i) According to [7] the disjoint union of a finite number of 1-realizable graphs is 1-realizable, too, and the desired result follows since K_{l+1} is a 1-realization of K_l.

(ii) Let G be a 1-realization of $H \cup nK_1$ and let H_1 be the graph obtained from H by deleting all of its isolated vertices. Then the deletion of all edges of G belonging to no triangle yields a 1-realization of H_1. As $H \cong H_1 \cup mK_1$ for a suitable non-negative integer m, the graph H is 1-realizable by (i).

2.5. Theorem. (i) *If H is a non-empty 1-realizable graph, then for every positive integer n the graph $\bar{H} + K_n$ is 2-realizable.*

(ii) *If H is a graph and there exists a positive integer n such that the graph $\bar{H} + K_n$ is 2-realizable, then H is 1-realizable.*

Proof. Use Theorems 2.1 and 2.3, Lemma 2.4, the isomorphism of $\overline{H \cup nK_1}$ and $\bar{H} + K_n$, the disconnectedness of $H \cup nK_1$ (for $H \neq K_0$ or $n \geqq 2$) and the 1-realizability of K_0 (for $H = K_0$ and $n = 1$).

2.6. Remark. Theorem 2.5 shows that the problem of 2-realizability is more complicated than the original Zykov problem — to solve the latter one it is sufficient to know which graphs of radius 1 are 2-realizable.

2.7. Theorem. *If $k, l, m_1, \ldots, m_l$ are positive integers and n is an integer, $3 \leqq n \leqq 6$, then the graphs $P_k, K_{m_1,\ldots,m_l}$ and C_n are 2-realizable by a graph with diameter 2.*

Proof. a) The graph $C_{k+3} + C_{k+3}$ is a suitable 2-realization of P_k.

b) The graph $\prod_{i=1}^{l} K_{m_i+1}$ is a 1-realization of the graph $\bigcup_{i=1}^{l} K_{m_i}$ (see [7]), hence by Theorem 2.1 the complete l-partite graph

$$K_{m_1,\ldots,m_l} \cong \overline{\bigcup_{i=1}^{l} K_{m_i}}$$

is 2-realizable by a graph with diameter 2.

c) For $C_3 \cong K_3$ and $C_4 \cong K_{2,2}$ recall b). Since C_5 is 1-realizable according to [3], it is sufficient to use Theorem 2.1 and the isomorphism of $\bar{C}_5$ and C_5. Fig. 1 depicts a 2-realization of C_6 with diameter 2.

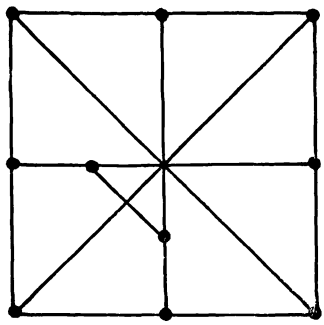

Fig. 1

Zelinka [10] proved that complements of paths are 1-realizable and that $\bar{C}_k$ is 1-realizable if and only if $k \leqq 6$. The following assertion is a generalization of this result.

2.8. Theorem. *If $\{c_k\}_{k=3}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ are sequences of non-negative integers such that $\sum_{k=3}^{\infty}(c_k + p_k + s_k)$ is finite, then the graph $\bigcup_{k=3}^{\infty} c_k\bar{C}_k \cup \bigcup_{k=1}^{\infty} p_k\bar{P}_k \cup \bigcup_{k=1}^{\infty} s_k\bar{S}_k$ is 1-realizable if and only if $\sum_{k=7}^{\infty} c_k = 0$.*

Proof. a) For a positive integer k the graph $\bar{S}_k = K_{k-1} \cup K_1$ is 1-realizable, and the same holds – in virtue of Theorems 2.1 and 2.7 – for the graph $\bar{P}_k$. For a positive integer k, $3 \leqq k \leqq 6$, the graph $\bar{C}_k$ is 1-realizable again by Theorems 2.1 and 2.7. Hence $\sum_{k=7}^{\infty} c_k = 0$ implies the desired 1-realizability.

