Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 101 (1976), No. 1, 28--44

Persistent URL: `http://dml.cz/dmlcz/108693`

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A NOTE ON A HEAT POTENTIAL AND THE PARABOLIC VARIATION

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(Received January 24, 1974)

INTRODUCTION

Let R^n stand for the n-dimensional Euclidean space (n positive integer). We shall deal with the plane R^2 in the sequel. Further let $^*R^1$ be the real axis together with the points $+\infty$ and $-\infty$. Whenever we say f is a function on a set M we mean f is a mapping from M into $^*R^1$; a real function on M is a mapping from M into R^1. If we speak about a continuous function we always consider a real function. Given I a compact interval in R^1, $\mathscr{C}(I)$ is defined to be the space of all continuous functions on I. We consider $\mathscr{C}(I)$ endowed with the supremum norm topology.

Let $\langle a, b\rangle$ be a compact interval in R^1 and let φ be a continuous function of bounded variation on $\langle a, b\rangle$. Conformably to [1] we shall introduce some notations. For any point $[x, t] \in R^2$ such that $t > a$ we define a function $\alpha_{x,t}$ on the interval $\langle a, \min\{t, b\})$ by

$$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2\sqrt{(t - \tau)}}\,.$$

$\alpha_{x,t}$ is always a continuous function of locally bounded variation on the interval $\langle a, \min\{t, b\})$. Further we define for each continuous function f on $\langle a, b\rangle$

$$Tf(x, t) = \frac{1}{2\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) \tag{0.1}$$

whenever $[x, t] \in R^2$, $t > a$ and the integral on the right hand side of (0.1) exists in the sense of the Lebesgue-Stieltjes integral and is finite. If $t \leqq a$ then we put $Tf(x, t) = 0$.

It turns out useful to investigate Tf considered as a function on $R^2 - \{[\varphi(t), t];\ t \in \langle a, b\rangle\}$ for a fixed f in connection with the boundary value problem of the heat equation in R^2, especially with the Fourier problem (see [1]).

A theorem concerning the limit value of Tf on the set $K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}$ has been proved in [1]. In this paper we shall show some complementary results on that matter and on the parabolic variation. The parabolic variation of the curve φ was defined in [1] and played the main role in the investigation of the potential Tf. In the same way as in [1] we define the so-called parabolic variation with a weight Q. Let Q be a nonnegative, lower-semicontinuous and bounded function on the interval $\langle a, b\rangle$. Let $[x, t] \in R^2$. For $\alpha, r > 0$, $\alpha < +\infty$ put

$$n^Q_{x,t}(r, \alpha) = \sum_\tau Q(\tau)\,, \tag{0.2}$$

where the sum on the right hand side is taken over all $\tau \in \langle a, b\rangle$ such that $0 < t - \tau < r$ and

$$t - \tau = \left(\frac{x - \varphi(\tau)}{2\alpha}\right)^2 .$$

The parabolic variation with the weight Q and the radius r of the curve φ at the point $[x, t]$ is defined by

$$V^Q_K(r; x, t) = \int_0^\infty e^{-\alpha^2}\, n^Q_{x,t}(r, \alpha)\, \mathrm{d}\alpha \tag{0.3}$$

(see [1], Definition 1.1). Further we denote

$$V^Q_K(\infty; x, t) = V^Q_K(x, t)\,, \quad V^1_K(r; x, t) = V_K(r; x, t)\,,$$
$$V^1_K(x, t) = V_K(x, t) \quad ([x, t] \in R^2)\,.$$

The function $V^Q_K(r; \cdot)$ (as a function on R^2) is a nonnegative lower-semicontinuous function on R^2 and is finite on $R^2 - K$ (see [1], Lemma 1.2). Further, it holds for each $r > 0$, $x \in R^1$, $t \in R^1$, $a < t < b + r$ that

$$V^Q_K(r; x, t) = \int_{\max\{a, t-r\}}^{\min\{t, b\}} Q(\tau) \exp\left(-\alpha^2_{x,t}(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) \tag{0.4}$$

(see [1], Lemma 1.1). If $t \leqq a$ or $t \geqq b + r$ then $V^Q_K(r; x, t) = 0$.

The parabolic variation is analogous to the cyclic variation introduced in [4] (or [3]). It has been found in [6] that there is a smooth curve which has infinite cyclic variation at its every point. Now an analogous question arises: is there a continuous function φ of bounded variation on $\langle a, b\rangle$ such that $V_K(x, t) = \infty$ for every point $[x, t] \in \{[\varphi(\tau), \tau];\ \tau \in (a, b\rangle\}$? This question is investigated in the second part of this paper.

1.

In this part of the present note we shall show some simple assertions concerning the parabolic variation and some complementary assertions concerning limits of the potential Tf on the curve φ.

Let φ be a continuous function of bounded variation on a compact interval $\langle a, b\rangle \subset R^1$ and let Q be a nonnegative, lower-semicontinuous, bounded function on the interval $\langle a, b\rangle$. Let the symbols $\alpha_{x,t}$, $n^Q_{x,t}$, Tf, K, V^Q_K, V_K denote the same as in the introduction.

