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APPLICATIONS OF THE SMOOTH INTEGRAL IN THE THEORY OF WEAK SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Summary. The paper deals with the existence and uniqueness of weak (in the distributional sense) solutions of linear and non-linear boundary value problems for ordinary differential equations. The main tools are the smooth integral and classical fixed point theorems.

Keywords: Ordinary differential equation, boundary value problem, smooth integral, weak solution, distributions, fixed point theorems.

AMS Classification: 34B15.

INTRODUCTION

Distributional solutions of ordinary differential equations have not been studied to a sufficient extent, due to certain difficulties in defining operations on distributions: some operations (e.g. multiplication, substitution, definite integral) cannot be defined for all distributions in a natural manner. In order to overcome these difficulties we apply the operational approach to differential equations using the smooth integral (see [1], p. 201).

In our paper we consider the existence and uniqueness of weak solutions of linear and non linear ordinary differential equations satisfying some additional conditions. The application of the smooth integral allows us to replace the given ordinary differential equations by special integral equations. Next, we apply the classical fixed point theorems to these equations to obtain, in particular, solutions in the Sobolev space $W^{s,p}(a, p)$ $(a, b \in R, 1 \leqq p \leqq \infty, s \geqq 0)$. In Chapter 2, we establish the main properties of the smooth integral. In Chapter 3, we discuss systems of non linear differential equations (of the first order) with some additional conditions which are expressed in the form of linear continuous functionals defined on the space $L^p(a, b)$. The solutions of these equations are vectors whose all coordinates are functions of the class $L^p(a, b)$. In Chapter 4, non linear differential equations of order n $(n \geqq 2)$ are studied with additional conditions in the form of linear continuous functionals on the space $W^{s,p}(a, b)$ $(s \geqq 1, 1 \leqq p \leqq \infty; a, b \in R)$. From the fact that the unique solution of the homogeneous problem is the trivial one we obtain in Chapter 5 the existence of solutions of the non homogeneous problem.

The study of distributional solutions of ordinary differential equations is still topical (see [7]). A particularly large number of papers have been devoted to linear differential equations with distributional coefficients (see [6], [9], [10], [17], [19], [20], [23], [26], [27], [29]). Other possibilities of generalization of the notion of a solution of an ordinary differential equation are considered in [8], [11], [12], [15], [16], [25], [28], [32].

Our considerations will be based on the sequential theory of distributions (see [1]).

1. NOTATION

Let $\mathbb{R}$ denote the set of all real numbers and N the set of all naturals. Let I denote a closed interval $[a, b]$ and I_0 the open interval (a, b) $(a, b \in \mathbb{R})$. By $L^p(I)$ we denote the space of all real Lebesgue measurable functions f defined on the interval I such that

$$\|f\|_p = (\int_I |f|^p (t) \, \mathrm{d}t)^{1/p} < \infty \quad \text{if} \quad 1 \leqq p < \infty ,$$

and

$$\|f\|_\infty = \sup_{t \in I} \operatorname{ess} |f(t)| < \infty \quad \text{if} \quad p = \infty .$$

We put

$$L^p_n(I) = \underbrace{L^p(I) \times \ldots \times L^p(I)}_{n \text{ times}}, \quad \|y\|_{p,n} = \sum_{i=1}^{n} \|y_i\|_p , \quad |I| = b - a ,$$

where $y = (y_1, \ldots, y_n)$ and $y_i \in L^p(I)$ for $i = 1, \ldots, n$. We adopt the following convention: if $p = 1$ and $q = \infty$, then $1/p + 1/q = 1$ and $1/q = 0$.

Let $W^{s,p}(I)$ or $W^{s,p}(I_0)$, $s \in N$, $p \geqq 1$ denote the set of all functions y possessing a continuous derivative of order $s - 1$ on the interval I or on I_0, respectively, and such that $y^{(s)} \in L^p(I)$. We introduce the following norm on these spaces:

$$\|z\| = \sum_{i=0}^{s-1} \|z^{(i)}\|_\infty + \|z^{(s)}\|_p .$$

The spaces $(W^{s,p}(I), \|\cdot\|)$ and $(W^{s,p}(I_0), \|\cdot\|)$ are Banach spaces. For $s = 0$, we put

$$W^{0,p}(I_0) = L^p(I_0) \quad \text{and} \quad \|z\|_p = \|z\| ,$$

where $z \in L^p(I_0)$.

If $s \in N \cup \{0\}$ and L is a linear continuous functional in $(W^{s,p}(I), \|\cdot\|)$, then we write $L \in (W^{s,p}(I), \|\cdot\|)^*$.

The symbol $L^{p(k)}(I_0)$ denotes the set of all the k-th derivatives (in the distributional sense) of functions of the class $L^p(I_0)$.

By $C^k(I)$ we denote the space of all real functions with a continuous k-th derivative on I, and by $C(I)$ we denote the set of real continuous functions on I.

Throughout the paper ω and $\bar{\omega}$ stand for infinitely differentiable functions with bounded carriers inside I_0 such that

$$\int_I \omega(t)\,\mathrm{d}t = \int_I \bar{\omega}(t)\,\mathrm{d}t = 1\,.$$

We adopt the convention that $a, b \in \mathbb{R}$ and $p \geqq 1$.

2. SMOOTH INTEGRAL

In the theory of differential equations the solving of various problems leads to integral equations. However, for distributions the definite integral does not exist in general. Therefore we introduce the operation $^\vee$, which assigns to a distribution which is the k-th derivative of a function of the class $L^p(I_0)$ one of its k-th primitives. In this chapter we establish some properties of the operation $^\vee$ while in the next chapters we present several applications.

We suppose that $\varphi = \Phi^{(k)}$, where Φ is a locally integrable function on the interval I_0 and the derivative is understood in the distributional sense. By the smooth integral of φ we mean a distribution $\varphi^\vee_{\omega,1}$ defined as follows:

$$\varphi^\vee_{\omega,1} = \Phi^{(k-1)} + (-1)^k \int_I \Phi(t)\,\omega^{(k-1)}(t)\,\mathrm{d}t \quad \text{(see [1])}\,. \tag{2.1}$$

The smooth integral of order r $(r \geqq 2)$ of a distribution φ is defined by

$$\varphi^\vee_{\omega,r} = \psi^\vee_{\omega,1}\,, \tag{2.2}$$

where $\psi = \varphi^\vee_{\omega,r-1}$.

It is easy to see that

$$(\varphi^\vee_{\omega,r})^{(r)} = \varphi \tag{2.2)'$$

and

$$\lambda_1 f^\vee_{\omega,r} + \lambda_2 g^\vee_{\omega,r} = h^\vee_{\omega,r}\,, \tag{2.3}$$

where f and g are distributions defined on I_0, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $h = \lambda_1 f + \lambda_2 g$.

We shall use the notation $f^\vee_{\omega,1}(y)$ instead of $(f(y))^\vee_{\omega,1}$ and $g^\vee_{\omega,k}(y)$ instead of $(g(y))^\vee_{\omega,k}$, where $f\colon L^p_n(I_0) \to L^{p(1)}(I_0)$, $g\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_0)$ and $2 \leqq 2k \leqq n$.

Now we shall give the fundamental properties of the smooth integral.

