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NOTE ON WEAK EPIMORPHISMS OF 3-NETS WITHOUT SINGULAR POINTS

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In the present Note we introduce weak epimorphisms of nets (of degree 3, without singular points). We shall consider conditions of regularity and of parallelity-preserving which guarantee that a weak epimorphism of nets is, up to parastrophies, an usual epimorphism. For nets of order at least 3, every weak epimorphism preserves parallelity. For arbitrary nets every weak isomorphism preserves parallelity. So weak epimorphisms are essentially the same as usual epimorphisms. If the image of every line contains at least five points then a surjective join-preserving map of nets must be a weak epimorphism. As a special case we obtain the known fact ([1], Lemma 5.3, pp. 73–74) that every "collineation" of a net of order at least 5 is necessarily "proper". Although the results of this Note are quite elementary we believe that they can be useful for further detailed study of fundamental properties of nets. Finally, let us mention that V. D. Belousov's homotopy of nets (conciding with "weak homomorphism" of nets in our sense) is introduced and studied in [1], pp. 18–19, with some inaccuracy. It was this circumstances which stimulated the origin of this Note.

The *net* (of degree 3, without singular points) is defined here as a triple $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ where $\mathscr{P}$ is a set having at least two elements, $\mathscr{L}$ is a set of some subsets of $\mathscr{L}$ and $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ are mutually disjoint subsets of $\mathscr{L}$ the union of which is $\mathscr{L}$, satisfying the following conditions:

(i) $\forall P \in \mathscr{P},\ i = \{1, 2, 3\}\quad \exists!\, l \in \mathscr{L}_i \quad P \in l,$

(ii) $\forall i, j \in \{1, 2. 3\};\ i \neq j \quad \forall a \in \mathscr{L}_i,\ b \in \mathscr{L}_j \quad \#(a \cap b) = 1,$

(iii) $\forall i \in \{1, 2, 3\} \quad \forall a, b \in \mathscr{L}_i;\ a \neq b \quad a \cap b = \emptyset.$

Elements of $\mathscr{P}$ are called *points*, elements of $\mathscr{L}$ are called *lines*, the sets $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ are called *first*, *second* and *third pencil*. Lines a, b of the same pencil are called *parallel* (notation: $a \parallel b$), lines a, b from distinct pencils are called *non-parallel* (notation: $a \nparallel b$). Points P, Q are termed *joinable* if they lie on the same line; if moreover $P \neq Q$ then this line is called *the join of* P, Q and is denoted by PQ.

The common point of non-parallel lines a, b is called the *intersection point* and is denoted by $a \sqcap b$. A set of points is said to be *collinear* if all its points lie on the same line. For any two lines l_1, l_2 one verifies easily that $\# l_1 = \# l_2$; the common cardinality of lines of the net is called *order* of the net. If $\mathcal{N}$, $\mathcal{N}'$ are nets we shall put standardly $\mathcal{N} =: (\mathcal{P}, \mathcal{L}, (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3))$, $\mathcal{N}' =: (\mathcal{P}', \mathcal{L}', (\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3))$.

Now let $\mathcal{N}$, $\mathcal{N}'$ be nets and $\pi : \mathcal{P} \to \mathcal{P}'$ a surjective map. Then we say that π is

1) *join-preserving* if for any joinable points P, Q also P^π, Q^π are joinable,
2) *collinearity-preserving* (or a *weak epimorphism* of $\mathcal{N}$ onto $\mathcal{N}'$) if to every $l \in \mathcal{L}$ there is an $l^\wedge \in \mathcal{L}'$ such that $\{x^\pi \mid X \in l\} \subseteq l^\wedge$,
3) a *weak isomorphism* of $\mathcal{N}$ onto $\mathcal{N}'$ if it is a bijective weak epimorphism,
4) an *epimorphism* of $\mathcal{N}$ onto $\mathcal{N}'$ if for every $i \in \{1, 2, 3\}$ and every $l \in \mathcal{L}_i$ there is an $l^\wedge \in \mathcal{L}'_i$ such that $\{X^\pi \mid X \in l\} \in l^\wedge$.
5) an *isomorphism* of $\mathcal{N}$ onto $\mathcal{N}'$ if it is a bijective epimorphism,
6) *line-preserving* if $\{X^\pi \mid X \in l\} \in \mathcal{L}'$ whenever $l \in \mathcal{L}$,
7) *regular* if $\#\{X^\pi \mid X \in l\} \geqq 2$ whenever $l \in \mathcal{L}$.