By the assumptions there is a constant $c \in R^1$ such that $Q \leqq c$ on $\langle a, b\rangle$. It is seen from the definition of the parabolic variation that

$$V^Q_K(r; x, t) \leqq c\, V_K(r; x, t)$$

for every $[x, t] \in R^2$, $r > 0$. In particular,

$$V_K(r; x, t) < \infty \quad \text{iff} \quad V^Q_K(r; x, t) < \infty\,.$$

Similarly if

$$\sup_{[x,t]\in M} V_K(r; x, t) < \infty \quad \text{then} \quad \sup_{[x,t]\in M} V^Q_K(r; x, t) < \infty$$

for any nonvoid set $M \subset R^2$. The converse statement is not valid. Nevertheless, one may formulate the following assertion:

Let $t_0 \in \langle a, b\rangle$ and suppose that $Q(t_0) > 0$. Then

$$V_K(r; \varphi(t_0), t_0) < \infty \quad \text{iff} \quad V^Q_K(r; \varphi(t_0), t_0) < \infty$$

for any $r > 0$. There is, in addition, an interval $I \subset \langle a, b\rangle$ which is open in $\langle a, b\rangle$ such that $t_0 \in I$ and

$$\sup_{t\in I} V_K(r; \varphi(t), t) < \infty \Leftrightarrow \sup_{t\in I} V^Q_K(r; \varphi(t), t) < \infty\,.$$

One may prove this assertion by means of the equality (0.4) regarding the fact that the function Q is lower-semicontinuous.

Lemma 1.1. *Let $t_0 \in (a, b\rangle$, $x_0 = \varphi(t_0)$ and suppose that*

$$\limsup_{t\to t_0} \frac{|x - \varphi(t)|}{\sqrt{(t_0 - t)}} < \infty\,. \tag{1.1}$$

Then $V^Q_K(x_0, t_0) < \infty$ if and only if

$$\int_a^{t_0} Q(\tau)\,\mathrm{d}\,\mathrm{var}_\tau \left[\frac{x_0 - \varphi(\tau)}{\sqrt{(t_0 - \tau)}}\right] < \infty\,. \tag{1.2}$$

Particularly: $V_K(x_0, t_0) < \infty$ if and only if

$$\mathrm{var}_\tau \left[\frac{x_0 - \varphi(\tau)}{\sqrt{(t_0 - \tau)}};\ \langle a, t_0)\right] < \infty\,. \tag{1.3}$$

Proof. If (1.2) holds then surely $V_K^Q(x_0, t_0) < \infty$ since

$$V_K^Q(x_0, t_0) \leqq \int_a^{t_0} Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{x_0,t_0}(\tau)$$

according to (0.4).

Suppose now that $V_K^Q(x_0, t_0) < \infty$. It is seen from (1.1) that

$$c_0 = \inf \left\{ \exp\left(- \frac{(x_0 - \varphi(\tau))^2}{4(t_0 - \tau)} \right); \quad \tau \in \langle a, t_0) \right\} > 0$$

so that

$$V_K^Q(x_0, t_0) = \int_a^{t_0} Q(\tau) \exp\left(-\alpha_{x_0,t_0}^2(\tau)\right) \mathrm{d} \operatorname{var} \alpha_{x_0,t_0}(\tau) \geqq c_0 \int_a^{t_0} Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{x_0,t_0}(\tau)$$

and thus (1.2) holds.

Now it suffices to note that if Q is the function which assumes the constant value 1 on $\langle a, b\rangle$ then the terms in (1.2) and (1.3) are equal.

In the same way as we defined the parabolic variation we may define a function $W_K^Q(r; \cdot)$ on R^2 putting

$$(1.4) \qquad W_K^Q(r; x, t) = \int_0^\infty n_{x,t}^Q(r, \alpha) \,\mathrm{d}\alpha$$

$(r > 0)$. Similarly we define W_K^Q, $W_K(r; \cdot)$ and W_K.

In the same way as (0.4) has been proved (see [1]) one may prove that

$$W_K^Q(r; x, t) = \int_{\max\{a, t-r\}}^{\min\{t, b\}} Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{x,t}(\tau)$$

for any $r > 0$, $[x, t] \in R^2$ with $a < t < b + r$; particularly

$$W_K(x, t) = \tfrac{1}{2} \operatorname{var}_\tau \left[\frac{x - \varphi(\tau)}{\sqrt{(t - \tau)}} ; \quad \langle a, \min\{t, b\}) \right],$$

whenever $[x, t] \in R^2$, $t > a$.

It follows from Lemma 1.1 that if φ is moreover $\frac{1}{2}$-Hölder on $\langle a, b\rangle$ then it holds

$$V_K^Q(x, t) < \infty \quad \text{iff} \quad W_K^Q(x, t) < \infty$$

for any point $[x, t] \in K$.

Lemma 1.2. *Let the function φ be $\frac{1}{2}$-Hölder on the interval $\langle a, b\rangle$. Then*

$$(1.5) \qquad \sup\{V_K^Q(x, t); [x, t] \in K\} < \infty$$

if and only if

$$\sup \{W_K^Q(x, t);\ [x, t] \in K\} < \infty . \tag{1.6}$$

Proof. There is a constant $k \in R^1$ such that

$$|\varphi(t_1) - \varphi(t_2)| \leqq k \sqrt{|t_1 - t_2|}$$

for each pair of points $t_1, t_2 \in \langle a, b \rangle$. It is seen from this and from the form of the function $\alpha_{x,t}$ that there is a $c_1 > 0$ such that

$$e^{-\alpha^2_{x,t}(\tau)} \geqq c_1$$

for every $[x, t] \in K$, $t > a$, $\tau \in \langle a, t)$. Hence

$$V_K^Q(x, t) \geqq c_1 \int_a^t Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{x,t}(\tau) = c_1\, W_K^Q(x, t) . \tag{1.7}$$

If $c \in R^1$ is a constant such that $Q \leqq c$ on $\langle a, b \rangle$, then

$$V_K^Q(x, t) \leqq c\, W_K^Q(x, t) .$$

From this and from (1.7) the assertion now follows.