Lemma 2.1. *Let f be a mapping, $f\colon L^p_n(I_0) \to L^{p(1)}(I_0)$, and let $\lim_{\nu\to\infty} \|y_\nu - y\|_{p,n} = 0$, where $y_\nu, y \in L^p_n(I_0)$. Moreover, let*

$$\lim_{\nu\to\infty} \|f^\vee_{\omega,1}(y_\nu) - f^\vee_{\omega,1}(y)\|_p = 0\,.$$

Then

$$\lim_{\nu\to\infty} \|f^\vee_{\bar{\omega},1}(y_\nu) - f^\vee_{\bar{\omega},1}(y)\|_p = 0\,.$$

Proof. In fact, by (2.1) and (2.2)′ we have

(2.4) $$f^{\vee}_{\bar\omega,1}(y_\nu) = f^{\vee}_{\omega,1}(y_\nu) - \int_I f^{\vee}_{\omega,1}(y_\nu)(t)\,\bar\omega(t)\,\mathrm{d}t$$

and

(2.5) $$f^{\vee}_{\bar\omega,1}(y) = f^{\vee}_{\omega,1}(y) - \int_I f^{\vee}_{\omega,1}(y)(t)\,\bar\omega(t)\,\mathrm{d}t\,.$$

Let $1/p + 1/q = 1$. Then, by (2.4)–(2.5) and by the Hölder inequality, we obtain

(2.6) $$\|f^{\vee}_{\bar\omega,1}(y_\nu) - f^{\vee}_{\bar\omega,1}(y)\|_p \leqq (1 + \|1\|_p\,\|\bar\omega\|_q)\,\|f^{\vee}_{\omega,1}(y_\nu) - f^{\vee}_{\omega,1}(y)\|_p\,,$$

which proves our assertion.

Similarly, we can prove the following lemmas:

Lemma 2.2. *Let f be a mapping, $f\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_0)$ $(2 \leqq 2k \leqq n)$, and let $\lim_{\nu\to\infty} \|y_\nu - y\| = 0$, where y_ν, $y \in W^{n-k,p}(I_0)$. Moreover, let*

$$\lim_{\nu\to\infty} \|f^{\vee}_{\omega,k}(y_\nu) - f^{\vee}_{\omega,k}(y)\|_p = 0\,.$$

Then

$$\lim_{\nu\to\infty} \|f^{\vee}_{\bar\omega,k}(y_\nu) - f^{\vee}_{\bar\omega,k}(y)\|_p = 0\,.$$

Lemma 2.3. *Let $f\colon L^p_n(I_0) \to L^{p(1)}(I_0)$ and let*

$$\|f^{\vee}_{\omega,1}(y)\|_p \leqq \alpha\|y\|_p + \beta\,,$$

where $\alpha, \beta \in \mathbb{R}$ and $y \in L^{p(1)}(I_0)$.

Then there exist non negative numbers α_1, β_1 such that

$$\|f^{\vee}_{\bar\omega,1}(y)\|_p \leqq \alpha_1\|y\|_{p,n} + \beta_1 \quad \text{for} \quad y \in L^p_n(I_0)\,.$$

Lemma 2.4. *Let $f\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_0)$ $(2 \leqq 2k \leqq n)$ and let*

$$\|f^{\vee}_{\omega,k}(y)\|_p \leqq \alpha\|y\| + \beta \quad for \quad y \in W^{n-k,p}(I_0) \quad (\alpha, \beta \in \mathbb{R})\,.$$

Then there exist non negative numbers α_1, β_1 such that

$$\|f^{\vee}_{\bar\omega,k}(y)\|_p \leqq \alpha_1\|y\| + \beta_1 \quad for \quad y \in W^{n-k,p}(I_0)\,.$$

Lemma 2.5. *Let $f\colon L^p_n(I_0) \to L^{p(1)}(I_0)$ and let*

$$\|f^{\vee}_{\omega,1}(y) - f^{\vee}_{\omega,1}(\bar y)\|_p \leqq \alpha\|y - \bar y\|_{p,n} \quad (\alpha \in \mathbb{R})\,.$$

Then

$$\|f^{\vee}_{\bar\omega,1}(y) - f^{\vee}_{\bar\omega,1}(\bar y)\|_p \leqq \alpha_1\|y - \bar y\|_{p,n}\,,$$

where $y, \bar y \in L^p_n(I_0)$, $\alpha_1 \in \mathbb{R}$.

Lemma 2.6. *Let $f\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_C)$ $(2 \leqq 2k \leqq n)$ and let*

$$\|f^{\vee}_{\omega,k}(y) - f^{\vee}_{\omega,k}(\bar y)\|_p \leqq \alpha\|y - \bar y\| \quad (\alpha \in \mathbb{R})\,.$$

Then

$$\|f^{\vee}_{\bar{\omega},k}(y) - f^{\vee}_{\bar{\omega},k}(\bar{y})\|_p \leqq \alpha_1 \|y - \bar{y}\|$$

for $y, \bar{y} \in W^{n-k,p}(I_0)$.

Lemma 2.7. *Assume that* $f^{\vee}_{\omega,1}: L^p_n(I_0) \to L^p(I_0)$ *and* $f^{\vee}_{\omega,1}$ *is a compact mapping. Then* $f^{\vee}_{\bar{\omega},1}$ *is also a compact mapping.*

Lemma 2.8. *If* $f^{\vee}_{\omega,k}: W^{n-k,p}(I) \to L^p(I)$ *and* $f^{\vee}_{\omega,k}$ *is a compact mapping, then* $f^{\vee}_{\bar{\omega},k}$ *is also a compact mapping.*

3. WEAK SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

We consider the problem

$$y_i' = f_i(y)\,, \tag{3.1}$$

$$\tilde{L}_i(y_i) = r_i\,, \quad r_i \in \mathbb{R}\,, \quad i = 1, \ldots, n\,, \tag{3.2}$$

where f_i are operations, $\tilde{L}_i$ are functionals and all derivatives are understood in the distributional sense.

Let $y = (y_1, \ldots, y_n) \in L^p_n(I_0)$, $f_i: L^p_n(I_0) \to L^{p(1)}(I_0)$, $\tilde{L}_i \in (L^p(I_0), \|\cdot\|_p)^*$ for $i = 1, \ldots, n$ and let y satisfy the system (3.1) on I_0 with the conditions (3.2). Then we say that y is a weak solution of the problem (3.1)–(3.2).

Theorem 3.1. *Assume that*

$$f_i: L^p_n(I_0) \to L^{p(1)}(I_0)\,, \quad i = 1, \ldots, n\,; \tag{3.3}$$

there exist a function ω *and* $\alpha \in \mathbb{R}$ *such that*

$$\|f^{\vee}_{i\omega,1}(y) - f^{\vee}_{i\omega,1}(\bar{y})\|_p \leqq \alpha \|y - \bar{y}\|_{p,n} \tag{3.4}$$

for all $i = 1, \ldots, n$ *and* $y, \bar{y} \in L^p_n(I_0)$;

$$\tilde{L}_i \in (L^p(I_0), \|\cdot\|_p)^* \quad \textit{for} \quad i = 1, \ldots, n\,; \tag{3.5}$$

$$\tilde{L}_i(1) = 1 \quad \textit{for} \quad i = 1, \ldots, n\,; \tag{3.6}$$

$$\lambda = \alpha n(1 + M_0 \|1\|_p) < 1\,, \tag{3.7}$$

where $M_0 = \sup\limits_{1 \leqq i \leqq n} \|\tilde{L}_i\|_p$.

