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SOME PROPERTIES OF LATTICE HOMOMORPHISMS

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Summary. Let L be a chain and K, K_1 be lattices. We show that an isomorphism of powers L^K, L^{K_1} does not imply an isomorphism of lattices K, K_1. In particular: for any lattice K there exists a distributive lattice K_1 such that the ordered sets L^K, L^{K_1} are isomorphic.

Keywords: Lattice, distributive lattice, homomorphism, ideal, prime ideal, power of lattices.

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B. Zástěra proved ([6]) the following assertion: Let L and L_1 be lattices. If the sets of join homomorphisms of L and L_1 into reals are isomorphic as ordered sets by pointwise ordering, then the lattices L, L_1 are isomorphic. In this note we study the set of homomorphisms of a lattice K into a chain L, i.e. the power L^K.

1. INTRODUCTORY CONCEPTS AND ASSERTIONS

The cardinality of a set A is denoted by $|A|$. Throughout the paper, any set G will be called nontrivial iff $|G| \geqq 2$.

Let G be an ordered (=partially ordered) set. For any $a \in G$ denote

$$(a] = \{x \in G;\ x \leqq a\}\,, \quad (a) = \{x \in G;\ x < a\}\,.$$

Let G be an ordered set and $H \subseteq G$. We call H *dense* in G iff it has the property

$$x, y \in G\,,\ x < y \Rightarrow \text{there exist } u, v \in H \text{ with } x \leqq u < v \leqq y\,.$$

1.1. Lemma. *Let G be an ordered set which is a join-semilattice, let $H \subseteq G$ be dense in G. Then $a = \inf\{x \in H;\ x \geqq a\}$ for any $a \in G$.*

Proof. Let $a \in G$ and denote $H(a) = \{x \in H;\ x \geqq a\}$. Clearly, a is a lower bound of $H(a)$. Let b be any lower bound of $H(a)$ and suppose $b \nleqq a$. Then $a \vee b > a$ and thus there exist $u, v \in H$ such that $a \leqq u < v \leqq a \vee b$. This means $u \in H(a)$ which implies $a \leqq u$, $b \leqq u$. Hence $a \vee b \leqq u$ which is a contradiction. Thus $b \leqq a$ and $a = \inf H(a)$.

Let G be a set, H an ordered set and $f\colon G \to H$ a mapping. We denote by Q_f the mapping of H into $\exp G$ defined by

$$Q_f(a) = f^{-1}((a]) = \{x \in G;\ f(x) \leqq a\} \quad \text{for any} \quad a \in H\,.$$

Analogously we define the mapping $R_f\colon H \to \exp G$ as

$$R_f(a) = f^{-1}((a)) = \{x \in G;\ f(x) < a\} \quad \text{for any} \quad a \in H\,.$$

Let L be a lattice, $I \subseteq L$. I is called an *ideal* in L iff it has the properties

$$x, y \in I \Rightarrow x \vee y \in I\,; \quad x \in L, \quad y \in I\,, \quad x \leqq y \Rightarrow x \in I\,.$$

An ideal I in a lattice L is called *prime* iff

$$x, y \in L, \quad x \wedge y \in I \Rightarrow x \in I \quad \text{or} \quad y \in I\,.$$

We denote by $\mathscr{I}(L)$ the set of all ideals of a lattice L and by $\mathscr{P}(L)$ the set of all prime ideals of L. Both sets $\mathscr{I}(L)$, $\mathscr{P}(L)$ are ordered by set inclusion.

Note that, according to our definition, $\emptyset \in \mathscr{P}(L)$, $L \in \mathscr{P}(L)$ for any lattice L.

If L is a lattice and $a \in L$, then $(a] \in \mathscr{I}(L)$; it is called a principal ideal of L. As $a \in (a]$, the necessary condition for $(a] \in \mathscr{P}(L)$ is that a is meet irreducible. An element a of a lattice L is *meet irreducible* iff

$$x, y \in L, \quad a = x \wedge y \Rightarrow x = a \quad \text{or} \quad y = a\,.$$

However, as is well known, $(a] \,\bar{\in}\, \mathscr{P}(L)$ may also hold when a is meet irreducible.