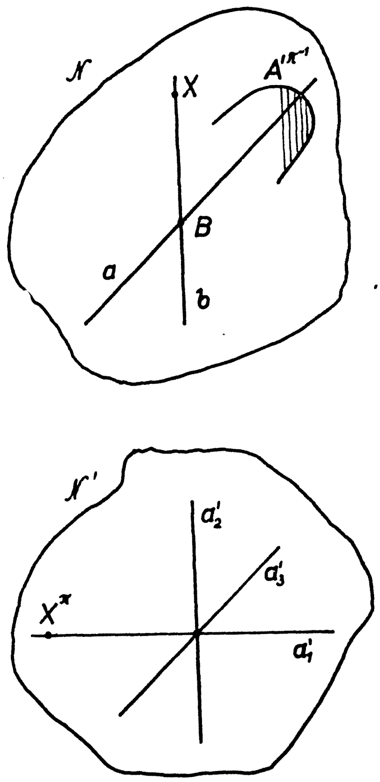


Fig. 1
(to the proof of Proposition 1).

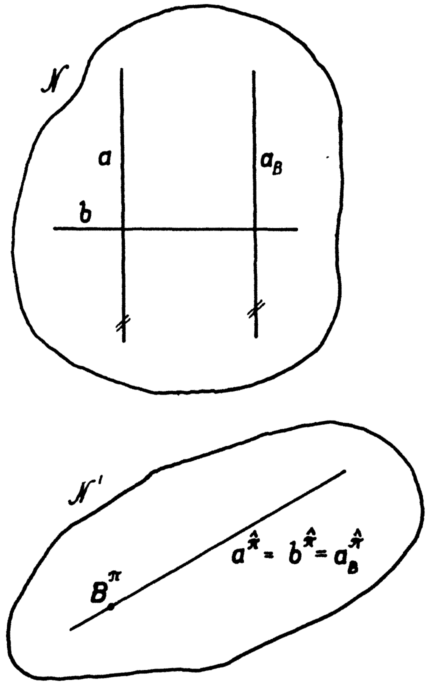


Fig. 2
(to the first part of the proof of Proposition 2).

Let π be a regular weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Then for every $l \in \mathcal{L}$ there is exactly one $l^{\wedge} \in \mathcal{L}'$ such that $\{X^{\pi} \mid X \in l\} \subseteq l^{\wedge}$. Thus $l \mapsto l^{\wedge}$ is a map of $\mathcal{L}$ onto $\mathcal{L}'$. This map will be denoted by $\hat{\pi}$. If π is a regular weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$ then π is said to be *parallelity-preserving* or *non-parallelity-preserving*, if $l_1 \parallel l_2 \Rightarrow l_1^{\hat{\pi}} \parallel l_2^{\hat{\pi}}$ or $l_1 \nparallel l_2 \Rightarrow l_1^{\hat{\pi}} \nparallel l_2^{\hat{\pi}}$ respectively.

Proposition 1. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets of order at least 3 and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Then π is regular.*

Proof. Let there exist a line $a \in \mathcal{L}$ such that $\{X^{\pi} \mid X \in a\} = \{A'\}$ for some $A' \in \mathcal{P}'$. For every $X \in \mathcal{P} \setminus A'^{\pi^{-1}}$ take a line $b \in \mathcal{L}$ such that $X \in b \nparallel a$. Further let $B := a \sqcap b$. As π is collinearity-preserving and $X, B \in b$ so $X^{\pi}, B^{\pi}(=A')$ must also lie on the same line and this line is one of the three lines a_1', a_2', a_3' through A'. Consequently $\{X^{\pi} \mid X \in \mathcal{P}\} \subseteq a_1' \cup a_2' \cup a_3'$, contrary to the hypothesis that π is surjective and $\mathcal{N}'$ has order greater than 2. ∎

Proposition 2. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a regular weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Then π preserves parallelity if and only if it preserves non parallelity.*