Then the problem (3.1)–(3.2) *has exactly one weak solution.*

Before giving the proof of Theorem 3.1 we formulate the following lemma:

Lemma 3.1. *Let us assume that the conditions* (3.3), (3.5)–(3.6) *are satisfied. Then* $y \in L^p_n(I_0)$ *is a weak solution of the problem* (3.1)–(3.2) *if and only if* y *is* *a fixed point of the operation* $G = (G_1, \ldots, G_n)$, *where*

$$(3.8)\qquad G_i(y) = \check{f}_{i\omega,1}(y) + r_i - \tilde{L}_i(\check{f}_{i\omega,1}(y)), \quad i = 1, \ldots, n,$$

and

$$G(y) = (G_1(y), \ldots, G_n(y)).$$

Proof of Lemma 3.1. Let $y \in L^p_n(I_0)$ be a fixed point of the transformation G. Then y is a solution of the equation (3.1) in I_0 and (by (3.5)–(3.6))

$$\tilde{L}_i(y_i) = \tilde{L}_i(\check{f}_{i\omega,1}(y)) + \tilde{L}_i(v_i) - \tilde{L}_i(\check{f}_{i\omega,1}(y)) = r_i, \quad i = 1, \ldots, n.$$

On the other hand, if $y \in L^p_n(I_0)$ is a weak solution of the problem (3.1)–(3.2), then

$$y_i = \check{f}_{i\omega,1}(y) + c_i,$$

where $c_i \in R$ and $i = 1, \ldots, n$.

Hence, by (3.5)–(3.6) we have

$$c_i = r_i - \tilde{L}_i(\check{f}_{i\omega,1}(y)),$$

which proves the lemma.

Proof of Theorem 3.1. By Lemma 3.1 and the assumption (3.4) we have

$$\|G(y) - G(\bar{y})\|_{p,n} \leqq \lambda \|y - \bar{y}\|_{p,n}.$$

We conclude by (3.7) that G is a contractive mapping. By virtue of the Banach fixed theorem our assertion follows.

Example 3.1. Let $D = I_0 \times R^n$. We say that a function $g: D \to R$ satisfies the condition (C) in D if

(3.9) the function $g(t, v_1, \ldots, v_n)$ is continuous with respect to $(v_1, \ldots, v_n)$ for every fixed t,

(3.10) the function $g(t, v_1, \ldots, v_n)$ is Lebesgue measurable with respect to t for fixed $(v_1, \ldots, v_n)$.

Let functions k_i $(i = 1, \ldots, n)$ satisfy the condition (C) in D and let

$$(3.11)\qquad |k_j(t, v_1, \ldots, v_n) - k_j(t, \bar{v}_1, \ldots, \bar{v}_n)| \leqq \sum_{i=1}^{n} q_i(t)\,|v_i - \bar{v}_i| \quad \text{for} \quad j = 1, \ldots, n;$$

$$(3.12)\qquad |k_j(t, 0, \ldots, 0)| \leqq p_j(t) \quad \text{for} \quad j = 1, \ldots, n,$$

where $p_i \in L^1(I_0)$, $q_i \in L^q(I_0)$, $1/p + 1/q = 1$ and $i, j = 1, \ldots, n$. Next, we assume that

$$h_{ij}: I \to I, \quad h_{ij} \in C^1(I), \quad h'_{ij}(t) > 0 \quad \text{for} \quad t \in I.$$

We define

$$(3.13)\qquad (f_i(y))(t) = k_i(t, y_1(h_{1i}(t)), \ldots, y_n(h_{ni}(t))) + Q'_i(t),$$

where $Q_i \in L^p(I_0)$, $y_i \in L^p(I_0)$, $i = 1, \ldots, n$.

Then

$$y_i(h_i) \in L^p(I_0) \quad \text{and} \quad f_i\colon L^p_n(I_0) \to L^{p(1)}(I_0) \quad \text{for} \quad i = 1, \ldots, n\,.$$

Let

$$(3.14) \qquad R_i(y)(t) = \int_a^t k_i(s, y_1(h_{1i}(s)), \ldots, y_n(h_{ni}(s)))\,\mathrm{d}s + Q_i(t) \quad \text{for} \quad i = 1, \ldots, n\,.$$

Then, applying (2.1) and the Hölder inequality we can write

$$(3.15) \qquad \|f^{\vee}_{i\omega,1}(y) - f^{\vee}_{i\omega,1}(\bar{y})\|_p \leqq (1 + \|1\|_p \|\omega\|_q) \|R_i(y) - R_i(\bar{y})\|_p \leqq \alpha \|y - \bar{y}\|_{p,n}\,,$$

where

$$\alpha = \sum_{i,j=1}^{n} \|q_{ij}\|_q (1 + \|1\|_p \|\omega\|_q)(|I| \, \|(h'_{ij})^{-1}\|_\infty)^{1/p}\,.$$

Hence the operations f_i satisfy the assumptions (3.3)–(3.4) for $i = 1, \ldots, n$.

We adopt the following convention: for $g \in L^p(I_0)$ we put $g(t + \tau) = 0$ for $t + \tau \notin I_0$. Let $p \in [1, \infty)$. We say that an operation $F\colon L^p_n(I_0) \to L^p(I_0)$ ($F\colon W^{m,p}(I_0) \to L^p(I_0)$, $m \in N$) has the property R on I_0 if for every ball $B \subset (L^p_n(I_0), \|\cdot\|_{p,n})$ ($B \subset (W^{m,p}(I_0), \|\cdot\|)$) and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(3.16) \qquad \int_a^b |F(y)(t + \tau) - F(y)(t)|^p \,\mathrm{d}t < \varepsilon$$

for every $0 < \tau < \delta$ and every $y \in B$.

From the relation (2.4) we obtain the following corollary:

Corollary 3.1. *Let $f^{\vee}_{i\omega,1}\colon L^p_n(I_0) \to L^p(I_0)$ and let $\|f^{\vee}_{i\omega,1}(y)\|_p \leqq \alpha\|y\|_{p,n} + \beta$, where $p \in [1, \infty)$ and $\alpha, \beta \in \mathbb{R}$. Moreover, let $f^{\vee}_{i\omega,1}$ have the property R on I_0. Then $f^{\vee}_{i\bar{\omega},1}$ also has this property on I_0 for $i = 1, \ldots, n$.*

Corollary 3.2. *Let $f^{\vee}_{\omega,k}\colon W^{n-k,p}(I_0) \to L^p(I_0)$ and let $\|f^{\vee}_{\omega,k}(y)\|_p \leqq \alpha\|y\| + \beta$, where $2 \leqq 2k \leqq n$; $\alpha, \beta \in \mathbb{R}$ and $p \in [1, \infty)$. Moreover, let $f^{\vee}_{\omega,k}$ have the property R on I_0. Then $f^{\vee}_{\bar{\omega},k}$ also has the property R on I_0.*

Theorem 3.2. *Let us assume that*

(3.17) *conditions (3.3), (3.5)–(3.7) are satisfied and $p \in [1, \infty)$; there exists function ω and $\alpha, \beta \in \mathbb{R}$ such that*

$$(3.18) \qquad \|f^{\vee}_{i\omega,1}(y)\|_p \leqq \alpha\|y\|_{p,n} + \beta$$

for all $y \in L^p_n(I_0)$ and $i = 1, \ldots, n$;

(3.19) *$f^{\vee}_{i\omega,1}$ are continuous operations on $L^p_n(I_0)$ for $i = 1, \ldots, n$;*

(3.20) *the operations $f^{\vee}_{i\omega,1}$ have the property R on I_0 for $i = 1, \ldots, n$.*

Then problem (3.1)–(3.2) has at least one weak solution.