Let us call an element a of a lattice L $(\vee, \wedge)$ – *distributive*, iff

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \quad \text{for any} \quad x, y \in L\,.$$

1.2. Lemma. *Let L be a lattice and $a \in L$ a $(\vee, \wedge)$ – distributive element. Then $(a] \in \mathscr{P}(L)$ if and only if a is meet irreducible.*

Proof. The necessity of the condition is clear; we prove its sufficiency. Thus, let a be meet irreducible and suppose $b \in (a]$, $b = x \wedge y$. Then $a = a \vee b = a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ and hence $a = a \vee x$ or $a = a \vee y$, i.e. $x \leqq a$ or $y \leqq a$. Thus $x \in (a]$ or $y \in (a]$ and $(a] \in \mathscr{P}(L)$.

Especially, if L is a distributive lattice and $a \in L$, then $(a] \in \mathscr{P}(L)$ iff a is meet irreducible ([1], p. 67 or [2], p. 28).

1.3. Remark. Let L be a chain. Then L is a distributive lattice and any element of L is meet irreducible. Thus $(a] \in \mathscr{P}(L)$ for any $a \in L$. Further, it is easy to see that also $(a) \in \mathscr{P}(L)$ for any $a \in L$.

Let K, L be lattices. We denote by $\operatorname{Hom}(K, L)$ the set of all homomorphisms of K into L.

1.4. Lemma. *Let K, L be lattices and $f \in \operatorname{Hom}(K, L)$. If $P \in \mathscr{P}(L)$ then $f^{-1}(P) \in P(K)$.*

Proof. Let $P \in \mathscr{P}(L)$ and $x, y \in f^{-1}(P)$. Then $f(x) \in P$, $f(y) \in P$, $f(x \vee y) = f(x) \vee f(y) \in P$ and $x \vee y \in f^{-1}(P)$. Let $x \in K$, $y \in f^{-1}(P)$, $x \leqq y$. Then $f(y) \in P$

and $f(x) \leqq f(y)$ as f is monotone. Thus $f(x) \in P$ and $x \in f^{-1}(P)$. Let $x, y \in K$, $x \wedge y \in f^{-1}(P)$. Then $f(x \wedge y) = f(x) \wedge f(y) \in P$, hence $f(x) \in P$ or $f(y) \in P$ and $x \in f^{-1}(P)$ or $y \in f^{-1}(P)$.

1.5. Lemma. *Let L be a lattice. Put $\mathscr{P}(x) = \{P \in \mathscr{P}(L);\ x \in P\}$ for any $x \in L$ and $\mathscr{R} = \{\mathscr{P}(x);\ x \in L\}$. Then $\mathscr{R}$ is a ring of sets (thus a distributive lattice with respect to set operations) and $\mathscr{P}$ is a surjective dual homomorphism of L onto $\mathscr{R}$.*

Proof. Clearly, $\mathscr{P}$ is a surjective mapping of L onto $\mathscr{R}$. Let $x, y \in L$. Then $\mathscr{P}(x \vee y) = \{P \in \mathscr{P}(L);\ x \vee y \in P\} = \{P \in \mathscr{P}(L);\ x \in P \text{ and } y \in P\} = \{P \in \mathscr{P}(L);\ x \in P\} \cap \{P \in \mathscr{P}(L);\ y \in P\} = \mathscr{P}(x) \cap \mathscr{P}(y)$, $\mathscr{P}(x \wedge y) = \{P \in \mathscr{P}(L);\ x \wedge y \in P\} = \{P \in \mathscr{P}(L);\ x \in P \text{ or } y \in P\} = \{P \in \mathscr{P}(L);\ x \in P\} \cup \{P \in \mathscr{P}(L);\ y \in P\} = \mathscr{P}(x) \cup \mathscr{P}(y)$. Thus $\mathscr{P}$ is a dual homomorphism and simultaneously we obtain that $\mathscr{R}$ is a ring of sets.

