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NATURAL TRANSFORMATIONS OF HIGHER ORDER TANGENT BUNDLES AND JET SPACES

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Dedicated to Professor Otakar Borůvka on the occasion of his ninetieth birthday

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Summary. We deduce that all natural transformations of the functor of the r-th order tangent vectors into itself are the homotheries only. We also determine all natural transformations of the r-th order jet functor into itself.

Keywords: Natural transformation, r-th order tangent vector, r-jet.

AMS Classication: 58A20.

Using a general method developed in [5], we first deduce that all natural transformations of the r-th order tangent functor T^r into itself are the homotheties only. From the general point of view it is worth pointing out that this property is related with the fact that T^r does not preserve products, and to contrast it with a recent result by G. Kainz and P. Michor, [3], which describes all natural transformations of the product-preserving differential geometric functors in terms of the homomorphisms of the related Weil algebras. Then we prove in a similar way that for $r \geqq 2$ the only natural transformations of the r-th jet functor J^r into itself are the identity and the contraction, while in the first order case, in which we deal with vector bundles, we have the one-parameter family of all homotheties. The authors hope that this interesting fact on a certain rigidity of the higher order jet spaces will lead to a deeper understanding of some general features of the higher order differential geometry. — All manifolds and maps are assumed to be infinitely differentiable.

1. Let $T^{r*}M = J^r(M, \mathbf{R})_0$ be the space of all r-jets of a manifold M into $\mathbf{R}$ with target 0. Since $\mathbf{R}$ is a vector space, $T^{r*}M$ has a canonical structure of a vector bundle over M. The dual vector bundle $T^rM := (T^{r*}M)^*$ is called the r-th order tangent bundle of M, [8]. Given a map $f: M \to N$, the jet composition $V \mapsto V \circ j^r_x f$, $V \in T^{r*}_{f(x)}N$, determines a linear map $T^{r*}_{f(x)}N \to T^{r*}_x M$. The dual map $T^r_x M \to T^r_{f(x)}N$ will be denoted by $T^r_x f$ and called the r-th order tangent map of f at x. This defines a functor T^r from the category $\mathcal{M}f$ of all manifolds and maps into the category $\mathcal{VB}$ of vector bundles.

If x^i are some local coordinates on M, then the induced fibre coordinates $u_i, u_{i_1 i_2}, \ldots, u_{i_1 \ldots i_r}$ (symmetric in all indices) on $T^{r*}M$ correspond to the polynomial representant $u_i x^i + u_{i_1 i_2} x^{i_1} x^{i_2} + \ldots + u_{i_1 \ldots i_r} x^{i_1} \ldots x^{i_r}$ of any element $U \in T^{r*}M$.

A linear functional on $T^{r*}_x M$ with the fibre coordinates $X^i, X^{i_1 i_2}, \dots, X^{i_1 \dots i_r}$ (symmetric in all indices) has the form

(1) $$X^i u_i + X^{i_1 i_2} u_{i_1 i_2} + \dots + X^{i_1 \dots i_r} u_{i_1 \dots i_r}.$$

Let y^p be some local coordinates on N, let $Y^p, Y^{p_1 p_2}, \dots, Y^{p_1 \dots p_r}$ be the induced fibre coordinates on $T^r N$ and let $y^p = f^p(x^i)$ be the coordinate expression of a map $f: M \to N$. Evaluating the jet composition $V \circ j^r_x f$, $V \in T^{r*}_{f(x)} N$, we deduce by (1) the following coordinate expression of $T^r f$, cf. [4],

(2) $$\begin{aligned} Y^p &= \frac{\partial f^p}{\partial x^i} X^i + \frac{1}{2!} \frac{\partial^2 f^p}{\partial x^{i_1} \partial x^{i_2}} X^{i_1 i_2} + \dots + \frac{1}{r!} \frac{\partial^r f^p}{\partial x^{i_1} \dots \partial x^{i_r}} X^{i_1 \dots i_r} \\ &\vdots \\ Y^{p_1 \dots p_s} &= \frac{\partial f^{p_1}}{\partial x^{i_1}} \dots \frac{\partial f^{p_s}}{\partial x^{i_s}} X^{i_1 \dots i_s} + \dots \\ &\vdots \\ Y^{p_1 \dots p_r} &= \frac{\partial f^{p_1}}{\partial x^{i_1}} \dots \frac{\partial f^{p_r}}{\partial x^{i_r}} X^{i_1 \dots i_r} \end{aligned}$$

where the dots in the middle row denote a polynomial expression, each term of which contains at least one partial derivative of f^p of an order at least two.

Since T^r is a functor with values in the category $\mathscr{V}\mathscr{B}$, for every $k \in \mathbf{R}$ the homotheties

(3) $$(k)^r_M : T^r M \to T^r M\,, \quad X \mapsto kX$$

represent natural transformations of T^r into itself.