b) Suppose that l is an integer, $l \geqq 7$, and that the graph $\bigcup_{k=3}^{\infty} c_k\bar{C}_k \cup \bigcup_{k=1}^{\infty} p_k\bar{P}_k \cup \bigcup_{k=1}^{\infty} s_k\bar{S}_k = \bigcup_{m=1}^{n} \bar{H}_m$ with $H_1 = C_l$ and each H_m, $m \in \{2, \ldots, n\}$ being a cycle, a path or a star, is 1-realizable. Then by Theorem 2.1 the graph

$$H = \overline{\bigcup_{m=1}^{n} \bar{H}_m} \cong \sum_{m=1}^{n} H_m$$

is 2-realizable by a graph G with $d(G) = 2$. For any $v \in V(G)$ there exists a decomposition $\{V_m(v): m = 1, \ldots, n\}$ of $V(N_2(v, G))$ such that $N_2(v, G)\langle V_m(v)\rangle \cong H_m$, $m = 1, \ldots, n$, and $\{x, y\} \in E(N_2(v, G))$ whenever $x \in V_i(v)$, $y \in V_j(v)$, $i \neq j$. Denote the vertices of $V_1(v)$ by $v_1, \ldots, v_l$ in such a way that $\{v_i, v_{i+1}\} \in E(N_2(v, G))$, $i = 1, \ldots$ $\ldots, l-1$; then necessarily $\{v_l, v_1\} \in E(N_2(v, G))$. As $\{v_2, v_i\} \notin E(G)$, $i = 4, \ldots, l$, $\{v_2, v\} \notin E(G)$ and $d(G) = 2$, we have $\{v, v_4, \ldots, v_l\} \subseteq V(N_2(v_2, G))$, and if $v \in V_j(v_2)$, $j \in \{1, \ldots, n\}$, then evidently $\{v_4, \ldots, v_l\} \subseteq V_j(v_2)$ – otherwise $\{v, v_m\} \in E(G)$ for some $m \in \{4, \ldots, l\}$ in contradiction with $v_m \in V(N_2(v, G))$. The connected graph $G_j = N_2(v_2, G)\langle V_j(v_2)\rangle$ is clearly not a star (it has at least $l - 5 \geqq 2$ vertices of degree $\geqq 2$), hence for $q = \min\{d_{G_j}(v, v_p): p = 4, \ldots, l\}$ we have $q \geqq 2$ and $d_{G_j}(v, v_p) > q$, $p = 5, \ldots, l - 1$ (if not, $\deg_{G_j}(v_p) \geqq 3$). Suppose therefore without loss of generality $d_{G_j}(v, v_l) = q$ and denote by $v_l = w_0, \ldots, w_q = v$ the vertices of a shortest path in G_j between v and v_l. Since $\deg_{G_j}(w_1) = 2$ and $\{w_1, \ldots, w_q\} \cap \cap \{v_4, \ldots, v_l\} = \emptyset$, $\{w_1, v_4\}$ is not an edge of G and $d_G(w_1, v_4) = 2$, $w_1 \in V_s(v_4)$ for a suitable $s \in \{1, \ldots, n\}$. Take $t_i \in \{1, \ldots, n\}$ so that $v_i \in V_{t_i}(v_4)$, $i = 1, 2, 6, \ldots, l$; then for $i \neq j$ with $\{v_i, v_j\} \notin E(G)$ necessarily $t_i = t_j$ and consequently $t_1 = t_2 = = t_6 = \ldots = t_l$. We can also assert $s = t_{l-1}$ – in the opposite case $\{w_1, v_{l-1}\} \in \in E(G_j)$ results in $\deg_{G_j}(v_{l-1}) \geqq 3$. Now $N_2(v_4, G) \langle V_s(v_4) \rangle$ has at least three vertice) of degree $\geqq 2$ and at least one vertex of degree $\geqq 3$ (v_l is adjacent to v_1, v_{l-1} and w_1s and it is neither a star nor a path nor a cycle.