Lemma 1.3. *Given* $\alpha \in (\frac{1}{2}, 1\rangle$, $t \in (a, b\rangle$ *suppose that*

$$\limsup_{\tau \to t-} \frac{|\varphi(t) - \varphi(\tau)|}{(t - \tau)^\alpha} < \infty .$$

Then $V_K^Q(\varphi(t), t) < \infty$ *if and only if*

$$\int_a^t \frac{Q(\tau)}{\sqrt{(t - \tau)}} \,\mathrm{d} \operatorname{var} \varphi(\tau) < \infty . \tag{1.8}$$

If φ *is even* α*-Hölder on the interval* $\langle a, b \rangle$, *then*

$$\sup \{V_K^Q(x, t);\ [x, t] \in K\} < \infty \tag{1.9}$$

if and only if

$$\sup \left\{ \int_a^t \frac{Q(\tau)}{\sqrt{(t - \tau)}} \,\mathrm{d} \operatorname{var} \varphi(\tau);\ t \in (a, b\rangle \right\} < \infty . \tag{1.10}$$

Particularly: *if* φ *is a Lipschitz function on* $\langle a, b \rangle$ *then* (1.9) *holds.*

Proof. Suppose that

$$|\varphi(t) - \varphi(t')| \leqq k(t - t')^\alpha$$

(where k is a suitable real constant) for each $t' \in \langle a, t)$.

Then we have

$$W_K^Q(\varphi(t), t) = \int_a^t Q(\tau) \, \mathrm{d} \, \mathrm{var}_\tau \left[\frac{\varphi(t) - \varphi(\tau)}{2\sqrt{(t - \tau)}} \right] \leqq$$

$$\leqq \int_a^t \frac{Q(\tau)}{2\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var}_\tau \, (\varphi(t) - \varphi(\tau)) + \int_a^t Q(\tau) \, |\varphi(t) - \varphi(\tau)| \, \mathrm{d} \, \mathrm{var}_\tau \left[\frac{1}{2\sqrt{(t - \tau)}} \right] =$$

$$= \int_a^t \frac{Q(\tau)}{2\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var} \, \varphi(\tau) + \int_a^t \frac{|\varphi(t) - \varphi(\tau)|}{4(t - \tau)^{3/2}} \, Q(\tau) \, \mathrm{d}\tau \leqq$$

$$\leqq \int_a^t \frac{Q(\tau)}{2\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var} \, \varphi(\tau) + \int_a^t \frac{k}{4(t - \tau)^{3/2 - \alpha}} \, Q(\tau) \, \mathrm{d}\tau \, .$$

Since Q is a bounded function and $\alpha > \frac{1}{2}$ by the assumption, the last integral is finite. Hence (1.8) implies $W_K^Q(\varphi(t), t)$ is finite (and $V_K^Q(\varphi(t), t)$ is finite, too).

In a similar way we obtain the following estimate:

$$\int_a^t \frac{Q(\tau)}{\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var} \, \varphi(\tau) \leqq \int_a^t \frac{Q(\tau)}{\sqrt{(t - \tau)}} \sqrt{(t - \tau)} \, \mathrm{d} \, \mathrm{var}_\tau \left[\frac{\varphi(t) - \varphi(\tau)}{\sqrt{(t - \tau)}} \right] +$$

$$+ \int_a^t \frac{Q(\tau)}{\sqrt{(t - \tau)}} \, \frac{|\varphi(t) - \varphi(\tau)|}{\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var}_\tau \sqrt{(t - \tau)} = 2W_K^Q(\varphi(t), t) +$$

$$+ \int_a^t \frac{|\varphi(t) - \varphi(\tau)|}{2(t - \tau)^{3/2}} \, Q(\tau) \, \mathrm{d}\tau \leqq 2 \, W_K^Q(\varphi(t), t) + \int_a^t \frac{k \, Q(\tau)}{2(t - \tau)^{3/2 - \alpha}} \, \mathrm{d}\tau \, .$$

The last integral is finite.

We obtain together that (1.8) is valid if and only if $W_K^Q(\varphi(t), t) < \infty$ but this is equivalent with $V_K^Q(\varphi(t), t) < \infty$ in our case (see Lemma 1.1).

One may prove the second part of the assertion by analogous estimates.

Now let φ be a Lipschitz function on $\langle a, b \rangle$ – suppose that

$$|\varphi(t_1) - \varphi(t_2)| \leqq k |t_1 - t_2|$$

for any $t_1, t_2 \in \langle a, b \rangle$. Let $t \in (a, b \rangle$. Then

$$\int_a^t \frac{1}{\sqrt{(t - \tau)}} \, \mathrm{d} \, \mathrm{var} \, \varphi(\tau) = \int_a^t \frac{|\varphi'(\tau)|}{\sqrt{(t - \tau)}} \, \mathrm{d}\tau \leqq k \int_a^t \frac{1}{\sqrt{(t - \tau)}} \, \mathrm{d}\tau =$$

$$= 2k \sqrt{(t - a)} \leqq 2k \sqrt{(b - a)} \, .$$

Thus the condition (1.8) with $Q = 1$ on $\langle a, b \rangle$ is fulfilled and, in fact, (1.9) is valid. This completes the proof.

Let us now define the space $\mathscr{C}_Q(\langle a, b \rangle)$ in the same way as in [1]. Let Q be always a nonnegative lower-semicontinuous and bounded function on the interval $\langle a, b \rangle$.