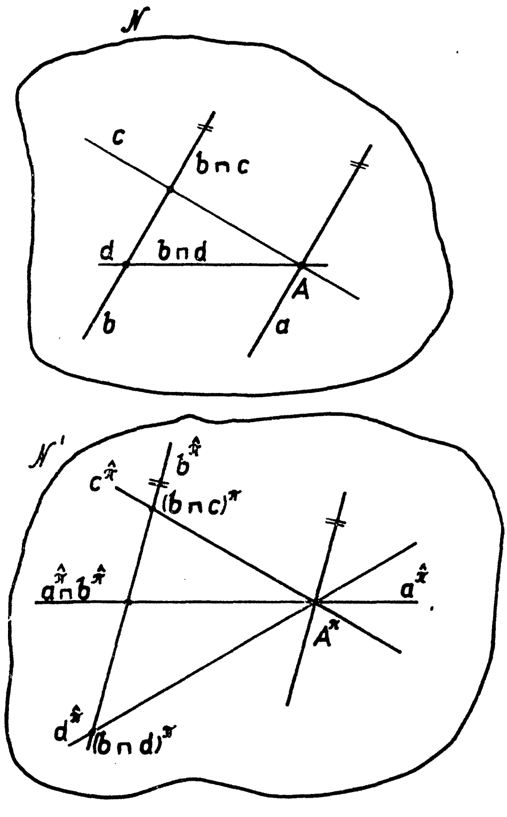
Proof. First let π preserve parallelity. We shall proceed indirectly supposing the existence of non-parallel lines $a, b \in \mathcal{L}$ such that $a^{\hat{\pi}} \parallel b^{\hat{\pi}}$. It follows $a^{\hat{\pi}} = b^{\hat{\pi}}$. We take an arbitrary point $B \in b$ and consider the line a_B such that $B \in a_B \parallel a$. Then $B^{\pi} \in a_B^{\pi}$ so that $a_B^{\hat{\pi}} = a^{\hat{\pi}}$. From this we get $\{X^{\pi} \mid X \in \mathcal{P}\} \subseteq a^{\hat{\pi}}$ which contradicts the surjectivity of π. Secondly let π preserve non-parallelity. For indirect proof suppose the existence of parallel lines $a, b \in \mathcal{L}$ such that $a^{\hat{\pi}} \nparallel b^{\hat{\pi}}$. Then there is a point $A \in a$ such that $A^{\pi} \in a^{\hat{\pi}} \setminus \{a^{\hat{\pi}} \sqcap b^{\hat{\pi}}\}$. Let c, d be the remaining lines through A. Then $(b \sqcap c)^{\pi}, (b \sqcap d)^{\pi} \in b^{\hat{\pi}} \setminus \{(a \sqcap b)^{\pi}\}$ are distinct points, each of them being joinable with A^{π}. However for both lines $A^{\pi}(b \sqcap c)^{\pi}$, $A^{\pi}(b \sqcap d)^{\pi}$ only one possibility remains, namely the line through A^{π} distinct to $a^{\hat{\pi}}$ and non-parallel to $b^{\hat{\pi}}$, a contradiction. ∎

Proposition 3. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a regular weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}''$ which preserves parallelity. Then there is a permutation σ of the set $\{1, 2, 3\}$ such that π is an epimorphism of $\mathcal{N}$ onto $(\mathcal{P}', \mathcal{L}', (\mathcal{L}'_{1\sigma}, \mathcal{L}'_{2\sigma}, \mathcal{L}'_{3\sigma}))$*

Proof. An immediate corollary of Proposition 2. ∎

Proposition 4. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets of order at least 3 and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$. Then π preserves non-parallelity.*

Proof. By Proposition 1, π is regular and we can work with the map $\hat{\pi}$. Suppose, on the contrary, that there exist non-parallel lines $a, c \in \mathcal{L}$ such that $a^{\hat{\pi}} \parallel c^{\hat{\pi}}$. Consequently, it must be $a^{\hat{\pi}} = c^{\hat{\pi}}$.



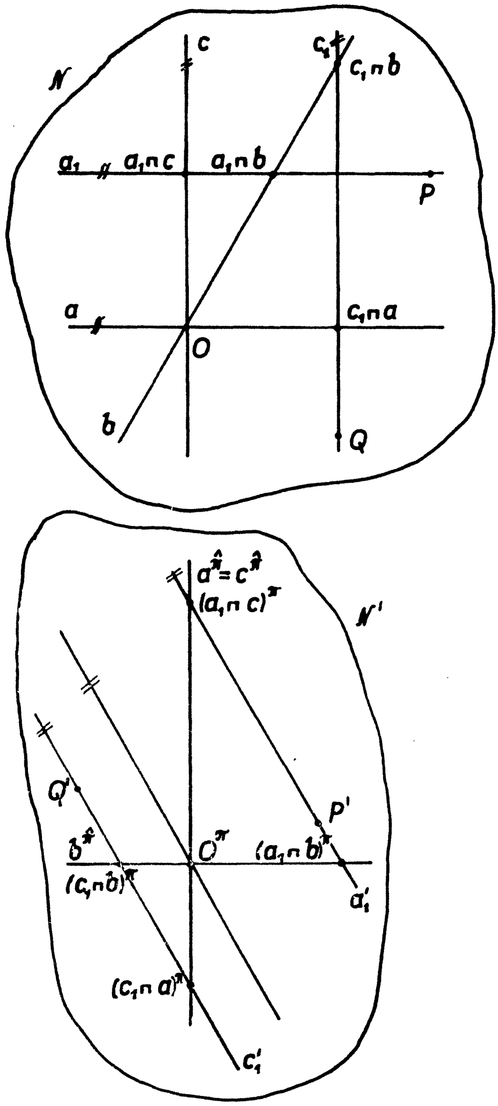


Fig. 3
(to the second part of the proof of Proposition 2).