2.9. Theorem. *If T is a tree, then $\bar{T}$ is 1-realizable if and only if T is a path or a star.*

Proof. a) If T is a non-empty path or star, $\bar{T}$ is 1-realizable according to Theorem 2.8. For any positive integer n the graph nK_1 is a 1-realization of the empty graph $\bar{P}_0 = \bar{S}_0$.

b) Suppose that a tree T is neither a path nor a star and that $\bar{T}$ is 1-realizable. Let G be a 2-realization of T with $d(G) = 2$ (existing by Theorem 2.1). If $v \in V(G)$, then $N_2(v, G) \cong T$ and in $V(N_2(v, G))$ we can find vertices x, y such that $\deg_{N_2(v,G)}(x) = = m \geqq 3$, $\deg_{N_2(v,G)}(y) = 1$ and $\{x, y\} \notin E(N_2(v, G))$. Since $d(G) = 2$, $V(N_2(x, G))$ consists of v, the vertices of $N_2(v, G)$ non-adjacent to x, and of m remaining vertices $w_1, \ldots, w_m \in V(G) - \{v\} - V(N_2(v, G))$ necessarily adjacent to v in G. As y is adjacent to exactly one vertex u of $N_2(v, G)$, $V(N_2(y, G)) \supseteq \{v\} \cup V(N_2(v, G)) - - \{u, y\}$, hence $V(N_2(y, G))$ contains at most one of the vertices $w_1, \ldots, w_m$. Thus $|V(N_1(y, G)) \cap \{w_1, \ldots, w_m\}| \geqq m - 1 \geqq 2$ and if, without loss of generality, $w_1, w_2 \in V(N_1(y, G))$, then $N_2(x, G)$ contains as a subgraph the cycle of length 4 passing through the vertices v, w_1, y, w_2, and we have obtained a contradiction with the structure of $T \cong N_2(x, G)$.

3. $\bar{k}$-REALIZABILITY OF GRAPHS

In this part we deal with $\bar{k}$-realizability of graphs. In view of the following two obvious facts we can restrict our analysis to connected $\bar{k}$-realizations of non-empty graphs.

3.1. Proposition. *If k is a positive integer, then*

(i) *the graph K_0 is k-realizable and is not $\bar{k}$-realizable;*

(ii) *the graph $G_1 \cup G_2$ is a $\bar{k}$-realization of a graph G if and only if G_1, G_2 are $\bar{k}$-realizations of G.*

The question of $\bar{1}$-realizability has rested practically untouched; this is probably due to the following simple results.

3.2. Lemma. *A graph $H \neq K_0$ is* 1*-realizable if and only if the graph $H + K_1$ is $\bar{1}$-realizable.*

3.3. Lemma. *If k is a positive integer and $H \neq K_0$ is a graph with $d(H) \leqq k$, then H is $\bar{k}$-realizable and a graph G is a connected $\bar{k}$-realization of H if and only if $G \cong H$.*

Proof. H is clearly a connected $\bar{k}$-realization of H. If G is another one, non-isomorphic to H, it must be (up to an isomorphism) equal to a supergraph of H with $V(G) - V(H) \neq \emptyset$. The connectednes of G yields the existence of vertices $v \in V(H)$ and $w \in V(G) - V(H)$ such that $\{v, w\} \in E(G)$, and then $V(N_{\bar{k}}(v, G)) \supseteq V(H) \cup \{w\}$ in contradiction with $N_{\bar{k}}(v, G) = H$.

The main result of this part is

3.4. Theorem. *If k is a positive integer and H is a $\bar{k}$-realizable graph with $d(H) > k$, then*

(i) $r(H) = k$,

(ii) $P(H) \neq \emptyset$,

(iii) $q = |V(H)| / |Z(H)|$ is an integer and there exists a decomposition $\mathscr{V} = \{V_1, \ldots, V_q\}$ of $V(H)$ such that $Z(H) \in \mathscr{V}$ and $H\langle V_i \rangle \cong H\langle Z(H) \rangle$, $i = 1, \ldots, q$.

Proof. Let G be a connected supergraph of H representing a $\bar{k}$-realization of H.

(i) For any vertex v of G we have $r(H) = r(N_{\bar{k}}(v, G)) \leqq k$. Since $V(N_{\bar{k}}(v, G)) \neq V(H)$ for each $v \in V(H)$ with $e_H(v) > k$ and $d(H) > k$, $V(H)$ is a proper subset of $V(G)$ and we can choose $w \in V(G) - V(H)$. Taking $z \in Z(H)$ we get $N_{\bar{k}}(z, G) = H$, hence $d = d_G(z, w) > k$: if $z = w_0, w_1, \ldots, w_d = w$ are the vertices forming a shortest path in G between z and w, then $w_k \in V(N_{\bar{k}}(z, G))$ and $r(H) \geqq k$.