The space $\mathscr{C}_Q(\langle a, b\rangle)$ is defined to be the space of all functions $f \in \mathscr{C}(\langle a, b\rangle)$ for which there is a real constant c (dependent on the function f) such that

$$|f| \leqq cQ$$

on the interval $\langle a, b\rangle$ and with the property that

$$|f(t_0) - f(t)| = o(Q(t))$$

for every point $t_0 \in \langle a, b\rangle$. We endow the space $\mathscr{C}_Q(\langle a, b\rangle)$ with the norm defined by

$$\|f\|_Q = \inf\{c \in R^1;\ |f| \leqq cQ \text{ on } \langle a, b\rangle\}$$

($f \in \mathscr{C}_Q(\langle a, b\rangle)$). Then the space $\mathscr{C}_Q(\langle a, b\rangle)$ is a Banach space (see [1]).

In [1] we have shown an assertion concerning the limits of the form

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} Tf(x, t)\,, \tag{1.11}$$

where M was a set in R^2 such that $[x_0, t_0] \in K \subset \overline{M}$ and either $M \subset \{[x, t];\ t \in \langle a, b\rangle,\ x > \varphi(t)\}$ or $M \subset \{[x, t];\ t \in \langle a, b\rangle,\ x < \varphi(t)\}$. Provided $Q(a) = 0$ it was proved that the limit (1.11) exists and is finite for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$ if and only if

$$\limsup_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} V_K^Q(x, t) < \infty\,. \tag{1.12}$$

The condition (1.12) is fulfilled, for instance, when there is a $\delta > 0$ such that

$$\sup\{V_K^Q(x, t);\ [x, t] \in K,\ t \in \langle a, b\rangle \cap (t_0 - \delta,\ t_0 + \delta)\} < \infty\,.$$

Let us now consider the case when the condition $Q(a) = 0$ is not supposed.

Proposition 1.1. *Let us suppose that*

$$\lim_{t\to a+} \frac{\varphi(t) - \varphi(a)}{\sqrt{(t - a)}} = 0\,. \tag{1.13}$$

Let β be a continuous function on $\langle a, b\rangle$ such that $\beta(a) = 0$ and

$$|\varphi(t) - \varphi(a)| < \beta(t)\sqrt{(t - a)}$$

for all $t \in (a\ b\rangle$ (according to (1.13) such a function β exists). Put

$$M_1 = \{[x, t];\ t \in (a, b\rangle,\ \varphi(t) < x < \varphi(a) + \beta(t)\sqrt{(t - a)}\}\,,$$
$$M_2 = \{[x, t];\ t \in (a, b\rangle,\ \varphi(a) - \beta(t)\sqrt{(t - a)} < x < \varphi(t)\}\,.$$

Then there are finite limits

$$\lim_{\substack{[x,t]\to[\varphi(a),a]\\ [x,t]\in M_1}} Tf(x, t)\,, \tag{1.14}$$

$$\lim_{\substack{[x,t]\to[\varphi(a),a]\\ [x,t]\in M_2}} Tf(x, t) \tag{1.15}$$

for each function $f \in \mathscr{C}_Q(\langle a, b\rangle)$ *if and only if there is a* $\delta > 0$ *such that*

$$\sup\{V_K^Q(\varphi(t), t);\ t \in (a, a + \delta)\} < \infty\,. \tag{1.16}$$

Proof. One can prove the necessity of the condition (1.16) for the existence of the limits (1.14), (1.15) in the same way as Lemma 2.1 in [1] and Theorem 2.1 in [1] were proved.

Assume now that the condition (1.16) is fulfilled and let a function $f \in \mathscr{C}_Q(\langle a, b\rangle)$ be given.

In the case $f(a) = 0$ the existence of limits (1.14), (1.15) may be proved in exactly the same way as in [1] (making use, of course, of Theorem 1.1 in [1]). In that case even

$$\lim_{[x,t]\to[\varphi(a),a]} Tf(x, t) = 0\,.$$

Now it suffices to show that the limits (1.14), (1.15) exist for any constant function f. That may be proved even if we assume nothing about the parabolic variation. It holds namely for $t \in (a, b\rangle$, $x > \varphi(t)$ that

$$T1(x, t) = 2 - \frac{2}{\sqrt{\pi}}\, G\left(\frac{x - \varphi(a)}{2\sqrt{(t - a)}}\right)$$

(where G is the function on $^*R^1$ defined in [1], i.e. $G(-\infty) = 0$,

$$G(t) = \int_{-\infty}^{t} \mathrm{e}^{-x^2}\,\mathrm{d}x\,,\quad t > -\infty)\,.$$

Consequently, for $[x, t] \in M_1$ it holds (for G is increasing)

$$2 - \frac{2}{\sqrt{\pi}}\, G\left(\frac{\varphi(a) + \beta(t)\sqrt{(t - a)} - \varphi(a)}{2\sqrt{(t - a)}}\right) = 2 - \frac{2}{\sqrt{\pi}}\, G(\tfrac{1}{2}\,\beta(t)) <$$

$$< T1(x, t) < 2 - \frac{2}{\sqrt{\pi}}\, G\left(\frac{\varphi(t) - \varphi(a)}{2\sqrt{(t - a)}}\right).$$

Since

$$\lim_{t\to a+} \tfrac{1}{2}\,\beta(t) = \lim_{t\to a+} \frac{\varphi(t) - \varphi(a)}{2\sqrt{(t - a)}} = 0$$

we obtain immediately that

$$\lim_{\substack{[x,t]\to[\varphi(a),a] \\ [x,t]\in M_1}} T\,1(x, t) = 1 .$$

Similarly for the limit (1.15). The proof is complete .