Fig. 4
(to the first part of the proof of Proposition 4).

Let b be the remaining line through $a \sqcap c$ $(=: O)$. Now we shall distinguish two cases:

First let $b^{\hat{\pi}} \neq a^{\hat{\pi}}$. Taking a point $P' \in \mathscr{P}$ outside the lines through O^{π} and choosing one of its pre-images $P \in P'^{\pi^{-1}}$ we can deduce the following: The image $a_1^{\hat{\pi}}$ of the line a_1 such that $P \in a_1 \parallel a$ has to be equal to the line a_1' such that $P' \in a_1' \nparallel a^{\hat{\pi}}, b^{\hat{\pi}}$ (which follows by $a_1 \nparallel b, c$ from the fact that the points $(a_1 \sqcap b)^{\pi}$, $(a_1 \sqcap c)^{\pi}$ must lie on $a_1^{\hat{\pi}}$ so that the lines through P' parallel to $a^{\hat{\pi}}, b^{\hat{\pi}}$ cannot be equal to $a_1^{\hat{\pi}}$ and a_1'

remains the only possibility for $a_1^{\hat{\pi}}$ *). Now repeat the same reasoning for a point $Q' \in \mathscr{P}'$ lying outside the lines through O^{π} and outside a'_1, (such a point must exist if the order of $\mathscr{N}$ is greater than 2): Let $Q \in Q'^{\pi^{-1}}$ and $c_1 \in \mathscr{L}$ be such that $Q \in c_1 \parallel c$. Further denote by $c'_1 \in \mathscr{L}'$ the line through Q' non-parallel to $a^{\hat{\pi}}, b^{\hat{\pi}}$. Then $c_1^{\hat{\pi}} = c'_1$. But $a_1 \nparallel c_1$ and the point $a_1 \sqcap c_1$ cannot have its image under π because this image would lie simultaneously on a'_1 and on c'_1 which is impossible as a'_1, c'_1 are distinct and parallel.

In the remaining case let $a^{\hat{\pi}} = b^{\hat{\pi}} = c^{\hat{\pi}}(=: d')$. We shall start from a point $P' \in \mathscr{P}' \setminus a^{\hat{\pi}}$. We choose a point $P \in \mathscr{P}'^{\pi^{-1}}$ (which is not contained in $a \cup b \cup c$). Let $a_2 \parallel a$, $b_2 \parallel b$, $c_2 \parallel c$ be lines through P. Then the lines $a_2^{\hat{\pi}}, b_2^{\hat{\pi}}, c_2^{\hat{\pi}}$ cannot be mutually distinct (because each of them must intersect d' as a_2 intersects b, b_2 intersects c and c_2 intersects a), nor can just two of them be equal (because this contradicts the first part of the proof). So $a_2^{\hat{\pi}} = b_2^{\hat{\pi}} = c_2^{\hat{\pi}}(=: e') \nparallel d'$. We repeat the same argument for a point $Q' \in \mathscr{P}'$ outside $d' \cup e'$. After choosing a point $Q \in Q'^{\pi^{-1}}$ (which must lie outside $a \cup b \cup c \cup a_2 \cup b_2 \cup c_2$) we consider the lines $a_3 \parallel a$, $b_3 \parallel b$, $c_3 \parallel c$ through Q and obtain $a_3^{\hat{\pi}} = b_3^{\hat{\pi}} = c_3^{\hat{\pi}}(=: f') \nparallel d', e'$. Finally we repeat the same argument for a point $R' \in \mathscr{P}'$ outside $d' \cup e' \cup f'$ (which is possible as $\mathscr{N}$ has order greater than 2). We choose a point $R \in R'^{\pi^{-1}}$ (which lies outside $a \cup b \cup \cup c \cup a_2 \cup b_2 \cup c_2 \cup a_3 \cup b_3 \cup c_3$) and consider the line $a_4 \parallel a$, $b_4 \parallel b$, $c_4 \parallel c$ through R. Now it results $a_4^{\hat{\pi}} = b_4^{\hat{\pi}} = c_4^{\hat{\pi}} \nparallel d', e'\ f'$, a contradiction. ■

Proposition 5. *Let $\mathscr{N}, \mathscr{N}'$ be nets and let $\pi : \mathscr{P} \to \mathscr{P}'$ be a regular weak epimorphism of $\mathscr{N}$ onto $\mathscr{N}'$ which preserves parallelity. Then π is line-preserving.*