Proof. We consider the transformation G defined by (3.8). Let

$$B = \{y \in L^p_n(I_0)\colon \|y\|_{p,n} \leqq K\}$$

and let M_0, λ be defined as in Theorem 3.1. Then

$$\|G_i(y)\|_p \leqq \alpha(1 + M_0\|1\|_p) K + \beta_1$$

where

$$\beta_1 = \beta + (\max_{1 \leqq i \leqq n} |r_i| + M_0\beta) \|1\|_p .$$

Evidently G is continuous and

$$\|G(y)\|_{p,n} \leqq \lambda K + \beta_1 .$$

Thus, if $\lambda < 1$ and $K \geqq \beta_1/(1 - \lambda)$, then $G(B) \subset B$. The property R and the Riesz theorem imply that $G(B)$ is a compact set in $(L^p_n(I_0), \|\cdot\|_{p,n})$. Applying the Schauder fixed point theorem we conclude that the operation G has a fixed point, which completes the proof of the theorem.

Remark 3.1. It is easy to show that the operations defined in Example 3.1 satisfy the assumptions (3.18)–(3.20).

Example 3.2. Let k_i $(i = 1, \ldots, n)$ satisfy the condition (C) in D, where $D = I_0 \times \mathbb{R}^n$. Moreover, let

$$|k_i(t, v_1, \ldots, v_n)| \leqq \sum_{j=1}^{n} q_{ij}(t) |v_i| + p_i , \tag{3.21}$$

where p_i, q_{ij} are non negative functions on I_0, $p_i \in L^1(I_0)$, $q_{ij} \in L^q(I_0)$ for $i, j = 1, \ldots, n$ and $1/p + 1/q = 1$. Then the operations f_i defined by (3.13) satisfy the assumptions (3.18)–(3.20).

4. WEAK SOLUTIONS OF NON LINEAR DIFFERENTIAL EQUATIONS OF ORDER n $(n \geqq 2)$

In this chapter we are going to discuss the problem

$$y^{(n)} = f(y) , \tag{4.1}$$

$$L_i(y) = r_i , \quad r_i \in \mathbb{R} , \quad i = 1, \ldots, n , \tag{4.2}$$

where f is an operation, L_i are functionals and the derivative is understood in the distributional sense.

Let $f: W^{n-k,p}(I_0) \to L^{p(k)}(I_0)$, $2 \leqq 2k \leqq n$, $L_i \in (W^{n-k,p}(I), \|\cdot\|)^*$ for $i = 1, \ldots, n$ and let $y \in W^{n-k,p}(I)$ satisfy the equation (4.1) on I_0 with the condition (4.2). Then we say that y is a weak solution of the problem (4.1)–(4.2).

Before formulating a theorem, we introduce some notation. Let $Q = [q_{ij}]$, where $q_{ij} = L_i(t^{j-1})$, $i, j = 1, \ldots, n$, and let

$$U_\omega(y)(t) = \int_a^t \frac{(t - s)^{n-k-1}}{(n - k - 1)!} f^{\vee}_{\omega,k}(y)(s)\, ds . \tag{4.3}$$

Moreover, let

$$d_0 = (r_1 - L_1(U_\omega(y)), \ldots, r_n - L_n(U_\omega(y)))\,.$$

The symbol $Q_{j\omega}(y)$ will denote a matrix obtained from Q by replacing the j-th column by the column d_0. We put

$$W_{j\omega}(y) = \det Q_{j\omega}(y) \quad \text{and} \quad \overline{W} = \det Q\,.$$

Now, we introduce the following hypothesis:

Hypothesis H_{41}:

$$f\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_0)\,, \quad 2 \leqq 2k \leqq n\,, \tag{4.4}$$

there exists a function ω and $\alpha \in R$ such that

$$\|f^{\vee}_{\omega,k}(y) - f^{\vee}_{\omega,k}(\bar{y})\|_p \leqq \alpha\|y - \bar{y}\| \tag{4.5}$$

for all $y, \bar{y} \in W^{n-k,p}(I_0)$.

Theorem 4.1. *Let*

$$L_i \in (W^{n-k,p}(I), \|\cdot\|)^* \quad \text{for} \quad i = 1, \ldots, n\,, \tag{4.6}$$

$$\overline{W} \neq 0\,. \tag{4.7}$$

Then there exists a number $\alpha_0 \in (0, \infty)$ *such that the problem* (4.1)–(4.2) *has exactly one weak solution for every* $\alpha \in (0, \alpha_0)$ *and for every operation* f *satisfying* H_{41}.

Proof. We observe that $y \in W^{n-k,p}(I)$ is a weak solution of the problem (4.1)–(4.2) if and only if y is a fixed point of the operation T_ω defined by

$$T_\omega(y)(t) = U_\omega(y)(t) + \sum_{i=0}^{n-1} a_{i\omega}(y)\, t^i\,, \tag{4.8}$$

where

$$a_{j-1\omega}(y) = \frac{W_{j\omega}y}{\overline{W}}\,, \quad j = 1, \ldots, n\,. \tag{4.9}$$

Next, we shall introduce some notation. Let

$$M_0 = \max\left[\|L_1\|_p, \ldots, \|L_n\|_p\right],$$

$$\mu_0 = \frac{1}{|\overline{W}|} M_0^n n[(\max(|I|^{n-1}, 1))(n - k + 1)(n - 1)!]^{n-1}\,,$$

$$k_{i,0} = \max(1, |I|^i)\,, \quad k_{i,j} = [\max(1, |I|)]^{i-j}\, i \ldots (i - j + 1)\,,$$

$$i = 0, 1, \ldots, n - 1\,, \quad j = 1, \ldots . n - k\,, \quad 0 \leqq i - j\,,$$

and

$$L_0 = \sum_{j=0}^{n-k-1} \frac{|I|^{n-k-j-1+1/q}}{(n - k - j - 1)!} + 1\,.$$

It is clear that

$$\|t^s\| \leqq (n-1)!\,(n-k+1)\max\left(|I|^{n-1}, 1\right) \quad \text{for} \quad s = 0, 1, \ldots, n-1 .$$

Hence, by (4.5) and (4.9), we infer that

$$|a_{i\omega}(y) - a_{i\omega}(\bar{y})| \leqq \mu_0 \alpha L_0 \|y - \bar{y}\| \quad \text{for} \quad i = 0, 1, \ldots, n-1 . \tag{4.10}$$

These inequalities and the assumption (4.5) yield

$$\begin{gathered} |(T_\omega(y))^{(j)} - (T_\omega(\bar{y}))^{(j)}| \leqq \\ \leqq \left[\frac{|I|^{n-k-1-j+1/q}}{(n-k-1-j)!} + \mu_0 \alpha L_0 \left(\sum_{i=j}^{n-1} k_{i,j}\right)\right] \|y - \bar{y}\| , \\ j = 0, 1, \ldots, n-k-1 \end{gathered} \tag{4.11}$$

and

$$\begin{gathered} |(T_\omega(y))^{(n-k)} - (T_\omega(\bar{y}))^{(n-k)}| \leqq \\ \leqq |f^{\vee}_{\omega,k}(y) - f^{\vee}_{\omega,k}(\bar{y})| + \sum_{i=n-k}^{n-1} \mu_0 L_0 k_{i,n-k} \alpha \|y - \bar{y}\| . \end{gathered} \tag{4.12}$$

Denoting

$$N(I) = \sum_{i=j}^{n-1} \sum_{j=0}^{n-k+1} k_{i,j} + \|1\|_p \left(\sum_{i=n-k}^{n-1} k_{i,n-k}\right),$$

we have

$$\|T_\omega(y) - T_\omega(\bar{y})\| \leqq \alpha L_0 (1 + \mu_0 N(I)) \|y - \bar{y}\| .$$

We conclude that T_ω is a contractive mapping if $\alpha < \alpha_0$, where

$$\alpha_0 = [L_0(1 + \mu_0 N(I))]^{-1} . \tag{4.13}$$

By virtue of the Banach fixed point theorem our assertion follows.