Now let us present an assertion concerning limits of the form

$$\lim_{x\to\varphi(t)+} Tf(x, t) \quad \text{or} \quad \lim_{x\to\varphi(t)-} Tf(x, t) ,$$

where t is a fixed point of the interval $(a, b\rangle$.

Theorem 1.1. *Given $t \in (a, b\rangle$ suppose that*

$$\limsup_{\tau\to t-} \frac{|\varphi(t) - \varphi(\tau)|}{\sqrt{(t - \tau)}} < \infty . \tag{1.17}$$

Then there are finite limits

$$\lim_{x\to\varphi(t)+} Tf(x, t) , \tag{1.18}$$

$$\lim_{x\to\varphi(t)-} Tf(x, t) \tag{1.19}$$

for each function $f \in \mathscr{C}_Q(\langle a, b\rangle)$ if and only if

$$V_K^Q(\varphi(t), t) < \infty .$$

Proof. If there is, for example, a finite limit (1.18) for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$ then

$$\limsup_{x\to\varphi(t)+} V_K^Q(x, t) < \infty .$$

Since the function V_K^Q is a lower-semicontinuous function on R^2, this implies that $V_K^Q(\varphi(t), t) < \infty$.

Let $V_K^Q(\varphi(t), t) < \infty$. It is sufficient to show that

$$\limsup_{x\to\varphi(t)+} V_K^Q(x, t) < \infty \tag{1.20}$$

and

$$\limsup_{x\to\varphi(t)-} V_K^Q(x, t) < \infty . \tag{1.21}$$

For every $x \in R^1$ we have

$$V_K^Q(x, t) = \sup\left\{\int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau);\ f \in \mathscr{C}_Q,\ \|f\|_Q \leqq 1\right\} .$$

Then it suffices to prove that there are $c \in R^1$, $\delta > 0$ such that

$$(1.22) \qquad \left| \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) - \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) \right| \leqq c$$

for each $x \in (\varphi(t) - \delta, \varphi(t) + \delta)$ and for each function $f \in \mathscr{C}(\langle a, b \rangle)$ with $|f| \leqq Q$ on $\langle a, b \rangle$. Since φ is a continuous function by assumption it follows from the condition (1.17) that there is a constant $k \in R^1$ such that

$$|\varphi(t) - \varphi(\tau)| \leqq k \sqrt{(t - \tau)}$$

for each $\tau \in \langle a, t)$. Let $r > 0$ such that

$$t - \left(\frac{r}{2k}\right)^2 > a\,.$$

Putting $x = \varphi(t) + r$ and considering a function $f \in \mathscr{C}(\langle a, b \rangle)$ with $|f| \leqq Q$ we have

$$\begin{aligned}
(1.23) \qquad & \left| \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) - \int_a^t f(\tau) \exp\left(-\alpha_{\varphi(t),t}^2(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) \right| \leqq \\
& \leqq \left| \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{x,t}(\tau) - \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) \right| + \\
& + \left| \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) - \int_a^t f(\tau) \exp\left(-\alpha_{\varphi(t),t}^2(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) \right| = \\
& = \left| \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\left(\alpha_{x,t}(\tau) - \alpha_{\varphi(t),t}(\tau)\right) \right| + \\
& + \left| \int_a^t f(\tau) \left(\exp\left(-\alpha_{x,t}^2(\tau)\right) - \exp\left(-\alpha_{\varphi(t),t}^2(\tau)\right)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) \right| \leqq \\
& \leqq \int_a^t f(\tau) \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}_\tau \left(\frac{\varphi(t) + r - \varphi(\tau)}{2\sqrt{(t - \tau)}} - \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{(t - \tau)}} \right) + \\
& + \int_a^t |f(\tau)| \left|\exp\left(-\alpha_{x,t}^2(\tau)\right) - \exp\left(\alpha_{\varphi(t),t}^2(\tau)\right)\right| \mathrm{d} \operatorname{var} \alpha_{\varphi(t),t}(\tau) \leqq \\
& \leqq c_0 \frac{r}{4} \int_a^t \frac{1}{(t - \tau)^{3/2}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\tau + \int_a^t Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{\varphi(t),t}(\tau)\,,
\end{aligned}$$

where c_0 is a finite constant such that $Q \leqq c_0$ on $\langle a, b \rangle$. Since the condition (1.17) is fulfilled it follows from Lemma 1.1 that

$$(1.24) \qquad \int_a^t Q(\tau) \,\mathrm{d} \operatorname{var} \alpha_{\varphi(t),t}(\tau) < \infty\,.$$

It holds for each $\tau \in (t - (r/2k)^2, t)$ that

$$|\varphi(t) - \varphi(\tau)| \leqq k \sqrt{(t - \tau)} \leqq \frac{r}{2},$$

that is

$$|\varphi(t) + r - \varphi(\tau)| \geqq \frac{r}{2}$$

for this τ. Thus

$$(1.25) \qquad \frac{r}{4} \int_a^t \frac{1}{(t-\tau)^{3/2}} \exp\left(-\alpha_{x,t}^2(\tau)\right) \mathrm{d}\tau = \frac{r}{4} \int_a^{t-(r/2k)^2} \frac{\mathrm{d}\tau}{(t-\tau)^{3/2}} +$$
$$+ \frac{r}{4} \int_{t-(r/2k)^2}^t \frac{1}{(t-\tau)^{3/2}} \exp\left(-\frac{r^2}{16(t-\tau)}\right) \mathrm{d}\tau = \frac{r}{4} \left[\frac{1}{\sqrt{(t-\tau)}}\right]_a^{t-(r/2k)^2} +$$
$$+ 2 \int_{k/2}^{\infty} \mathrm{e}^{-z^2} \, \mathrm{d}z \leqq \frac{r}{2} \left(\frac{2k}{r} - \frac{1}{\sqrt{(t-a)}}\right) + \sqrt{\pi} \leqq k + \sqrt{\pi}\,.$$

On the right hand side of the estimate (1.25) we have a constant which is independent of the value $r > 0$ ($r < 2k \sqrt{(t - a)}$). Hence the condition (1.20) is fulfilled. Similarly for the condition (1.21). This completes the proof.