(ii) Using $w_k \in V(H)$ and $r(H) = k$ we can state that $d_H(v, w_k) \leqq k$ for every $v \in Z(H)$. On the other hand, $d_H(v, w_k) \geqq k$, for if this were not the case, $w_{k+1} \in V(N_{\bar{k}}(v, G))$ in contradiction with $N_{\bar{k}}(v, G) = H = N_{\bar{k}}(z, G)$ and $d_H(z, w_{k+1}) = k + 1$. Thus $d_H(v, w_k) = k$ and $w_k \in P(H)$.

(iii) As $r(H) = k$, any $v \in V(G)$ belongs to $Z(N_{\bar{k}}(v, G))$ and it is easy to see that $N_{\bar{k}}(u, G) = N_{\bar{k}}(v, G)$ for all $u \in Z(N_{\bar{k}}(v, G))$. If $w \in V(N_{\bar{k}}(v, G)) - Z(N_{\bar{k}}(v, G)) = W$, then by (i) $d_G(w, x) > k$ for a suitable $x \in W$, while for $w \in V(G) - V(N_{\bar{k}}(v, G))$ we have $d_G(v, w) > k$; hence $w \in V(G) - Z(N_{\bar{k}}(v, G))$ implies $N_{\bar{k}}(w, G) \neq N_{\bar{k}}(v, G)$ and consequently $Z(N_{\bar{k}}(w, G)) \cap Z(N_{\bar{k}}(v, G)) = \emptyset$. This proves that $r = |V(G)| / |Z(H)|$ is an integer and there exists a decomposition $\mathscr{U} = \{U_1, \ldots, U_r\}$ of $V(G)$ such that $Z(H) \in \mathscr{U}$ and $G\langle U_i \rangle \cong H\langle Z(H) \rangle$, $i = 1, \ldots, r$. Furthermore, since $Z(H) \subseteq V(N_{\bar{k}}(v, G))$ for $v \in V(H)$ and $V(N_{\bar{k}}(w, G)) \cap Z(H) = \emptyset$ for $w \in V(G) - V(H)$, we have $U_i \subseteq V(H)$ or $U_i \cap V(H) = \emptyset$ for each $i = 1, \ldots, r$, and q members of $\mathcal{U}$ form a decomposition of $V(H)$.

3.5. Corollary. *If H is a 1-realizable graph with $r(H) = 1$, then $|V(H)| + 1$ is divisible by $|Z(H)| + 1$.*

Proof. By Lemma 3.2 the graph $H + K_1 = H_1$ is 1-realizable; $r(H) = 1$ leads to $|Z(H_1)| = |Z(H)| + 1$. As $|V(H_1)| = |V(H)| + 1$, for $d(H) > 1$ (which implies $d(H_1) > 1$) apply Theorem 3.4 (iii), while for $d(H) = 1$, i.e. $H \cong K_l$, $l \geqq 2$, make use of the equality of $Z(H)$ and $V(H)$.

3.6. Remark. Corollary 3.5 is also a consequence of Theorem 1 in [7].

3.7. Theorem. *If k is a positive integer, H is a $\bar{k}$-realizable graph with $d(H) > k$ and G is a $\bar{k}$-realization of H, then $\{\deg_G(v): v \in V(G)\} = \{\deg_H(w): w \in Z(H)\}$.*

Proof. Since all vertices adjacent to $v \in V(G)$ belong to $N_{\bar{k}}(v, G)$, we get $\deg_G(v) = \deg_{N_{\bar{k}}(v,G)}(v)$. By Theorem 3.4 (i) we have $v \in Z(N_{\bar{k}}(v, G))$, hence the isomorphism of $N_{\bar{k}}(v, G)$ and H implies the existence of $w \in Z(H)$ such that $\deg_G(v) = \deg_H(w)$. The converse inclusion is obvious.

In what follows we deal with $\bar{k}$-realizability of trees.