2. CHARACTERIZATION OF LATTICE HOMOMORPHISMS

2.1. Theorem. *Let K, L be lattices and $f: K \to L$ a mapping. If there exists a subset $H \subseteq L$ dense in L such that $Q_f(y) \in \mathscr{P}(K)$ for any $y \in H$, then $f \in \operatorname{Hom}(K, L)$.*

Proof. Let $x_1, x_2 \in K$ and denote $f(x_1) = y_1$, $f(x_2) = y_2$. We prove first $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = y_1 \vee y_2$. Denote $y_1 \vee y_2 = y$, $f(x_1 \vee x_2) = z$ and assume $z \nleqq y$. Then $y < y \vee z$ and thus there exist $u_1, v_1 \in H$ with $y \leqq u_1 < v_1 \leqq y \vee z$. Then $y_1 \leqq u_1$, $y_2 \leqq u_1$, i.e. $x_1 \in Q_f(u_1)$, $x_2 \in Q_f(u_1)$, and as $Q_f(u_1) \in \mathscr{P}(K)$, we have $x_1 \vee x_2 \in Q_f(u_1)$, i.e. $f(x_1 \vee x_2) = z \leqq u_1$. Hence $y \vee z \leqq u_1$, a contradiction. Thus $z \leqq y$; assume that $z < y$. Then there exist $u_2, v_2 \in H$ such that $z \leqq u_2 < v_2 \leqq y$. As $x_1 \vee x_2 \in Q_f(u_2)$ and $Q_f(u_2) \in \mathscr{P}(K)$, we have $x_1 \in Q_f(u_2)$ and $x_2 \in Q_f(u_2)$. Hence $f(x_1) = y_1 \leqq u_2$, $f(x_2) = y_2 \leqq u_2$ and $y_1 \vee y_2 = y \leqq u_2$, a contradiction. Thus $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$. Further, we prove $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2) = y_1 \wedge y_2$. Denote $f(x_1 \wedge x_2) = u$, $y_1 \wedge y_2 = v$; we show first $u \leqq y_1$. If this is not the case then $y_1 < y_1 \vee u$ and thus there exist $u_3, v_3 \in H$ with $y_1 \leqq u_3 < v_3 \leqq y_1 \vee u$. As $x_1 \in Q_f(u_3)$ and $Q_f(u_3) \in \mathscr{P}(K)$, we have $x_1 \wedge x_2 \in Q_f(u_3)$, i.e. $f(x_1 \wedge x_2) = u \leqq u_3$. Then $y_1 \vee u \leqq u_3$, a contradiction. Thus $u \leqq y_1$ and similarly $u \leqq y_2$. Hence $u \leqq y_1 \wedge y_2 = v$; suppose that $u < v$. Then there exist $u_4, v_4 \in H$ with $u \leqq u_4 < v_4 \leqq v$. As $x_1 \wedge x_2 \in Q_f(u_4)$ and $Q_f(u_4) \in \mathscr{P}(K)$, we have $x_1 \in Q_f(u_4)$ or $x_2 \in Q_f(u_4)$, i.e. $f(x_1) = y_1 \leqq u_4$ or $f(x_2) = y_2 \leqq u_4$. But then $y_1 \wedge y_2 = v \leqq u_4$, which is a contradiction. Thus $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$ and $f \in \operatorname{Hom}(K, L)$.

2.2. Lemma. *Let K be a lattice, L a chain and $f: K \to L$ a mapping. If there exists a subset $H \subseteq L$ dense in L such that $R_f(y) \in \mathscr{P}(K)$ for any $y \in H$, then $f \in \operatorname{Hom}(K, L)$.*