Let us now show some complementary assertions concerning the operators $\tilde{T}_+$, $\tilde{T}_-$ which have been established in [1] in connection with the boundary value problem of the heat equation. In [1] we have defined a space of all continuous functions on $\langle a, b\rangle$ vanishing at the point a. This space may be considered a space $\mathscr{C}_Q(\langle a, b\rangle)$ where Q is a function on $\langle a, b\rangle$ for which $Q(a) = 0$ and $Q(t) = 1$ for each $t \in (a, b\rangle$. Provided the condition

$$(1.26) \qquad \sup \{V_K(\varphi(t), t);\; t \in \langle a, b\rangle\} < \infty$$

was fulfilled the operators $\tilde{T}_+$ and $\tilde{T}_-$ have been defined on that space by

$$(1.27) \qquad \tilde{T}_+ f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' > \varphi(t')}} Tf(x', t'),$$

$$(1.28) \qquad \tilde{T}_- f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' < \varphi(t')}} Tf(x', t')$$

($f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$). These operators map the space $\mathscr{C}_0(\langle a, b\rangle)$ into itself.

Now let Q be a nonnegative lower-semicontinuous and bounded function on $\langle a, b\rangle$ and suppose that

$$(1.29) \qquad \sup \{V_K^Q(x, t);\; [x, t] \in K\} < \infty\,.$$

Then the limits (1.27), (1.28) exist for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$ and for each $t \in (a, b\rangle$.

Proposition 1.2. *Suppose that the condition* (1.29) *is fulfilled and let* $Q(a) > 0$. *For each* $f \in \mathscr{C}_Q(\langle a, b\rangle)$ *let us define on the interval* $(a, b\rangle$ *functions* $\tilde{T}_+ f$, $\tilde{T}_- f$ *by* (1.27), (1.28). *Then the functions* $\tilde{T}_+ f$, $\tilde{T}_- f$ *may be continuously extended to the whole interval* $\langle a, b\rangle$ *for each* $f \in \mathscr{C}_Q(\langle a, b\rangle)$ *if and only if the limit (finite or infinite)*

$$\lim_{t \to a+} \frac{\varphi(t) - \varphi(a)}{\sqrt{(t - a)}} \tag{1.30}$$

exists.

Proof. Suppose, for instance, that $\tilde{T}_+ f$ has a continuous extension on the interval $\langle a, b\rangle$ for each $f \in \mathscr{C}_Q(\langle a, b\rangle)$. Since $Q(a) > 0$ (and Q is lower-semicontinuous) there are $\delta > 0$, $f_1 \in \mathscr{C}_Q(\langle a, b\rangle)$ such that $f_1(t) = 1$ for each $t \in \langle a, a + \delta)$. It is easily seen that for each $t \in (a, a + \delta)$

$$\tilde{T}_+ f_1(t) = 2 - \frac{2}{\sqrt{\pi}} G\left(\frac{\varphi(t) - \varphi(a)}{2\sqrt{(t - a)}}\right) \tag{1.31}$$

where G is the function defined above (see [1], proof of Lemma 2.1). The limit

$$\lim_{t \to a+} \tilde{T}_+ f_1(t)$$

exists by the assumption and since G is an increasing function the limit (1.30) exists as well.

Suppose that the limit (1.30) exists. If f_1 denotes the same as in the first part of this proof one may write any function $f \in \mathscr{C}_Q(\langle a, b\rangle)$ in the form $f = f_0 + kf_1$, where $f_0 \in \mathscr{C}_Q(\langle a, b\rangle)$, $f_0(a) = 0$ ($k = f(a)$). Operator T is linear (and so the operators $\tilde{T}_+$, $\tilde{T}_-$ are) and thus it suffices to show that $\tilde{T}_+ f_0$, $\tilde{T}_+ f_1$, $\tilde{T}_- f_0$, $\tilde{T}_- f_1$ may be continuously extended to $\langle a, b\rangle$. But

$$\lim_{t \to a+} \tilde{T}_+ f_0(t) = \lim_{t \to a-} \tilde{T}_- f_0(t) = 0$$

(for $\lim_{[x,t] \to [\varphi(a),a]} Tf_0(x, t) = 0$) and finite limits

$$\lim_{t \to a+} \tilde{T}_+ f_1(t), \lim_{t \to a+} \tilde{T}_- f_1(t)$$

exist according to (1.31) and to the assumption of the existence of the limit (1.30). This completes the proof.

Remark. Provided (1.26) holds the operators $\tilde{T}_+$, $\tilde{T}_-$ have been defined on the space $\mathscr{C}_0(\langle a, b\rangle)$. Conformably to Proposition 1.2 we may define operators $\tilde{T}_+$, $\tilde{T}_-$

on the space $\mathscr{C}_Q(\langle a, b\rangle)$ (provided the condition (1.29) is fulfilled) by (1.27), (1.28) for $t \in (a, b\rangle$. We define

$$\tilde{T}_+ f(a) = \lim_{t \to a+} \tilde{T}_+ f(t), \quad \tilde{T}_- f(a) = \lim_{t \to a-} \tilde{T}_- f(t).$$

Then the operators $\tilde{T}_+$, $\tilde{T}_-$ map $\mathscr{C}_Q(\langle a, b\rangle)$ into $\mathscr{C}(\langle a, b\rangle)$.