Proposition 1. *All natural transformations $T^r \to T^r$ form the one-parameter family* (3) *with any* $k \in \mathbf{R}$.

Proof. First, consider T^r as a functor on the subcategory $\mathscr{M}f_n \subset \mathscr{M}f$ of all n-dimensional manifolds and their local diffeomorphisms. Since T^r is an r-th order functor, its standard fibre $S = T^r_0 \mathbf{R}^n$ is a G^r_n-space, where G^r_n means the group of all invertible r-jets of $\mathbf{R}^n$ into $\mathbf{R}^n$ with source and target 0. By (2), the action of an element $(a^i_j, a^i_{j_1 j_2}, \dots, a^i_{j_1 \dots j_r}) \in G^r_n$ on $(X^i, X^{i_1 i_2}, \dots, X^{i_1 \dots i_r}) \in S$ is

(4) $$\begin{aligned} \bar{X}^i &= a^i_j X^j + a^i_{j_1 j_2} X^{j_1 j_2} + \dots + a^i_{j_1 \dots j_r} X^{j_1 \dots j_r} \\ &\vdots \\ \bar{X}^{i_1 \dots i_s} &= a^{i_1}_{j_1} \dots a^{i_s}_{j_s} X^{j_1 \dots j_s} + \dots \\ &\vdots \\ \bar{X}^{i_1 \dots i_r} &= a^{i_1}_{j_1} \dots a^{i_r}_{j_r} X^{j_1 \dots j_r} \end{aligned}$$

where the dots in the middle row denote a polynomial expression, each term of which contains at least one of the quantities $a^i_{j_1 j_2}, \dots, a^i_{j_1 \dots j_r}$. In the sequel we shall write shortly $(X^i, X^{i_1 i_2}, \dots, X^{i_1 \dots i_r}) = (X_1, X_2, \dots, X_r)$.

According to a general theory, cf. [2], [7], the natural transformations $T^r \to T^r$ are in bijection with G^r_n-equivariant maps $f: S \to S$. There is a canonical injection i:

$GL(n, \mathbf{R}) \to G_n^r$ transforming every matrix into the r-jet at 0 from the corresponding linear transformation of $\mathbf{R}^n$. The subgroup $i(GL(n, \mathbf{R})) \subset G_n^r$ is characterized by $a^i_{j_1 j_2} = 0, \ldots, a^i_{j_1 \ldots j_r} = 0$. First consider the equivariancy of $f = (f_1, \ldots, f_r)$ with respect to the homotheties $a^i_j = k\delta^i_j$. Using (4) we obtain

$$(5) \qquad \begin{aligned} kf_1(X_1, \ldots, X_s, \ldots, X_r) &= f_1(kX_1, \ldots, k^s X_s, \ldots, k^r X_r) \\ &\vdots \\ k^s f_s(X_1, \ldots, X_s, \ldots, X_r) &= f_s(kX_1, \ldots, k^s X_s, \ldots, k^r X_r) \\ &\vdots \\ k^r f_r(X_1, \ldots, X_s, \ldots, X_r) &= f_r(kX_1, \ldots, k^s X_s, \ldots, k^r X_r)\,. \end{aligned}$$

To discuss (5), we need the following simple property of the globally defined smooth homogeneous functions, a proof of which can be found e.g. in [9].

Lemma. *Let $g(x^i, y^p, \ldots, z^t)$ be a smooth function defined on $\mathbf{R}^m \times \mathbf{R}^n \times \ldots \times \mathbf{R}^p$, and let $a > 0$, $b > 0, \ldots, c > 0$, d be real numbers such that*

$$(6) \qquad k^d g(x^i, y^p, \ldots, z^t) = g(k^a x^i, k^b y^p, \ldots, k^c z^t)$$

for every real $k > 0$. Then g is a sum of polynomials of degrees ξ in x^i, η in $y^p, \ldots, \zeta$ in z^t satisfying

$$(7) \qquad a\xi + b\eta + \ldots + c\zeta = d\,.$$

If there are no non-negative integers $\xi, \eta, \ldots, \zeta$ with the property (7), then g is the zero function.

According to this lemma, f_1 is linear in X_1 and independent of $X_2, \ldots, X_r$, while $f_s = g_s(X_s) + h_s(X_1, \ldots, X_{s-1})$, where g_s is linear in X_s and h_s is a certain polynomial in $X_1, \ldots, X_{s-1}$, $2 \leqq s \leqq r$. Considering the equivariancy of f with respect to the whole subgroup $i(GL(n, \mathbf{R}))$, we find that g_s is a $GL(n, \mathbf{R})$-equivariant map of the s-th symmetric tensor power $S^s\mathbf{R}^n$ into itself. By the classical theory of the invariant tensors, $g_s = c_s X_s$ (or, explicitly, $g^{i_1 \ldots i_s} = c_s X^{i_1 \ldots i_s}$) with any $c_s \in \mathbf{R}$, cf. [1].