Proof. Let $x_1, x_2 \in K, f(x_1) = y_1, f(x_2) = y_2$. Then either $y_1 \leqq y_2$ or $y_2 \leqq y_1$; let us assume that $y_1 \leqq y_2$. Denote $f(x_1 \vee x_2) = y$ and assume that $y = y_2$ does not hold. If $y_2 < y$, then there exist $u_1, v_1 \in H$ with $y_2 \leqq u_1 < v_1 \leqq y$; then $x_1 \in R_f(v_1), x_2 \in R_f(v_1)$ and $x_1 \vee x_2 \in R_f(v_1)$, i.e. $f(x_1 \vee x_2) = y < v_1$, a contradiction. If $y < y_2$, then there exist $u_2, v_2 \in H$ with $y \leqq u_2 < v_2 \leqq y_2$; then $x_1 \vee x_2 \in \in R_f(v_2)$, thus $x_2 \in R_f(v_2)$, i.e. $f(x_2) = y_2 < v_2$, a contradiction. Thus $y = y_2$, i.e. $f(x_1 \vee x_2) = y_2 = y_1 \vee y_2 = f(x_1) \vee f(x_2)$.

Denote further $f(x_1 \wedge x_2) = z$ and assume that $z = y_1$ does not hold. Let $y_1 < z$; then there exist $u_3, v_3 \in H$ such that $y_1 \leqq u_3 < v_3 \leqq z$. As $x_1 \in R_f(v_3)$, we have $x_1 \wedge x_2 \in R_f(v_3)$, i.e. $f(x_1 \wedge x_2) = z < v_3$, a contradiction. Let $z < y_1$; then there exist $u_4, v_4 \in H$ with $z \leqq u_4 < v_4 \leqq y_1$. As $x_1 \wedge x_2 \in R_f(v_4)$ and $R_f(v_4) \in \mathscr{P}(K)$, we have $x_1 \in R_f(v_4)$ or $x_2 \in R_f(v_4)$, i.e. $f(x_1) = y_1 < v_4$ or $f(x_2) = y_2 < v_4$. As $y_1 \leqq y_2$, we have $y_1 < v_4$ and this is a contradiction. Hence $z = y_1$, i.e. $f(x_1 \wedge x_2) = = y_1 = y_1 \wedge y_2 = f(x_1) \wedge f(x_2)$ and $f \in \operatorname{Hom}(K, L)$.

2.3. Theorem. *Let K be a lattice, L a chain and $f: K \to L$ a mapping. Then the following statements are equivalent:*

(1) $f \in \operatorname{Hom}(K, L)$;

(2) $Q_f(y) \in \mathscr{P}(K)$ for any $y \in L$;

(3) *there exists a subset $H \subseteq L$ dense in L such that $Q_f(y) \in \mathscr{P}(K)$ for any $y \in H$;*

(4) $R_f(y) \in \mathscr{P}(K)$ for any $y \in L$;

(5) *there exists a subset $H \subseteq L$ dense in L such that $R_f(y) \in \mathscr{P}(K)$ for any $y \in H$.*

Proof. (1) $\Rightarrow$ (2) by 1.3 and 1.4. (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) by 2.1. (1) $\Rightarrow$ (4) by 1.3 and 1.4, (4) $\Rightarrow$ (5) is trivial and (5) $\Rightarrow$ (1) by 2.2.

2.4. Theorem. *Let K be a lattice, L a nontrivial chain, and let $x_1, x_2 \in K$. Then the following statements are equivalent:*

(1) $f(x_1) = f(x_2)$ *for any* $f \in \operatorname{Hom}(K, L)$;

(2) $x_1 \in P \Leftrightarrow x_2 \in P$ *for any* $P \in \mathscr{P}(K)$.