Proof. As π is regular we can deal with the mapping $\hat{\pi}$. Suppose, on the contrary, that there exist a point $A' \in \mathscr{P}'$ and a line $a \in \mathscr{L}$ such that $A' \in a^{\hat{\pi}}$ but $A'^{\pi^{-1}} \cap a = \emptyset$. Take an arbitrary point $A \in A'^{\pi^{-1}}$ and denote by $a_1 \in \mathscr{L}_1$, $a_2 \in \mathscr{L}_2$, $a_3 \in \mathscr{L}_3$ the lines through A. Without loss of generality let $a \parallel a_1$. Then $a^{\hat{\pi}} \parallel a^{\hat{\pi}}$. Putting $A_2 := := a_2 \sqcap a$, $A_3 := a_3 \sqcap a$ we get $a^{\hat{\pi}} = a_3^{\hat{\pi}}$ (since $A^{\pi} = A' \neq A_2^{\pi} \in a_1^{\hat{\pi}}$) and $a^{\hat{\pi}} = a_3^{\hat{\pi}}$ (since $A^{\pi} = A' \neq A_3^{\pi} \in a^{\hat{\pi}}$), a contradiction to parallelity-preserving. ■

Proposition 6. *Let $\mathscr{N}, \mathscr{N}'$ be nets and let $\pi : \mathscr{P} \to \mathscr{P}'$ be a weak isomorphism of $\mathscr{N}$ onto $\mathscr{N}'$. Then π preserves parallelity.*

Proof. With regard to Propositions 1, 2, 4 we could restrict ourselves to nets $\mathscr{N}'$ with order 2 but we shall give a proof which is independent on order of $\mathscr{N}'$. The mapping π under consideration is necessarily regular (as π is bijective) so that we can deal with the mapping $\hat{\pi}$. Suppose, contrary to the conclusion of Proposition 6, that there are parallel lines $a, b \in \mathscr{L}$ such that $a^{\hat{\pi}} \nparallel b^{\hat{\pi}}$ and put $C' := a^{\hat{\pi}} \sqcap b^{\hat{\pi}}$.

We shall distinguish two cases: First let $C \notin a \cup b$. Let $c \in \mathscr{L}$ be a line through C non-parallel to a. Putting $C_a := a \sqcap c$, $C_b := b \sqcap c$ we see that C, C_a, C_b are $X \in x \parallel a$. Further set $X_c := c \sqcap x$, $X_d := d \sqcap x$. Then $X_c \neq X_d$ so that also

*) This important step was found by J. Klouda. The author thaks him for his kindly communicating it and also for further helpful comments to this article.

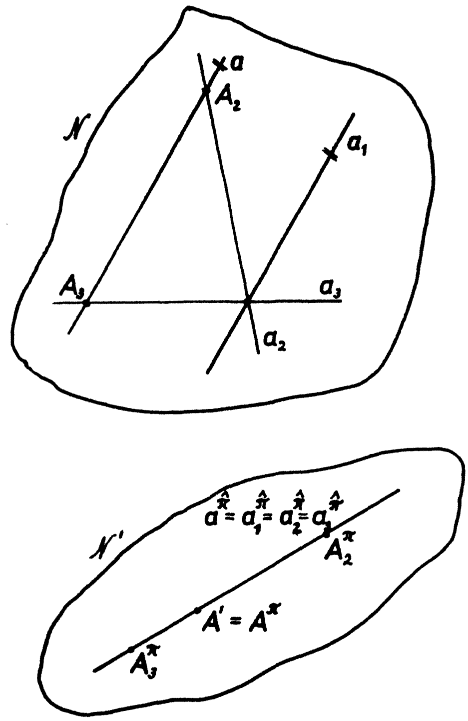


Fig. 6
(to the proof of Proposition 5).