Remark 4.1. Let $\Psi_i\,(i = 0, 1, \ldots, n-k-1)$ be a function of bounded variation on the interval I. Moreover, let $y \in W^{n-k,p}(I)$, $g \in L^q(I)$, where $1/p + 1/q = 1$. Then

$$L(y) = \sum_{i=0}^{n-k-1} \int_I y^{(i)}(t)\, d\psi_i + \int_I y^{(n-k)}(t)\, g(t)\, dt \in (W^{n-k,p}(I), \|\cdot\|)^* .$$

Thus, taking functions Ψ_i and g in a special form, we obtain the interpolation problem or the de la Vallée-Poussin problem as particular cases of the problems considered in our paper.

Example 4.1. Let functions g_1, g_2 satisfy the condition (C) in the set $D = I_0 \times R^2$ and let

$$|g_i(t, v_0, v_1) - g_i(t, \bar{v}_0, \bar{v}_1)| \leqq q_1(t)\,|v_0 - \bar{v}_0| + q_2(t)\,|v_1 - \bar{v}_1| \tag{4.14}$$

and

$$|g_i(t, 0, 0)| \leqq q_3(t) \quad (i = 1, 2) ,$$

where q_1, q_2, q_3 are non negative functions such that $q_1, q_3 \in L^1(I)$, $q_2 \in L^q(I)$ and $1/p + 1/q = 1$. Moreover, let

$$h_1, h_2 : I \to I\,, \quad h_1 \in C(I)\,, \quad h_2 \in C^1(I)\,, \quad h_2'(t) > 0 \quad \text{for} \quad t \in I\,.$$

We define

$$f(y) = Q'(t)\, y(t) + R_0'(t) \int_I g_1(t, y(h_1(t)), y'(h_2(t)))\, dt + \\ + g_2(t, y(h_1(t)), y'(h_2(t))) + A(t)\, y''(t)\,,$$

where $Q \in L^r(I)$, $r = \max(p, q)$, $R_0 \in L^p(I)$, $A \in W^{1,q}(I)$, $y \in W^{1,p}(I)$ and $Q'y = (Qy)' - Qy'$. We put

$$(4.15) \qquad F(y)(t) = Q(t)\, y(t) - \int_a^t Q(s)\, y'(s)\, ds + R_0(t) \int_I g_1(t, y(h_1(t)), \\ y'(h_2(t)))\, dt + \int_a^t g_2(s, y(h_1(s)), y'(h_2(s)))\, ds + A(t)\, y'(t) - \int_a^t A'(s)\, y'(s)\, ds\,.$$

Evidently

$$F: W^{1,p}(I_0) \to L^p(I_0)\,, \quad f: W^{1,p}(I_0) \to L^{p(1)}(I_0)$$

and

$$(4.16) \quad \|\check{f}_{\omega,1}(y) - \check{f}_{\omega,1}(\bar{y})\|_p \leqq (1 + \|1\|_p\, \|\omega\|_q)\, \|F(y) - F((\bar{y})\, \|_p \leqq \alpha \|y - \bar{y}\|$$

(by (2.1) and the Hölder inequality), where

$$\alpha = [\|Q\|_p + \|Q\|_q\, \|1\|_p + (\|R_0\|_p + \|1\|_p)(\|q_1\|_1 + \|q_2\|_q (\|(h_2')^{-1}\|_\infty)^{1/p}) + \\ + \|A\|_\infty + \|1\|_p\, \|A'\|_q]\,(1 + \|1\|_p\, \|\omega\|_q)\,.$$

Hence $\check{f}_{\omega,1}$ satisfies the assumption (4.5).

Before giving a theorem on existence of weak solutions of the problem (4.1)–(4.2), we formulate the following hypothesis:

Hypothesis H_{42}

$$(4.17) \qquad f: W^{n-k,p}(I_0) \to L^{p(k)}(I_0)\,, \quad 2 \leqq 2k \leqq n\,, \quad p \in [1, \infty)\,;$$

there exists a function ω such that

$$(4.18) \qquad \|\check{f}_{\omega,k}(y)\|_p \leqq \alpha \|y\| + \beta\,, \quad \text{where} \quad y \in W^{n-k,p}(I_0)\,, \quad \alpha, \beta \in \mathbb{R}\,;$$

(4.19) $\check{f}_{\omega,k}$ is a continuous operation on the space $W^{n-k,p}(I)$;

(4.20) the operation $\check{f}_{\omega,k}$ has the property R on I.

Theorem 4.2. *Assume that the conditions (4.6)–(4.7) are satisfied. Then there exists a number $\alpha_0 \in (0, \infty)$ such that the problem (4.1)–(4.2) has a weak solution for every $\alpha \in (0, \alpha_0)$ and for every operation f satisfying H_{42}.*

Proof. We use the Schauder theorem for the transformation T_ω defined by (4.8). Let B be the ball

$$\{y \in W^{n-k,p}(I): \|y\| \leqq K\}\,.$$

By (4.8)–(4.9), we have

$$(4.21)\qquad |a_{j-1\omega}(y)| \leqq \mu_0(r + \alpha M_0 K L_0 + \beta M_0 L_0),$$

where $r = \max(|r_1|, \ldots, |r_n|)$ and μ_0, M_0, L_0 are defined in the proof of Theorem 4.1.

Hence, we infer that

$$(4.22)\qquad \|T_\omega(y)\| \leqq \alpha\alpha_0^{-1}K + \beta_1,$$

where

$$\beta_1 = \beta L_0 + N(I)\,\mu_0(r + \beta M_0 L_0).$$

Let $\alpha < \alpha_0$ and let

$$K \geqq \max\left[1, \frac{\beta_1}{1 - \alpha\alpha_0}\right].$$

Then $T_\omega(B) \subset B$ and

$$(4.23)\qquad \|T_\omega(y_\nu) - T_\omega(y)\| \leqq \alpha_0^{-1}\|f^{\vee}_{\omega,k}(y_\nu) - f^{\vee}_{\omega,k}(y)\|_p.$$

Thus

$$T_\omega: W^{n-k,p}(I) \to W^{n-k,p}(I)$$

and T_ω is continuous. Let $x_\nu \in B$, i.e.

$$x_\nu = T_\omega(y_\nu), \quad y_\nu \in B.$$

Since $T_\omega(B)$ is a bounded set in $(W^{n-k,p}(I), \|\cdot\|)$, there exist subsequences (by the Arzela theorem) $\{x^{(j)}_{\nu_\mu}\}$ and $\{y^{(j)}_{\nu_\mu}\}$ of sequences $\{x^{(j)}_\nu\}$ and $\{y^{(j)}_\nu\}$, almost uniformly convergent to $x^{(j)}$ and $y^{(j)}$, respectively, for $j = 0, 1, \ldots, n - k - 1$. Without loss of generality we can assume that the sequences $\{x^{(j)}_\nu\}$ and $\{y^{(j)}_\nu\}$ are almost uniformly convergent to $x^{(j)}$ and $y^{(j)}$, respectively (for $j = 0, 1, \ldots, n - k - 1$). The property R of the operation $f^{\vee}_{\omega,k}$ implies that the sequence $\{x^{(n-k)}_\nu\}$ satisfies the assumptions of the Riesz theorem. There exists a subsequence $\{x^{(n-k)}_{\nu_\gamma}\}$ of the sequence $\{x^{(n-k)}_\nu\}$ convergent in $L^p(I)$ to a function $x^{(n-k)}$. Applying the Schauder theorem we can show that the problem (4.1)–(4.2) has a weak solution, which implies our assertion.