Proof. 1. Let (1) hold and let $P \in \mathscr{P}(K)$. Choose any $y_1, y_2 \in L$, $y_1 < y_2$ and define a mapping $f: K \to L$ by $f(x) = y_1$ for $x \in P$ and $f(x) = y_2$ for $x \in K - P$. It is easy to show that $f \in \operatorname{Hom}(K, L)$: if $u, v \in K$, $u, v \in P$, then $u \vee v \in P$ and $f(u \vee v) = y_1 = y_1 \vee y_1 = f(u) \vee f(v)$; if $u \bar{\in} P$ or $v \bar{\in} P$, then $u \vee v \bar{\in} P$ and $f(u \vee v) = y_2 = f(u) \vee f(v)$. If $u \bar{\in} P$, $v \bar{\in} P$, then $u \wedge v \bar{\in} P$ and $f(u \wedge v) = y_2 = = y_2 \wedge y_2 = f(u) \wedge f(v)$; if $u \in P$ or $v \in P$, then $u \wedge v \in P$ and $f(u \wedge v) = y_1 = = f(u) \wedge f(v)$. Thus $f \in \operatorname{Hom}(K, L)$ and by (1) $f(x_1) = f(x_2)$. But this implies $x_1 \in P \Leftrightarrow x_2 \in P$.

2. Let (2) hold and let $f \in \operatorname{Hom}(K, L)$. Denote $f(x_1) = y_1, f(x_2) = y_2$. By 2.3, we have $Q_f(y_2) \in \mathscr{P}(K)$ and as $x_2 \in Q_f(y_2)$, we have $x_1 \in Q_f(y_2)$, i.e. $f(x_1) = y_1 \leqq y_2$.

Similarly $Q_f(y_1) \in \mathscr{P}(K)$ and $x_1 \in Q_f(y_1)$, thus $x_2 \in Q_f(y_1)$, i.e. $f(x_2) = y_2 \leqq y_1$. We have $y_1 = y_2$, i.e. $f(x_1) = f(x_2)$.

3. FURTHER PROPERTIES OF LATTICE HOMOMORPHISMS

3.1. Lemma. *Let K be a lattice, L a distributive lattice and $f \in \operatorname{Hom}(K, L)$. Let there exist a subset $L_0 \subseteq f(K)$ dense in $f(K)$ and containing only meet irreducible elements in L. Then $Q_f: L_0 \to \mathscr{P}(K)$ is an isomorphic embedding and $f(x) = \inf\{z \in L_0;\ x \in Q_f(z)\}$ holds for any $x \in K$.*

Proof. By 1.2 and 1.4 we have $Q_f(y) \in \mathscr{P}(K)$ for any $y \in L_0$, so that Q_f maps L_0 into $\mathscr{P}(K)$. Let $y_1, y_2 \in L_0$, $y_1 \leqq y_2$. Then $x \in Q_f(y_1) \Rightarrow f(x) \leqq y_1 \Rightarrow f(x) \leqq y_2 \Rightarrow x \in Q_f(y_2)$ and thus $Q_f(y_1) \subseteq Q_f(y_2)$. Let $Q_f(y_1) \subseteq Q_f(y_2)$ and choose $x_1 \in K$, $x_2 \in K$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Then $x_1 \in Q_f(y_1)$, thus $x_1 \in Q_f(y_2)$ and $x_2 \in Q_f(y_2)$. As $Q_f(y_2) \in \mathscr{P}(K)$, we have $x_1 \vee x_2 \in Q_f(y_2)$ so that $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = y_1 \vee y_2 \leqq y_2$ which implies $y_1 \leqq y_2$. Thus $Q_f: L_0 \to \mathscr{P}(K)$ is an isomorphic embedding. Let $x \in K$ be any element and put $f(x) = y$. By 1.1 we have $y = \inf\{z \in L_0;\ y \leqq z\} = \inf\{z \in L_0; f(x) \leqq z\} = \inf\{z \in L_0;\ x \in Q_f(z)\}$.

3.2. Theorem. *Let K be a lattice, L a chain and $f: K \to L$ a mapping. If there exists a subset $L_0 \subseteq L$ such that $Q_f: L_0 \to \mathscr{P}(K)$ is an isomorphic embedding and $f(x) = \inf\{z \in L_0;\ x \in Q_f(z)\}$ for any $x \in K$, then $f \in \operatorname{Hom}(K, L)$.*