Further, consider the equivariancy with respect to the kernel of the jet projection $G_n^r \to G_n^1 = GL(n, \mathbf{R})$, which is characterized by $a^i_j = \delta^i_j$. Then the first line of (4) implies

$$(8) \qquad c_1 X^i + a^i_{j_1 j_2}(c_2 X^{j_1 j_2} + h^{j_1 j_2}(X_1)) + \ldots + a^i_{j_1 \ldots j_r}(c_r X^{j_1 \ldots j_r} + h^{j_1 \ldots j_r}(X_1, \ldots, X_{r-1})) = c_1(X^i + a^i_{j_1 j_2} X^{j_1 j_2} + \ldots + a^i_{j_1 \ldots j_r} X^{j_1 \ldots j_r})\,.$$

Setting $a^i_{j_1 \ldots j_s} = 0$ for all $s > 2$, we find $c_2 = c_1$ and $h^{j_1 j_2}(X_1) = 0$. By a recurrence procedure of this type we further deduce $c_s = c_1$ and $h^{j_1 \ldots j_s}(X_1, \ldots, X_{s-1}) = 0$ for all $s = 3, \ldots, r$.

This implies that the restriction of every natural transformation $T^r \to T^r$ to each subcategory $\mathcal{M}f_n \subset \mathcal{M}f$ is a homothety with a coefficient k_n. Taking into account the injection $\mathbf{R}^n \to \mathbf{R}^{n+m}$, $(x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0, \ldots, 0)$, we find $k_{n+m} = k_n$ for all m and n. This completes the proof of Proposition 1.

2. Let $f: M \to \bar{M}$ be a local diffeomorphism and let $g: N \to \bar{N}$ be any map. Then there is an induced map $J^r(f, g)$ from the space $J^r(M, N)$ of all r-jets of M into N into $J^r(\bar{M}, \bar{N})$ given by

$$J^r(f, g)(X) = (j^r_y g) \circ X \circ (j^r_{f(x)} f^{-1}) \tag{9}$$

where $x = \alpha X$ or $y = \beta X$ is the source or the target of $X \in J^r(M, N)$ and the inverse map f^{-1} is constructed locally, cf. [6], This defines a functor J^r from the product category $\mathcal{M}f_m \times \mathcal{M}f$ into the category of fibred manifolds (we consider $J^r(M, N)$ as a fibred manifold over $M \times N$).

Denote by $\hat{y}: M \to N$ the constant map of M into $y \in N$. Obviously, the assignment $X \mapsto j^r_{\alpha X} \widehat{\beta X}$ is a (trivial) natural transformation of J^r into itself called the contraction. For $r = 1$, $J^1(M, N)$ coincides with Hom (TM, TN), which is a vector bundle over $M \times N$.

Proposition 2. *For $r \geqq 2$ the only natural transformations $J^r \to J^r$ are the identity and the contraction. For $r = 1$ all natural transformations $J^1 \to J^1$ form the one-parameter family of homotheties $A \mapsto kA$, $k \in \mathbf{R}$.*

Proof. We shall consider the subcategory $\mathcal{M}f_m \times \mathcal{M}f_n \subset \mathcal{M}f_m \times \mathcal{M}f$ only, since the remaining part of the proof is quite similar to the end of the proof of Proposition 1. The standard fibre $S = J^r_0(\mathbf{R}^m, \mathbf{R}^n)_0$ is a $G^r_m \times G^r_n$-space, see [6]. The action of $(A, B) \in G^r_m \times G^r_n$ on $X \in S$ is given by the jet composition

$$\bar{X} = B \circ X \circ A^{-1}\,. \tag{10}$$

Quite analogously to the classical case, the natural transformations $J^r \to J^r$ are in bijection with the $G^r_m \times G^r_n$-equivariant maps $f: S \to S$.