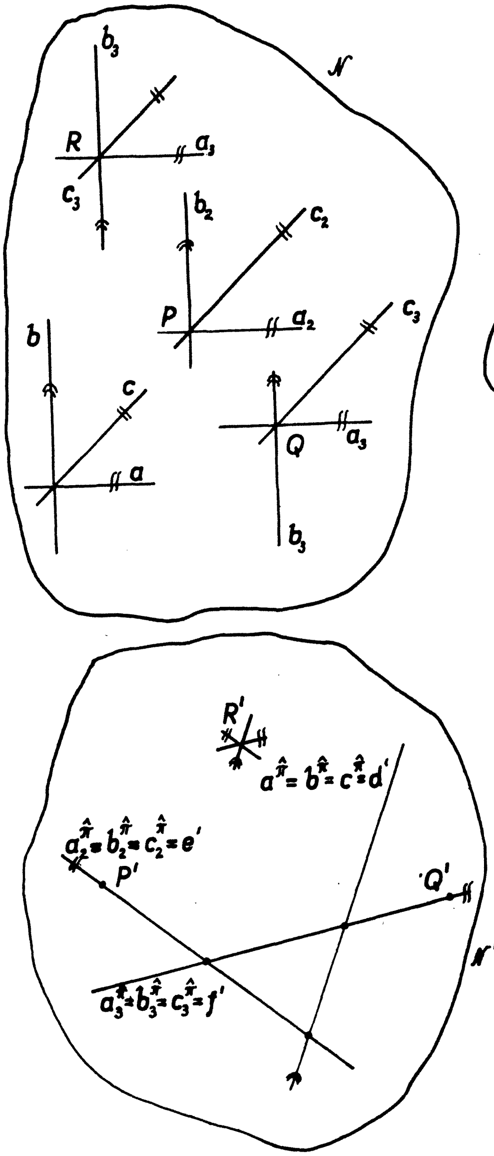


Fig. 5 (to the second part of the proof Proposition 4).

mutually different so that also C^π, C_a^π, C_b^π are mutually different. But $\{C, C_a, C_b\}$ is a collinear set whereas $\{C^\pi, C_a^\pi, C_b^\pi\}$ is not, a contradiction.

Secondly let $C \in a \cup b$. Without loss of generality let $C \in a$. Denote by $c, d \in \mathcal{L}$ the two remaining lines through C and put $C_b := b \sqcap c$, $C_d := b \sqcap d$. Then C, C_b, C_d are pairwise different so that also C^π, C_b^π, C_d^π are pairwise different points on the line $b^{\hat\pi}$. Thus $b^{\hat\pi} = c^{\hat\pi} = d^{\hat\pi}$. For every $X \in \mathcal{P} \setminus a$ let $x \in \mathcal{L}$ be such that

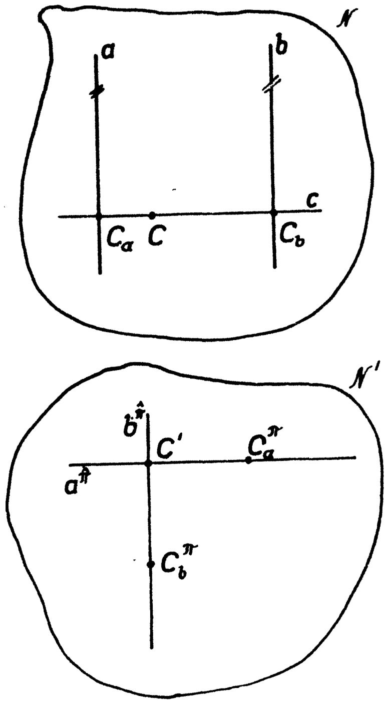


Fig. 7

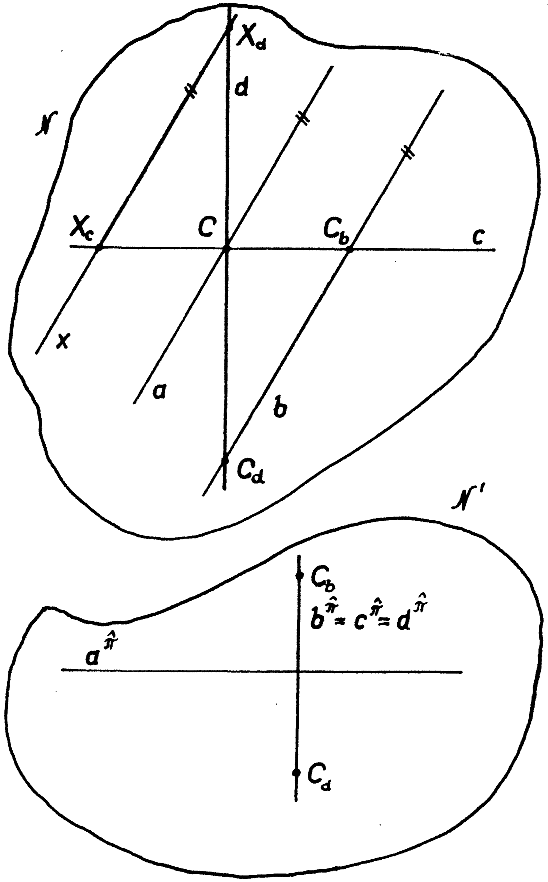


Fig. 8

(to the proof of Proposition 6)