Remark 4.2. It is easy to show that the operation f defined in Example 4.1 does not satisfy the assumption (4.20) (in general). If $A = 0$, then the operation f satisfies the assumptions (4.4), (4.18), (4.19) and (4.20).

5. APPLICATIONS OF THE ROTATION OF A VECTOR FIELD IN THE THEORY OF WEAK SOLUTIONS

Let $(E, |\cdot|)$ denote a Banach space, let $S_R = \{z \in E: |z| = R\}$ and $K_R = \{z \in E: |z| \leqq R\}$, where $R > 0$. Moreover, let the operation $F: E \to E$ be completely continuous (i.e. continuous and compact). Then functions of the form $\Phi(z) = z - F(z)$

are called completely continuous vector fields. If $\Phi(z) \neq 0$ on S_R, then to each system (Φ, S_R) there corresponds a certain integer $\gamma(\Phi, S_R)$, which we shall call the rotation of the vector field Φ (or the degree of the mapping Φ, see [13] and [5]). If $\gamma(\Phi, S_R) \neq 0$ on the sphere S_R, then there exists at least one solution of the quation

$$x = F(x) \quad (\text{see [14], p. 189}).$$

Let $E_1 = L_n^p(I) \times R^n$ and let $E_2 = W^{n-k,p}(I) \times R^n$ $(1 \leqq p \leqq \infty,\ 2 \leqq 2k \leqq n,$ $I \subset R)$ denote linear spaces. The sum of two elements and the product of a scalar and an element of E_i $(i = 1, 2)$ are defined in the usual way. We introduce the following norms on the spaces E_1, E_2:

$$|z_1|_1 = \max\left(\max_{1 \leqq i \leqq n} \|y_i\|_p,\ \max_{1 \leqq i \leqq n} |q_i|\right),$$

$$|z_2|_2 = \max\left(\|y\|,\ \max_{1 \leqq i \leqq n} |q_i|\right),$$

where

$z_1 = (y_1, \ldots, y_n, q_1, \ldots, q_n) \in E_1$ and $z_2 = (y, q_1, \ldots, q_n) \in E_2$. The spaces $(E_1, |\cdot|_1)$ and $(E_2, |\cdot|_2)$ are Banch spaces.

Theorem 5.1. *Assume*

(5.1) $$f_i, g_i\colon L_n^p(I_0) \to L^{p(1)}(I_0), \quad i = 1, \ldots, n;$$

(5.2) $$f_i(\lambda y) = \lambda f_i(y) \quad \textit{for all} \quad \lambda \in R, \quad y \in L_n^p(I_0) \quad \textit{and} \quad i = 1, \ldots, n;$$

(5.3) *the mappings* $\check{f}_{i\omega,1}, \check{g}_{i\omega,1}\colon L_n^p(I_0) \to L^p(I_0)$ *are completely continuous for a fixed function* ω $(i = 1, \ldots, n)$;

(5.4) $$\tilde{L}_i \in (L^p(I_0), \|\cdot\|_p)^*, \quad i = 1, \ldots, n;$$

(5.5) *the problem*

$$(*) \qquad \begin{cases} y_i' = f_i(y) \\ \tilde{L}_i(y_i) = 0, \quad i = 1, \ldots, n \end{cases}$$

has only the zero solution (in the class $L_n^p(I_0)$*),*

(5.6) $$\|\check{g}_{i\omega,1}(y)\|_p \leqq M < \infty \quad \text{for all} \quad y \in L_n^p(I_0).$$

Then the problem

(5.7) $$\begin{cases} y_i' = f_i(y) + g_i(y), \\ \tilde{L}_i(y_i) = r_i, \quad r_i \in R, \quad i = 1, \ldots, n \end{cases}$$

has at least one weak solution (in the class $L_n^p(I_0)$*).*

Proof. We consider two vector fields:

(5.8) $$\Phi(y, q) = (y_1 - \check{f}_{1\omega,1}(y) - q_1, \ldots, y_n - \check{f}_{n\omega,1}(y) - q_n, \tilde{L}_1(y_1), \ldots, \tilde{L}_n(y_n))$$

and

$$\Psi(y, q) = (y_1 - \check{f}_{1\omega,1}(y) - \check{g}_{1\omega,1}(y) - q_1, \dots, \tag{5.9}$$
$$y_n - \check{f}_{n\omega,1}(y) - \check{g}_{n\omega,1}(y) - q_n, \ \tilde{L}_1(y_1) - r_1, \dots, \tilde{L}_n(y_n) - r_n),$$

where $y = (y_1, \dots, y_n) \in L_n^p(I_0)$ and $q = (q_1, \dots, q_n) \in \mathbb{R}^n$. Obviously $\Phi: E_1 \to E_1$, $\Psi: E_1 \to E_1$ and the vector fields Φ and Ψ are completely continuous. $\Phi(y, q)$ is non zero on a sphere S_R in the space E_1 ($R > 0$). In fact, if $\Phi(\bar{y}, \bar{q}) = 0$ on S_R, where $(\bar{y}, \bar{q}) \in S_R$ and $\bar{q} = (\bar{q}_1, \dots, \bar{q}_n)$, then $\bar{y}$ is a solution of the problem (*). Thus, (5.5) implies $\bar{y} = 0$. Taking into account that

$$\Phi(0, \bar{q}) = (-\bar{q}, 0)$$

we obtain

$$|\Phi(0, \bar{q})|_1 = \max_{1 \leqq i \leqq n} |\bar{q}_i| \, \|1\|_p > 0 \quad (\text{because } R > 0)$$

and

$$\Phi(y, q) \neq 0 \quad \text{on} \quad S_R.$$

By [13] (p. 112) we get

$$\inf_{(y,q) \in S_R} |\Phi(y, q)|_1 = \alpha > 0. \tag{5.10}$$

Now, we shall show that $\gamma(\Phi, S_R) \neq 0$. For this purpose we apply the Borsuk theorem (the antipodal theorem, [13] p. 130). Therefore, it is enough to prove that

$$\frac{\Phi(y, q)}{|\Phi(y, q)|_1} \neq \frac{\Phi(-y, -q)}{|\Phi(-y, -q)|_1} \quad \text{on} \quad S_R.$$

Suppose the contrary, then there exists a number $\beta > 0$ satisfying the equality

$$\Phi(y, q) = \beta \Phi(-y, -q) \quad \text{on} \quad S_R. \tag{5.11}$$

By the assumptions (5.2), (5.4) and the relation (2.3), we infer that

$$\check{f}_{i\omega,1}(\beta y) = \beta \check{f}_{i\omega,1}(y) \quad \text{for} \quad i = 1, \dots, n$$

and

$$(1 + \beta) \, \Phi(y, q) = 0 \quad \text{on} \quad S_R,$$

which contradicts (5.10). Hence it follows that $\gamma(\Phi, S_R) \neq 0$. Let m be a real number such that

$$\alpha m > M + \max_{1 \leqq i \leqq n} |r_i|$$

and let S_{mR} be the sphere of radius mR. Then we have

$$|\Phi(y, q) - \Psi(y, q)|_1 = |(\check{g}_{1\omega,1}(y), \dots, \check{g}_{n\omega,1}(y), r_1, \dots, r_n)|_1 \leqq$$
$$\leqq M + \max_{1 \leqq i \leqq n} |r_i| < \inf_{(y,q) \in S_{mR}} |\Phi(y, q)|_1 \leqq |\Phi(y, q)|_1 \quad \text{on} \quad S_{mR}.$$

Using [13] (p. 128), we get

$$\gamma(\Psi, S_{mR}) \neq 0,$$

which completes the proof.