2.

In this part we shall show that there is a continuous function φ of bounded variation on an interval $\langle a, b\rangle$ such that

$$V_K(\varphi(t), t) = \infty$$

for almost all $t \in (a, b\rangle$ ($K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}$).

Let $a, b \in R^1$, $a < b$. The supremum norm on $\mathscr{C}(\langle a, b\rangle)$ is denoted by $\|\ldots\|$ or $\|\ldots\|_{\mathscr{C}}$. Let us define a space $\mathscr{B} = \mathscr{B}(\langle a, b\rangle)$. Put

$$\mathscr{B} = \{f \in C(\langle a, b\rangle);\ \operatorname{var}[f; \langle a, b\rangle] < \infty\}$$

and endow the space $\mathscr{B}$ with the norm $\|\ldots\|_{\mathscr{B}}$ defining

$$\|f\|_{\mathscr{B}} = \|f\|_{\mathscr{C}} + \operatorname{var}[f; \langle a, b\rangle], \quad (f \in \mathscr{B}).$$

It is well known that the space $\mathscr{B}$ with the norm $\|\ldots\|_{\mathscr{B}}$ is a Banach space.

For $f \in \mathscr{B}$ we define on $\langle a, b\rangle$ a function W^f by

$$W^f(t) = \begin{cases} 0 & \text{for } t = a \\ \displaystyle\int_a^t \frac{1}{\sqrt{(t - \tau)}} \, \mathrm{d} \operatorname{var} f(\tau) & \text{for } t \in (a, b\rangle . \end{cases}$$

For a positive integer k such that $1/k < b - a$ we set

$$M_k = \left\{f \in \mathscr{B};\ \text{there is a } t \in \left\langle a + \frac{1}{k}, b\right\rangle \text{ with } W^f(t) \leqq k\right\}.$$

Proposition 2.1. *The sets M_k are closed in $\mathscr{B}$.*

Proof. For $\varepsilon > 0$, $\varepsilon < b - a$, $f \in \mathscr{B}$ we put

$$W_\varepsilon^f(t) = \begin{cases} 0 & \text{for } t \in \langle a, a + \varepsilon\rangle \\ \displaystyle\int_a^{t-\varepsilon} \frac{1}{\sqrt{(t - \tau)}} \, \mathrm{d} \operatorname{var} f(\tau) & \text{for } t \in (a + \varepsilon, b\rangle . \end{cases}$$

It is easily verified that W_ε^f is a continuous function on $\langle a, b\rangle$ (since var $[f; \langle a, b\rangle] < \infty$) and it holds

$$W_\varepsilon^f \nearrow W^f \quad \text{as} \quad \varepsilon \searrow 0 .$$

Hence it immediately follows that W^f is a lower-semicontinuous function on $\langle a, b\rangle$.

Let $f_n \in M_k$ (where k is a fixed number, $n = 1, 2, \ldots$) and let $\|f_n - f\|_{\mathscr{B}} \to 0$. Then particularly

$$\lim_{n\to\infty} \operatorname{var}[f_n - f; \langle a, b\rangle] = 0$$

and thus

$$W_\varepsilon^{f_n} \to W_\varepsilon^f \quad \text{as} \quad n \to \infty$$

for any $\varepsilon > 0$, $\varepsilon < b - a$ and this convergence is uniform on the interval $\langle a, b\rangle$ (since the functions $1/\sqrt{(t - \tau)}$ are uniformly bounded on the intervals $\langle a, t - \varepsilon\rangle$ with respect to $t \in (a + \varepsilon, b\rangle$ and $W_\varepsilon^{f_n}(t) = 0$ for $t \in \langle a, \varepsilon\rangle$). According to the definition of the set M_k there are points $t_n \in \langle a + 1/k, b\rangle$ such that

$$W^{f_n}(t_n) \leqq k .$$

Let us suppose that the sequence $\{t_n\}$ converges to a point $t \in \langle a + (1/k), b\rangle$. We assert that $f \in M_k$. To this end it suffices to show that $W^f(t) \leqq k$.

Suppose that

$$k < W^f(t) = k + c .$$

Then there are $\varepsilon, \delta > 0$ such that

$$W_\varepsilon^f(t') > k + \frac{c}{2}$$

for each $t' \in \langle t - \delta, t + \delta\rangle \cap \langle a, b\rangle$ (we assume $c < \infty$; in the case $c = \infty$ one would proceed by analogy).

There is n_0 such that

$$|W_\varepsilon^f(t') - W_\varepsilon^{f_n}(t')| \leqq \frac{c}{4}$$

for each $n > n_0$ and each $t' \in \langle a, b\rangle$. But then

$$k \geqq W_\varepsilon^{f_n}(t_n) = W_\varepsilon^f(t_n) - \frac{c}{4} > k + \frac{c}{4}.$$

This is a contradiction which completes the proof.

Proposition 2.2. *There is a function $\varphi \in \mathscr{B}$ such that*

(2.1) $$V_K(\varphi(t), t) = \infty$$

(*where* $K = \{[x, t];\ t \in \langle a, b \rangle,\ x = \varphi(t)\}$) *for almost all* $t \in (a, b\rangle$. *The function* φ *may be even chosen to be absolutely continuous.*

Proof. Let $\mathscr{A}$ denote the closure in $\mathscr{B}$ of the family of all Lipschitz functions on $\langle a, b \rangle$ (it is clear that $\mathscr{A}$ is the set of all absolutely continuous functions on $\langle a, b \rangle$). $\mathscr{A}$ endowed with the norm restricted from $\mathscr{B}$ is a Banach space.