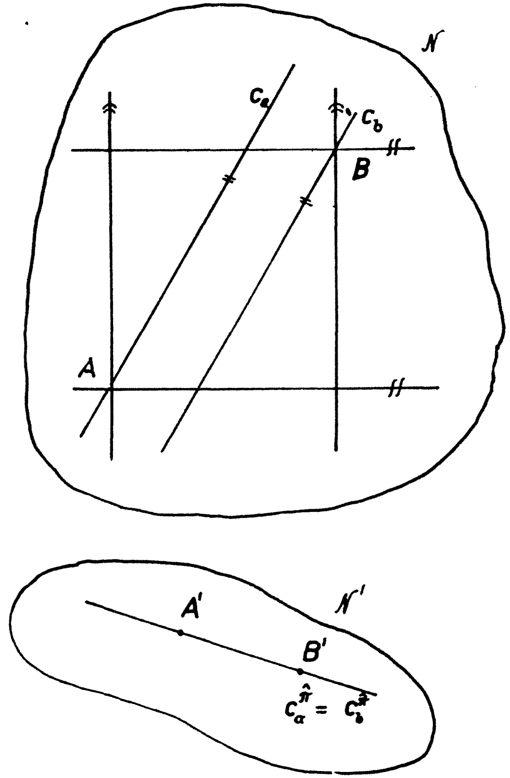


Fig. 9

(to the proof of Proposition 7).

$X_c^\pi \neq X_d^\pi$, where $X_c^\pi \in c^{\hat\pi} = b^{\hat\pi}$, $X_d^\pi \in d^{\hat\pi} = b^{\hat\pi}$. Consequently $\{X^\pi \mid X \in \mathscr{P}\} \subseteq a^{\hat\pi} \cup b^{\hat\pi}$, a contradiction to surjectivity of π. ∎

Proposition 7. *Let $\mathscr{N}$, $\mathscr{N}'$ be nets and let $\pi : \mathscr{P} \to \mathscr{P}'$ be a weak isomorphism of $\mathscr{N}$ onto $\mathscr{N}'$. Then $\pi^{-1} : \mathscr{P}' \to \mathscr{P}$ is a weak isomorphism of $\mathscr{N}'$ onto $\mathscr{N}$.*

Proof. Let there exist joinable points $A', B' \in \mathscr{P}'$ such that $A := A'^{\pi^{-1}}$, $B := B'^{\pi^{-1}}$ are not joinable. Then we see that there are parallel lines $c_a, c_b \in \mathscr{L}$ such that $A \in c_a$, $B \in c_b$, $c_a^{\hat\pi} = c_b^{\pi} = A'B'$, a contradiction to injectivity of π. So π^{-1} is join-preserving. Now let there exist a collinear set $\{A', B', C'\} \leqq \mathscr{P}'$ such that for $A := A'^{\pi^{-1}}$, $B := B'^{\pi^{-1}}$, $C := C'^{\pi^{-1}}$ the set $\{A, B, C\}$ is not collinear. Then, by the preceding A, B; A, C; B, C are joinable and the lines AB, AC, BC must belong to distinct pencils although they have the same image under $\hat\pi$. This yields a contradiction to non-parallelity-preserving. Consequently for each collinear set $Q' \subset \mathscr{P}'$ the set $\{X^{\pi^{-1}} \mid X \in Q'\}$ is collinear, too. ∎

Theorem 1. *Let $\mathscr{N}$, $\mathscr{N}'$ be nets and $\pi : \mathscr{P} \to \mathscr{P}'$ a weak epimorphism of $\mathscr{N}$ onto $\mathscr{N}'$. If $\mathscr{N}'$ is of order greater than 2 or if π is a weak isomorphism of $\mathscr{N}$ onto $\mathscr{N}'$ then there is a permutation σ of $\{1, 2, 3\}$ such that π is an epimorphism or an isomorphism of $\mathscr{N}$ onto $(\mathscr{P}', \mathscr{L}', (\mathscr{L}'_{1\sigma}, \mathscr{L}'_{2\sigma}, \mathscr{L}'_{3\sigma}))$. In both cases π is line-preserving, in the latter case π^{-1} is an isomorphism of $(\mathscr{P}', \mathscr{L}', (\mathscr{L}'_{1\sigma}, \mathscr{L}'_{2\sigma}, \mathscr{L}'_{3\sigma}))$ onto $\mathscr{N}$.*

Proof. A corollary of Propositions 1–7. ∎

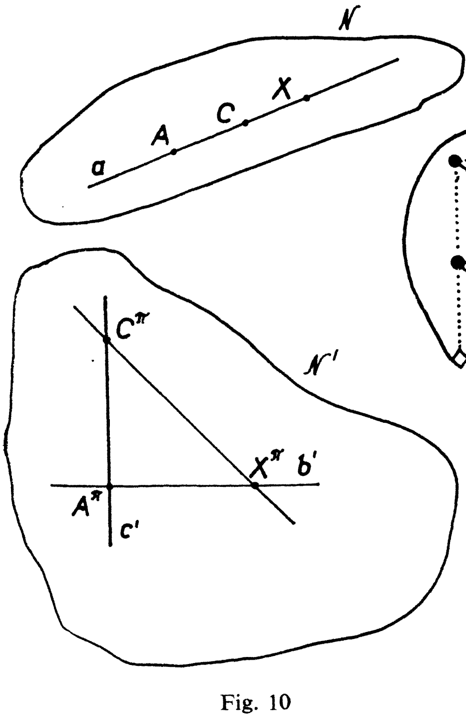


Fig. 10
(to the proof of Proposition 8).