Theorem 5.2. *Assume*

(5.12) $$f, g\colon W^{n-k,p}(I_0) \to L^{p(k)}(I_0)\,, \quad 2 \leqq 2k \leqq n\,;$$

(5.13) $$f(\lambda y) = \lambda f(y) \quad \text{for} \quad \lambda \in \mathbb{R} \quad \text{and} \quad y \in W^{n-k,p}(I_0)\,;$$

(5.14) *the mappings* $f^{\vee}_{\omega,k}, g^{\vee}_{\omega,k}\colon W^{n-k,p}(I_0) \to L^p(I_0)$ *are completely continuous for a fixed function* ω;

(5.15) $$L_i \in (W^{n-k,p}(I), \|\cdot\|)^* \quad \text{for} \quad i = 1, \ldots, n\,;$$

(5.16) *the problem*

(*, *) $$\begin{cases} y^{(n)} = f(y) \\ L_i(y) = 0\,, \quad i = 1, \ldots, n \end{cases}$$

has only the zero solution (in the class $W^{n-k,p}(I)$*);*

(5.17) $$\|g^{\vee}_{\omega,k}(y)\|_p \leqq M < \infty \quad \text{for all} \quad y \in W^{n-k,p}(I)\,.$$

Then the problem

(5.18) $$\begin{cases} y^{(n)} = f(y) + g(y)\,, \\ L_i(y) = r_i\,, \quad r_i \in \mathbb{R}\,, \quad i = 1, \ldots, n \end{cases}$$

has a weak solution (in the class $W^{n-k,p}(I)$*).*

Proof. We consider two vector fields

(5.19) $$\Phi_1(y, q) = \Big(y - U_\omega(y) - \sum_{i=0}^{n-1} q_i t^i\,, \quad L_1(y), \ldots, L_n(y)\Big)$$

and

(5.20) $$\Psi_1(y, q) = \Big(y - U_\omega(y) - g^{\vee}_{\omega,k}(y) - \sum_{i=0}^{n-1} q_i t^i\,,$$
$$L_1(y) - r_1, \ldots, L_n(y) - r_n\Big)\,,$$

where $y \in W^{n-k,p}(I)$, $q = (q_0, q_1, \ldots, q_{n-1}) \in \mathbb{R}^n$ and $U_\omega(y)$ is defined by (4.3).

It is clear that $\Phi_1\colon E_2 \to E_2$, $\Psi_1\colon E_2 \to E_2$ and the vector fields Φ_1 and Ψ_1 are completely continuous. Let S_R be a sphere in the space E_2 ($R > 0$). Then $\Phi_1(y, q) \neq 0$ on S_R. Indeed, if $\Phi_1(\bar{y}, \bar{q}) = 0$ on S_R ($(\bar{y}, \bar{q}) \in S_R$ and $\bar{q} = (\bar{q}_0, \bar{q}_1, \ldots, \bar{q}_{n-1})$), then $\bar{y}$ is a solution of the problem (*, *). By (5.16) we have

$$\bar{y} = 0 \quad \text{and} \quad \Phi_1(0, \bar{q}) = \Big(\sum_{i=0}^{n-1} \bar{q}_i t^i, 0\Big)\,.$$

Hence

$$|\Phi_1(0, \bar{q})|_2 = \Big\|\sum_{i=0}^{n-1} \bar{q}_i t^i\Big\| > 0 \quad \text{(because } R > 0\text{)}$$

and

(5.21) $$\inf_{(y,q) \in S_R} |\Phi_1(y, q)|_2 = \alpha > 0 \quad \text{(see [13], p. 112)}\,.$$

We shall prove that

$$\frac{\Phi_1(y, q)}{|\Phi_1(y, q)|_2} \neq \frac{\Phi_1(-y, -q)}{|\Phi_1(-y, -q)|_2}.$$

In fact, if there exists a number $\beta > 0$ satisfying the equality

$$\Phi_1(y, q) = \beta \Phi_1(-y, -q) \quad \text{on} \quad S_R,$$

then

$$(1 + \beta) \Phi_1(y, q) = 0 \quad \text{on} \quad S_R \quad (\text{by } (2.3), (5.13) \text{ and } (5.15)),$$

which contradicts (5.21). By the antipodal theorem (K. Borsuk) we have

$$\gamma(\Phi_1, S_R) \neq 0.$$

Now, we take a number m such that

$$\alpha m > M + \max_{1 \leqq i \leqq n} |r_i|$$

and consider the sphere S_{mR} in the space E_2 of radius mR. Evidently

$$|\Phi_1(y, q) - \Psi_1(y, q)|_2 = |(g^{\vee}_{\omega,k}(y), r_1, \ldots, r_n)|_2 \leqq$$
$$\leqq M + \max_{1 \leqq i \leqq n} |r_i| < \inf_{(y,q) \in S_{mR}} |\Phi_1(y, q)|_2 \leqq |\Phi_1(y, q)|_2 \quad \text{on} \quad S_{mR}$$

and

$$\gamma(\Psi_1, S_{mR}) \neq 0 \quad (\text{see [13], p. 128}),$$

which completes the proof.

Now, we shall give some examples and remarks.

Example 5.1. Let mappings $k_{ij}: I \times \mathbb{R}^n \to \mathbb{R}$ satisfy the condition (C) in the set $D = I \times \mathbb{R}^n$ and let

$$|k_{ij}(t, u_1, \ldots, u_n)| \leqq M \quad \text{for} \quad (t, u_1, \ldots, u_n) \in D, \quad M \in \mathbb{R},$$
$$i = 1, \ldots, n \quad \text{and} \quad j = 1, 2.$$

Moreover, let $h_j: I \to I$, $h_j \in C^1(I)$ and $h'_j(t) > 0$ for $t \in I$ and $j = 1, \ldots, 2n$. Then the operations

$$g_i(y)(t) = k_{i1}(t, y_1(h_1(t)), \ldots, y_n(h_n(t))) +$$
$$= Q'_i(t) \int_a^b k_{i2}(t, y_1(h_{n+1}(t)), \ldots, y_n(h_{2n}(t)))\, dt,$$

where $Q_i \in L^p(I)$, $y_i \in L^p(I)$, $p \in [1, \infty)$ and $i = 1, \ldots, n$ satisfy the assumptions (5.1), (5.3) and (5.6).