Proof. Put $\mathscr{K}_0 = \{Q_f(z);\ z \in L_0\} \subseteq \mathscr{P}(K)$; by assumption, $Q_f^{-1}: \mathscr{K}_0 \to L_0$ is an isomorphism. Denote $\mathscr{R}(x) = \{P \in \mathscr{K}_0;\ x \in P\}$ for any $x \in K$; by assumption we have $f(x) = \inf\{z \in L_0;\ x \in Q_f(z)\} = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x)\}$. For any $x_1, x_2 \in K$ we have $\mathscr{R}(x_1 \vee x_2) = \{P \in \mathscr{K}_0;\ x_1 \vee x_2 \in P\} = \{P \in \mathscr{K}_0;\ x_1 \in P \text{ and } x_2 \in P\} = \mathscr{R}(x_1) \cap \mathscr{R}(x_2)$, $\mathscr{R}(x_1 \wedge x_2) = \{P \in \mathscr{K}_0;\ x_1 \wedge x_2 \in P\} = \{P \in \mathscr{K}_0;\ x_1 \in P \text{ or } x_2 \in P\} = \mathscr{R}(x_1) \cup \mathscr{R}(x_2)$. Denote $f(x_1) = y_1$, $f(x_2) = y_2$. Then either $y_1 \leqq y_2$ or $y_2 \leqq y_1$; let us suppose that $y_1 \leqq y_2$. Let first $y_1 < y_2$ and $P_2 \in \mathscr{R}(x_2)$. As $y_1 = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1)\} < \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_2)\} = y_2$, there must exist $P_1 \in \mathscr{R}(x_1)$ such that $Q_f^{-1}(P_1) \leqq Q_f^{-1}(P_2)$. As Q_f^{-1} is an isomorphism, we have $P_1 \subseteq P_2$. Then $P_2 \in \mathrm{R}(x_1)$ and this shows $\mathscr{R}(x_2) \subseteq \mathscr{R}(x_1)$. This implies $f(x_1 \vee x_2) = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1 \vee x_2)\} = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1) \cap \mathscr{R}(x_2)\} = \inf\{Q_f^{-1}(P); P \in \mathscr{R}(x_2)\} = f(x_2) = y_2 = y_1 \vee y_2 = f(x_1) \vee f(x_2)$, $f(x_1 \wedge x_2) = \inf\{Q_f^{-1}(P); P \in \mathscr{R}(x_1 \wedge x_2)\} = \inf\{Q_f^{-1}(P); P \in \mathscr{R}(x_1) \cup \mathscr{R}(x_2)\} = \inf\{Q_f^{-1}(P); P \in \mathscr{R}(x_1)\} = f(x_1) = y_1 = y_1 \wedge y_2 = f(x_1) \wedge f(x_2)$. Now suppose that $y_1 = y_2$ holds. If for any $P_2 \in \mathscr{R}(x_2)$ there exists $P_1 \in \mathscr{R}(x_1)$ with $Q_f^{-1}(P_1) \leqq Q_f^{-1}(P_2)$, then repeating the preceding consideration we obtain $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$, $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$. Thus let there exist $P_2 \in \mathscr{R}(x_2)$ such that $Q_f^{-1}(P) > Q_f^{-1}(P_2)$ for any $P \in \mathscr{R}(x_1)$. As Q_f^{-1} is an isomorphism, this means $P \supseteq P_2$ for any $P \in \mathscr{R}(x_1)$. As $x_2 \in P_2$, we have $x_2 \in P$ for any $P \in \mathscr{R}(x_1)$ and hence $\mathscr{R}(x_1) \subseteq$

$\subseteq \mathscr{R}(x_2)$. Now we obtain $f(x_1 \vee x_2) = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1 \vee x_2)\} = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1) \cap \mathscr{R}(x_2)\} = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1)\} = f(x_1) = y_1 = y_1 \vee y_2 = f(x_1) \vee f(x_2)$, $f(x_1 \wedge x_2) = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1 \wedge x_2)\} = \inf\{Q_f^{-1}(P);\ P \in \mathscr{R}(x_1) \cup \mathscr{R}(x_2)\} = \inf\{Q_f^{-1}(P); P \in \mathscr{R}(x_2)\} = f(x_2) = y_2 = y_1 \wedge y_2 = f(x_1) \wedge f(x_2)$. Thus $f \in \mathrm{Hom}(K, L)$.