Let us prove that the set

$$A = \{f \in \mathscr{A};\ t \in (a, b\rangle \Rightarrow W^f(t) = \infty\}$$

is of the second category in $\mathscr{A}$. From this the assertion will follow.

Since

$$A = \mathscr{A} - \bigcup_{k > 1/(b-a)} M_k\,,$$

it suffices to show that the sets $M_k \cap \mathscr{A}$ are nowhere dense in $\mathscr{A}$. Those sets are closed and thus it suffices to prove that no set $M_k \cap \mathscr{A}$ contains any interior point (with respect to $\mathscr{A}$). We assume for the simplicity that $\langle a, b \rangle = \langle 0, 1 \rangle$. Let us define functions $f_n \in \mathscr{A}$ in the following way.

For a positive integer n we put $b_n = 1/(2n^6)$ and

$$\varphi_n(t) = \begin{cases} 0 & \text{for } t \in \left\langle 0, \dfrac{1}{n} - b_n \right) \\ \dfrac{1}{n^2 b_n}\ (= 2n^4) & \text{for } t \in \left\langle \dfrac{1}{n} - b_n, \dfrac{1}{n} \right). \end{cases}$$

We extend the function φ_n periodically with the period $1/n$ on the whole interval $\langle 0, 1 \rangle$. Further, we put

$$f_n(t) = \int_0^t \varphi_n(\tau)\,\mathrm{d}\tau \quad (t \in \langle a, b \rangle)\,.$$

With respect to the fact that the function f_n is nondecreasing (for φ_n is nonnegative) and $f_n(0) = 0$ we have

$$\|f_n\|_{\mathscr{B}} = 2 f_n(1) = 2n \frac{1}{n^2 b_n} b_n = \frac{2}{n}\,.$$

Let $t_0 \in (0, 1/n\rangle$. Then

$$W^{f_n}\left(\frac{1}{n} + t_0\right) = \int_0^{1/n + t_0} \frac{\varphi_n(\tau)}{\sqrt{\left(\frac{1}{n} + t_0 - \tau\right)}}\,\mathrm{d}\tau = \frac{1}{n^2 b_n} \int_{1/n - b_n}^{1/n} \frac{\mathrm{d}\tau}{\sqrt{\left(\frac{1}{n} + t_0 - \tau\right)}} =$$

$$= \frac{2}{n^2 b_n}\left[-\sqrt{\left(\frac{1}{n} + t_0 - \tau\right)}\right]_{1/n - b_n}^{1/n} = \frac{2}{n^2 b_n}\left(\sqrt{(t_0 + b_n)} - \sqrt{t_0}\right) =$$

$$= \frac{2}{n^2 b_n} \frac{b_n}{\sqrt{(b_n + t_0)} + \sqrt{t_0}} \geqq \frac{1}{n^2 \sqrt{(2b_n)}} = n .$$

In virtue of the fact that the function φ_n is $1/n$ – periodic one sees that

$$W^{f_n}(t) \geqq n$$

for any $t \in (1/n, 1\rangle$.

Suppose now that for a positive integer k (with $1/k < b - a$) the set $M_k \cap \mathscr{A}$ has an interior point (in $\mathscr{A}$). Then there are $f_0 \in M_k \cap \mathscr{A}$, $\varepsilon > 0$ such that

$$(2.2) \qquad (f \in \mathscr{A},\ \|f_0 - f\|_{\mathscr{B}} < \varepsilon) \Rightarrow f \in M_k .$$

Since the set of all Lipschitz functions on $\langle 0, 1\rangle$ is dense in $\mathscr{A}$ (by the definition of the set $\mathscr{A}$) one may suppose that the function f_0 is a Lipschitz function. Then there is a positive integer k_0 such that

$$W^{f_0}(t) \leqq k_0$$

for each $t \in \langle a, b\rangle$ (see Lemma 1.3). Choose n to be a positive integer such that

$$n > 2 \max\{k, k_0\}, \quad \|f_n\|_{\mathscr{B}} = \frac{2}{n} < \varepsilon .$$

Then for each $t \in \langle 1/k, 1\rangle$,

$$W^{(f_0 + f_n)}(t) = \int_0^t \frac{1}{\sqrt{(t - \tau)}} \,\mathrm{d}\operatorname{var}(f_0 + f_n)(\tau) \geqq$$

$$\geqq \int_0^t \frac{1}{\sqrt{(t - \tau)}} \,\mathrm{d}\operatorname{var} f_n(\tau) - \int_0^t \frac{1}{\sqrt{(t - \tau)}} \,\mathrm{d}\operatorname{var} f_0(\tau) = W^{f_n}(t) - W^{f_0}(t) \geqq$$

$$\geqq n - k_0 > k .$$

It follows from this that $f_0 + f_n \notin M_k$ which contradicts (2.2) (where we put $f = f_0 + f_n$). Thus, in fact, the sets $M_k \cap \mathscr{A}$ are nowhere dense in $\mathscr{A}$.

We conclude that there is a function $\varphi \in \mathscr{A}$ such that $W^{\varphi}(t) = \infty$ for each $t \in (a, b\rangle$. But φ has a finite derivative at almost all points $t \in (a, b)$ and at every such point t it holds

$$W^{\varphi}(t) = \infty \Leftrightarrow V_K(\varphi(t), t) = \infty$$

(where $K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}$) according to Lemma 1.3.

The proof is complete.

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