Write $A^{-1} = (a^i_j, \ldots, a^i_{j_1 \ldots j_r})$, $B = (b^p_q, \ldots, b^p_{q_1 \ldots q_r})$, $X = (X^p_i, \ldots, X^p_{i_1 \ldots i_r}) = (X_1, \ldots, X_r)$. First, consider the equivariancy of $f = (f_1, \ldots, f_r)$ with respect to the homotheties $a^i_j = k^{-1} \delta^i_j$ in $i(GL(m, \mathbf{R}))$. This gives the homogeneity conditions of type (5). Taking into account the homotheties $b^p_q = k \delta^p_q$ in $i(GL(n, \mathbf{R}))$, we further find

$$\begin{aligned} k f_1(X_1, \ldots, X_r) &= f_1(k X_1, \ldots, k X_r) \\ &\;\vdots \\ k f_r(X_1, \ldots, X_r) &= f_r(k X_1, \ldots, k X_r)\,. \end{aligned} \tag{11}$$

Applying our lemma to both (5) and (11), we deduce that f_s is linear in X_s and independent of the other coordinates, $s = 1, \ldots, r$. Further, consider the equivariancy with respect to the subgroup $i(GL(m, \mathbf{R})) \times i(GL(n, \mathbf{R}) \subset G^r_m \times G^r_n$. This yields that f_s corresponds to a $GL(m, \mathbf{R}) \times GL(n, \mathbf{R})$-equivariant map of $\mathbf{R}^n \otimes S^s \mathbf{R}^{m*}$ into itself. By Lemma 3 of [5], we have $f_s = c_s X_s$ (or, explicitly, $f^p_{i_1 \ldots i_s} = c_s X^p_{i_1 \ldots i_s}$) with any $c_s \in \mathbf{R}$.

For $r = 1$ we have deduced $f^p_i = c_1 X^p_i$, which proves Proposition 2. For $r = 2$ consider the equivariancy with respect to the kernel of the jet projection $G^2_m \times$

$\times\, G^2_n \to G^1_m \times G^1_n$. Taking into account the coordinate form of the jet composition, we find that the action of an element $((\delta^i_j, a^i_{jk}), (\delta^p_q, b^p_{qr}))$ on (X^p_i, X^p_{ij}) is $\bar{X}^p_i = X^p_i$ and

$$\bar{X}^p_{ij} = X^p_{ij} + b^p_{qr} X^q_i X^r_j + X^p_k a^k_{ij}\,. \tag{12}$$

Then the equivariancy condition for f^p_{ij} reads

$$c_2 X^p_{ij} + c_1^2 b^p_{qr} X^q_i X^r_j + c_1 X^p_k a^k_{ij} = c_2 (X^p_{ij} + b^p_{qr} X^q_i X^r_j + X^p_k a^k_{ij})\,. \tag{13}$$

This implies $c_1 = c_2 = 0$ or $c_1 = c_2 = 1$. Assume by induction that Proposition 2 holds for the order $r - 1$. Consider the equivariancy with respect to the kernel of the jet projection $G^r_m \times G^r_n \to G^{r-1}_m \times G^{r-1}_n$. The action of an element $((\delta^i_j, 0, \ldots, 0, a^i_{j_1 \ldots j_r}), (\delta^p_q, 0, \ldots, 0, b^p_{q_1 \ldots q_r}))$ leaves $X_1, \ldots, X_{r-1}$ unchanged and

$$\bar{X}^p_{i_1 \ldots i_r} = X^p_{i_1 \ldots i_r} + b^p_{q_1 \ldots q_r} X^{q_1}_{i_1} \ldots X^{q_r}_{i_r} + X^p_j a^j_{i_1 \ldots i_r}\,. \tag{14}$$

Then the equivariancy condition for $f^p_{i_1 \ldots i_r}$ requires

$$c_r X^p_{i_1 \ldots i_r} + c_1^r b^p_{q_1 \ldots q_r} X^{q_1}_{i_1} \ldots X^{q_r}_{i_r} + c_1 X^p_j a^j_{i_1 \ldots i_r} = \tag{15}$$
$$= c_r (X^p_{i_1 \ldots i_r} + b^p_{q_1 \ldots q_r} X^{q_1}_{i_1} \ldots X^{q_r}_{i_r} + X^p_j a^j_{i_1 \ldots i_r})\,.$$

This implies $c_r = c_1 = 0$ or 1, QED.

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Souhrn

PŘIROZENÉ TRANSFORMACE TEČNÝCH VEKTORŮ VYŠŠÍHO ŘÁDU A JETOVÝCH PROSTORŮ

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Dokazuje se, že všechny přirozené transformace funktoru tečných vektorů r-tého řádu do sebe jsou pouze homotetie. Určují se rovněž všechny přirozené transformace funktoru jetů r-tého řádu do sebe.

Резюме

НАТУРАЛЬНЫЕ ПРЕОБРАЗОВАНИЯ РАССЛОЕНИЙ КАСАТЕЛЬНЫХ ВЕКТОРОВ ВЫСШЕГО ПОРЯДКА И ПРОСТРАНСТВ СТРУЙ

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Показывается, что гомотетии являются единственными естественными преобразованиями функтора касательных векторов высшего порядка в себя. Определяются также все естественные преобразования функтора струй любого порядка в себя.

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