Example 5.2. We assume that

1° the function $k_0: I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded,

2° the functions $k_1, k_2: I \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the condition (C) in the set $D = I \times \mathbb{R}^2$ and $|k_0(t, u)| \leqq M$,

$$|k_i(t, u, v)| \leqq M \quad \text{for} \quad (t, u, v) \in D, \quad i = 1, 2$$

and $M \in \mathbb{R}$,

3° the functions $h_j: I \to I$ for $j = 0, 1, 2, 3, 4$; $h_0, h_1, h_3 \in C(I)$, $h_2, h_4 \in C^1(I)$, $h_2'(t) > 0$ and $h_4'(t) > 0$, where $t \in I$;

4° Q is the function of bounded variation on I,

5° $S \in L^p(I)$, $p \in [1, \infty)$.

We define

$$(5.22)\qquad g(y)(t) = Q'(t)\, k_0(t, y(h_0(t))) + S'(t) \int_a^b k_1(t, y(h_1(t)), y'(h_2(t)))\, dt + \\ + k_2(t, y(h_3(t)), y'(h_4(t)))\,,$$

where $y \in W^{1,p}(I)$, the product $Q'(t)\, k_0(t, y(h_0(t)))$ is understood as a generalized operation (see [1], p. 256 and [2]) and the derivative is understood in the distributional sense. It is worth noting that the product $Q'(t)\, k_0(t, y(h_0(t)))$ exists and is a measure (see [2]). Hence

$$g\colon W^{1,p}(I_0) \to L^{p(1)}(I_0) \quad \text{and} \quad g^{\vee}_{\omega,1}\colon W^{1,p}(I_0) \to L^p(I_0)\,.$$

Now, we shall show that the operation $g^{\vee}_{\omega,1}$ is continuous and bounded. For this purpose we shall define the definite integral of a measure $\bar{p}$ as

$$\int_{t_1}^{t_2} \bar{p}(t)\, dt = P^*(t_2) - P^*(t_1)\,,$$

where

$$P' = \bar{p} \text{ on } I_0\,, \quad t_1, t_2 \in I_0\,, \quad P^*(t_i) = \tfrac{1}{2}(P(t_i+) + P(t_i-))\,,$$

$P(t_i+)$ and $P(t_i-)$ denote the right and left hand side limits of the function P at the point t_i (for $i = 1, 2$). It is known (see [3], [22]) that

$$(5.23)\qquad \left|\int_{t_1}^{t_2} Q'(t)\, k_0(t, y(h_0(t)))\, dt\right| \leqq M\left|U^*(t_2) - U^*(t_1)\right|$$

and

$$(5.24)\qquad \lim_{n\to\infty} \int_{t_1}^{t} Q'(t) \left[k_0(s, y_n(h_0(s))) - k_0(s, y(h_0(s)))\right] ds = 0$$

(almost uniformly on I), where $U' = |Q'|$, $|Q'|$ is defined as a generalized operation (see [1] and [3]) and $\lim_{n\to\infty} y_n = y$ (uniformly on I, $y_n, y \in C(I)$). By the relations (5.22)–(5.24) we conclude that $g^{\vee}_{\omega,1}$ is a continuous and bounded operation and $g^{\vee}_{\omega,1}$ has the property R in I. Next, from the Riesz theorem we infer that $g^{\vee}_{\omega,1}$ is a completely continuous operation. S_0 g and $g^{\vee}_{\omega,1}$ satisfy the assumptions (5.12), (5.14) and (5.17) (for $n = 2$ and $k = 1$).

Remark 5.1. Let $g^{\vee}_{i\omega,1}$ be mappings such that

(a) $g^{\vee}_{i\omega,1}\colon L^p(I_0) \to L^p(I_0)$ for $i = 1, \ldots, n$;

(b) $g^{\vee}_{i\omega,1}$ are bounded and continuous for $i = 1, \ldots, n$;

(c) $g^{\vee}_{i\omega,1}$ have the property R in I_0 for $i = 1, \ldots, n$.

Then $g^{\vee}_{i\omega,1}$ are completely continuous mappings for $i = 1, \ldots, n$.

Remark 5.2. If $g^{\vee}_{\omega,k}$ satisfies the conditions

(a′) $g^{\vee}_{\omega,k}\colon W^{n-k,p}(I_0) \to L^p(I_0)$ $(2 \leqq 2k \leqq n)$,

(b′) $g^{\vee}_{\omega,k}$ is bounded and continuous,
(c′) $g^{\vee}_{\omega,k}$ has the property R in I_0,
then $g^{\vee}_{\omega,k}$ is a completely continuous mapping.

Example 5.3. We consider "homogeneous problems" of the form

$$\begin{aligned} y'(x) &= z(x) \\ z'(x) &= -y(x) \\ \int_0^{2\pi} y(x)\,\mathrm{d}x &= \int_0^{2\pi} z(x)\,\mathrm{d}x = 0 \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} y''(x) &= -y(x) \\ y(0) &= y(2\pi) = 0\,. \end{aligned} \tag{5.26}$$

It is easy to observe that the problems (5.25)—(5.26) have non trivial solutions (for instance $y(x) = \sin x$, $z(x) = \cos x$).

Remark 5.3. In the papers [4], [12], [17], [18], [19], [21], [24], [29], [31] we can find some conditions which guarantee that the trivial solution is the unique solution of the homogeneous problems.

Remark 5.4. A. Filippov in [7] considers some ordinary (linear and non linear) differential equations with distributions as coefficients, which can be replaced by a system of equations satisfying Caratheodory's conditions.

Remark 5.5. Let $g \in L^{p(k)}(I_0)$, where $I_0 \subset \mathbb{R}$, $r \in \mathbb{N}$ and $p \in (1, \infty)$. Then there exists exactly one element $g^\wedge \in L^p(I_0)$ such that

$$\|g^\wedge\|_p = \inf\{\|G\|_p\colon G \in L^p(I_0),\ G^{(r)} = g\} \tag{5.27}$$

(by the uniform convexity of the space $L^p(I_0)$ for $p \in (1, \infty)$). In [33] an operation $^\wedge$ is considered which assigns to a distribution $g \in L^{2(r)}(I_0)$ one of its r-th primitives (satisfying the condition (5.27)). If $1 < p < \infty$, the operation $^\wedge$ is not linear unless $p = 2$. Hence we conclude that

$$\|g_1^\wedge - g_2^\wedge\|_2 \leqq \|G_1 - G_2\|_2\,, \tag{5.28}$$

where $g_1 = G_1^{(r)}$, $g_2 = G_2^{(r)}$, $G_1, G_2 \in L^2(I_0)$ and $r \in \mathbb{N}$. Taking into account (5.28), we obtain a better estimate for the coefficient α in Examples 3.1—3.2 and 4.2 (for $p = 2$) using the operation $^\wedge$ than for the operation $^\vee$.

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Souhrn

APLIKACE HLADKÉHO INTEGRÁLU V TEORII SLABÝCH ŘEŠENÍ OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC

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Článek se zabývá existencí a jednoznačností slabých (distributivních) řešení lineárních a nelineárních okrajových úloh pro obyčejné diferenciální rovnice. Hlavními prostředky jsou pojem hladkého integrálu a klasické věty o pevném bodě.

Резюме

ПРИЛОЖЕНИЯ ГЛАДКОГО ИНТЕГРАЛА В ТЕОРИИ СЛАБЫХ РЕШЕНИЙ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

JAN LIGĘZA

В статье изучается существование и однозначность слабых (в смысле обобщенных функций) решений линейных и нелинейных краевых задач для обыкновенных дифференциальных уравнений. Основными средсвами являются понятие гладкого интеграла и классические теоремы о неподвижной точке.

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