3.3. Theorem. *Let K be a lattice, L a chain and $f: K \to L$ a mapping. Then the following statements are equivalent:*

(1) $f \in \mathrm{Hom}(K, L)$;

(2) $Q_f: f(K) \to \mathscr{P}(K)$ *is an isomorphic embedding and* $f(x) = \inf\{y \in f(K);\ x \in Q_f(y)\}$ *for any* $x \in K$;

(3) *there exists a subset* $L_0 \subseteq f(K)$ *dense in* $f(K)$ *such that* $Q_f: L_0 \to \mathscr{P}(K)$ *is an isomorphic embedding and* $f(x) = \inf\{y \in L_0;\ x \in Q_f(y)\}$ *for any* $x \in K$.

Proof. (1) $\Rightarrow$ (2) by 3.1, (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) by 3.2.

4. POWER OF LATTICES

4.1. Lemma. *Let K be a lattice, L a nontrivial chain and let $x_1, x_2 \in K$. Let the mapping $\mathscr{P}$ have the same meaning as in 1.5. Then the following statements are equivalent:*

(1) $\mathscr{P}(x_1) = \mathscr{P}(x_2)$;

(2) $f(x_1) = f(x_2)$ *for any* $f \in \mathrm{Hom}(K, L)$.

Proof. $\mathscr{P}(x_1) = \mathscr{P}(x_2)$ means $\{P \in \mathscr{P}(K);\ x_1 \in P\} = \{P \in \mathscr{P}(K);\ x_2 \in P\}$ which means $x_1 \in P \Leftrightarrow x_2 \in P$ for any $P \in \mathscr{P}(K)$. But by 2.4 this statement is equivalent to $f(x_1) = f(x_2)$ for any $f \in \mathrm{Hom}(K, L)$.

4.2. Definition. Let K, L be lattices. The *power* L^K is the set $\mathrm{Hom}(K, L)$ equipped with an order $\leqq$ given by $f \leqq g \Leftrightarrow f(x) \leqq g(x)$ for any $x \in K$.

The power L^K of lattices L, K is thus a subset of a cardinal power $(L, \leqq)^{(K, \leqq)}$ of ordered sets $(L, \leqq)$, $(K, \leqq)$ which consists of all monotonic mappings of K into L. The *cardinal power* $(L, \leqq)^{(K, \leqq)}$ is a lattice in which $f \vee g: x \to f(x) \vee g(x)$, $f \wedge g: x \to f(x) \wedge g(x)$, $x \in K$. L^K is, however, not a sublattice of $(L, \leqq)^{(K, \leqq)}$ as $f \vee g$, $f \wedge g$ need not be homomorphisms of K into L whenever f, g are such homomorphisms.

4.3. Theorem. *Let K be a lattice, L a chain. Then there exists a distributive lattice K_1 such that the ordered sets L^K, L^{K_1} are isomorphic.*

Proof. If L is trivial, then the assertion is clear; thus let $|L| \geqq 2$. For the lattice K, let us construct the lattice $\mathscr{R}$ and the mapping $\mathscr{P}$ as given in 1.5 and let $\mathscr{R}^*$ be a dual of $\mathscr{R}$. Then $\mathscr{P}$ is a surjective homomorphism of L onto $\mathscr{R}^*$ and $\mathscr{R}^*$ is a distributive lattice. We show that the ordered sets L^K and $L^{\mathscr{R}^*}$ are isomorphic. Let us define a mapping φ: Hom $(\mathscr{R}^*, L) \to$ Hom (K, L): for $g \in$ Hom $(\mathscr{R}^*, L)$ let $\varphi(g) = g \circ \mathscr{P}$, i.e. $\varphi(g): K \to L$ is such a mapping f that $f(x) = g(\mathscr{P}(x))$ for any $x \in K$. As $\varphi(g)$ is a composition of two homomorphisms $\mathscr{P}$ and g, it is a homomorphism of K into L so that really φ: Hom $(\mathscr{R}^*, L) \to$ Hom (K, L).