3.8. Lemma. *If T is a tree with $P(T) \neq \emptyset$, then*
(i) $|Z(T)| = 1$,
(ii) *$T \cong K_1$ or $\deg_T(v) = 1$ for every $v \in P(T)$.*

Proof. (i) The assumption $|Z(T)| > 1$ leads to $Z(T) = \{z_1, z_2\} \in E(T)$ (see [6]). If $v \in V(T)$ and $v_0 = v, v_1, \ldots, v_m = z_1$ are vertices of the (unique) path joining v and z_1 in T, then $z_2 = v_{m-1}$ with $d_T(v, z_2) = d_T(v, z_1) - 1$ or $z_2 \notin \{v_0, \ldots, v_m\}$ with $d_T(v, z_2) = d_T(v, z_1) + 1$, both cases resulting in $v \notin P(T)$. Thus $P(T) \neq \emptyset$ implies $Z(T) = \{z\}$.

(ii) If $v \in V(T)$ and $\deg_T(v) \geqq 2$, take a vertex $w \in V(T)$ adjacent to v and not belonging to the path joining v and z. Since $d_T(w, z) = d_T(v, z) + 1$, v is not a peripheral vertex.

For positive integers k, l, let $T_{k,l}$ be a tree with radius k and one-vertex centre whose all vertices except the peripheral ones are of degree 1.

3.9. Theorem. *If k is a positive integer, then a tree $T \neq K_0$ is $\bar{k}$-realizable if and only if $d(T) \leqq k$ or $T \cong T_{k,l}$ for a suitable integer $l \geqq 2$.*

Proof. In view of Lemma 3.3 it is sufficient to analyze the case $d(T) > k$.

(a) If T is a $\bar{k}$-realizable tree with $d(T) > k$, then $r(T) = k$ and $P(T) \neq \emptyset$ by Theorem 3.4. Hence using Lemma 3.8 we get $|Z(T)| = 1$. If $Z(T) = \{z\}$ and $\deg_T(z) = l$, then clearly $l \geqq 2$.

Take a $\bar{k}$-realization G of T which is a supergraph of T; by Theorem 3.7 it is an l-regular graph. For any $v \in V(T) - P(T)$ we have $d_T(z, v) < k$, hence all vertices adjacent to v in G belong to $N_{\bar{k}}(z, G)$ and $\deg_T(v) = \deg_G(v) = l$. Thus we have proved that $T \cong T_{k,l}$.

(b) Let G be an l-regular graph whose girth is $2k + 2$ — its existence is guaranteed by Sachs [8] for $l \geqq 3$, for $l = 2$ take a cycle with $2k + 2$ vertices. It is easy to see that for every $v \in V(G)$ the closed k-neighbourhood of v is a tree isomorphic to $T_{k,l}$.

A $\bar{k}$-realizable graph serves as a basis for a wide class of $\bar{k}$-realizable graphs.

3.10. Theorem. *If k is an integer, $k \geqq 2$, and H is a $\bar{k}$-realizable graph with $|V(H)| \geqq 2$, then for any graph $K \neq K_0$ the graph $H[K]$ is $\bar{k}$-realizable, too.*

Proof. As a $\bar{k}$-realization of $H[K]$ we can take $G[K]$ where G is a $\bar{k}$-realization of H.

3.11. Theorem. *If $k \in \{1, \bar{1}\}$, H is a k-realizable graph and G is a k-realization of H, then the graph $H + G$ is k-realizable, too.*

Proof. The graph $G + G$ is a k-realization of $H + G$.

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Súhrn

O GRAFOCH S DANÝMI OKOLIAMI

PETER BUGATA, MIRKO HORŇÁK, STANISLAV JENDROĽ

Je nájdený vzťah medzi 1- a 2-realizovateľnosťou grafov. Zykovov problém 1-realizovateľnosti je vyriešený pre dve triedy grafov. Je zavedený pojem $\bar{k}$-realizovateľnosti grafov; sú uvedené isté nutné a isté postačujúce podmienky $\bar{k}$-realizovateľnosti.

Резюме

О ГРАФАХ С ДАННЫМИ ОКРЕСТНОСТЯМИ

PETER BUGATA, MIRKO HORŇÁK, STANISLAV JENDROĽ

Устанавливается связь между 1- и 2-реализуемостью графов. Проблема Зыкова об 1-реализуемости решается для двух классов графов. Вводится понятие $\bar{k}$-реализуемости; приводятся некоторые необходимые и некоторые достаточные условия $\bar{k}$-реализуемости.

Authors' address: Katedra geometrie a algebry PF UPJŠ, Jesenná 5, 041 54 Košice.