We show that φ is surjective. Let $f \in$ Hom (K, L). Let us define a mapping $g: \mathscr{R}^* \to L$ by $g(\mathscr{P}(x)) = f(x)$ for any $\mathscr{P}(x) \in \mathscr{R}^*$. This definition is correct, for if $\mathscr{P}(x_1) = \mathscr{P}(x_2)$ for some $x_1, x_2 \in K$, then $f(x_1) = f(x_2)$ by 4.1. Now, if $\mathscr{P}(x_1), \mathscr{P}(x_2) \in \in \mathscr{R}^*$, then $g(\mathscr{P}(x_1) \vee \mathscr{P}(x_2)) = g(\mathscr{P}(x_1 \vee x_2)) = f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = = g(\mathscr{P}(x_1)) \vee g(\mathscr{P}(x_2))$ and similarly we see that $g(\mathscr{P}(x_1) \wedge \mathscr{P}(x_2)) = g(\mathscr{P}(x_1)) \wedge \wedge g(\mathscr{P}(x_2))$. Thus $g \in$ Hom $(\mathscr{R}^*, L)$ and from its definition we conclude $\varphi(g) = f$. We show further that φ is injective. Let $g_1, g_2 \in$ Hom $(\mathscr{R}^*, L)$, $g_1 \neq g_2$. Then there exists a $\mathscr{P}(x) \in \mathscr{R}^*$ such that $q_1(\mathscr{P}(x)) \neq g_2(\mathscr{P}(x))$ and then $\varphi(g_1)(x) = g_1(\mathscr{P}(x)) \neq \neq g_2(\mathscr{P}(x)) = \varphi(g_2)(x)$, i.e. $\varphi(g_1) \neq \varphi(g_2)$.

Thus φ is a bijection of Hom $(\mathscr{R}^*, L)$ onto Hom (K, L). For any two elements $g_1, g_2 \in$ Hom $(\mathscr{R}^*, L)$ we now have $g_1 \leqq g_2$ in $L^{\mathscr{R}^*} \Leftrightarrow g_1(\mathscr{P}(x)) \leqq g_2(\mathscr{P}(x))$ for any $\mathscr{P}(x) \in \mathscr{R}^* \Leftrightarrow \varphi(g_1)(x) \leqq \varphi(g_2)(x)$ for any $x \in K \Leftrightarrow \varphi(g_1) \leqq \varphi(g_2)$ in L^K. Hence φ is an isomorphism of $L^{\mathscr{R}^*}$ onto L^K.

Note that 4.3 in particular implies that the isomorphism of ordered sets L^K, L^{K_1} does not generally imply the isomorphism of the lattices K, K_1.

4.4. Problem. Let K, K_1 be distributive lattices and L a nontrivial chain. Does the isomorphism of ordered sets L^K, L^{K_1} imply the isomorphism of the lattices K, K_1?

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Souhrn

NĚKTERÉ VLASTNOSTI SVAZOVÝCH HOMOMORFISMŮ

VÍTĚZSLAV NOVÁK

Nechť L je řetězec a K, K_1 jsou svazy. V práci je ukázáno, že z izomorfismu mocnin L^K, L^{K_1} obecně neplyne izomorfismus svazů K, K_1. Zejména platí: pro každý svaz K existuje distributivní svaz K_1 tak, že uspořádané množiny L^K, L^{K_1} jsou izomorfní.

Резюме

НЕКОТОРЫЕ СВОЙСТВА ГОМОМОРФИЗМОВ РЕШЕТОК

VÍTĚZSLAV NOVÁK

Пусть L — цепь и K, K_1 — решетки. В статье показано, что из изоморфизма степеней L^K, L^{K_1} не следует изоморфизм решеток K, K_1. В частности: для всякой решетки K существует дистрибутивная решетка K_1 такая, что упорядоченные множества L^K, L^{K_1} изоморфны.

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