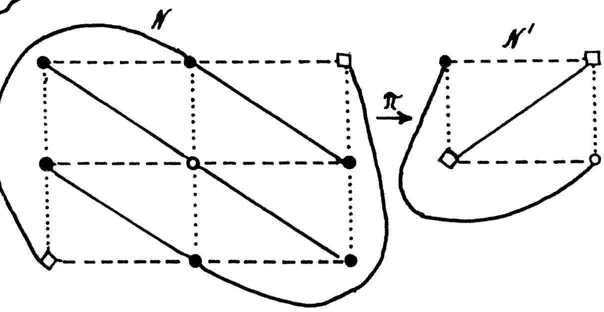


Fig. 11
(to the final Remark).

Proposition 8. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a surjective join-preserving mapping satisfying the condition $\#\{X^\pi \mid X \in l\} \geqq 5$ for all $l \in \mathcal{L}$. Then π is a weak epimorphism of $\mathcal{N}$ onto $\mathcal{N}'$.*

Proof. Let there exist, on the contrary, a line $a \in \mathcal{L}$ such that $\{X^\pi \mid X \in a\}$ is not collinear. We shall start with an arbitrary point $A \in a$. Then for all $X \in a \setminus (A^\pi)^{\pi^{-1}}$ the images X^π lie on the lines through A^π and on every such line it lies at most one point of $\{X^\pi \mid X \in a\} \setminus \{A^\pi\}$. Thus let $b' \in \mathcal{L}'$ be a line through A^π. Then there exists a point $C \in a$ such that $C^\pi \notin b'$. Let $c' := A^\pi C^\pi$. Here $c' \neq b'$ and for every $X \in a \setminus (A^\pi)^{\pi^{-1}}$ the image X^π is joinable with C^π. The line $X^\pi C^\pi$ is different from c' and non parallel to b' and thus it is only one possibility for it. Consequently they are at most $3.1 + 1 = 4$ possibilities for points X^π where $X \in a$, contrary to the hypothesis $\#\{X^\pi \mid X \in a\} \geqq 5$. ∎

Proposition 9. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets or orders at least 5 and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a bijective join-preserving mapping. Then π is a weak isomorphism of $\mathcal{N}$ onto $\mathcal{N}'$.*

Proof. Follows immediately from Proposition 8. ∎

Proposition 10. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets and let $\pi : \mathcal{P} \to \mathcal{P}'$ be a bijective join-preserving mapping. Then $\pi^{-1} : \mathcal{P}' \to \mathcal{P}$ is also join-preserving.*

Proof. If order of $\mathcal{N}'$ is at least 5, then the conclusion follows from Propositions 9 and 7. Thus it would suffice to restrict the proof to nets $\mathcal{N}'$ of order less than 5, but we give a proof for all nets of finite order, say n. In every net of order n there are just $n^2(\frac{3}{2}) . (n - 1)$ couples (we mean not ordered couples) of joinable points. As the mapping $\{X, Y\} \mapsto \{X^\pi, Y^\pi\rangle$ of the set $\{\{X, Y\} \mid X, Y \in \mathcal{P}; X, Y$ distinct and joinable$\}$ into the set $\{\{X', Y'\} \mid X', Y' \in \mathcal{P}$ distinct and joinable$\}$ is bijective we see that there are no joinable points $A', B' \in \mathcal{P}'$ such that $A'^{\pi^{-1}}, B'^{\pi^{-1}}$ are not joinable. ∎

Theorem 2. *Let $\mathcal{N}$, $\mathcal{N}'$ be nets and let $\pi : \mathcal{P} \to \mathcal{P}'$ be join-preserving mapping. If $\#\{x^\pi \mid X \in l\} \geqq 5$ for all $l \in \mathcal{L}$ then there is a permutation σ of $\{1, 2, 3\}$ such that π is an epimorphism of $\mathcal{N}$ onto $(\mathcal{P}', \mathcal{L}', (\mathcal{L}'_{1\sigma}, \mathcal{L}'_{2\sigma}, \mathcal{L}'_{3\sigma}))$.*

Proof. A consequence of Proposition 8 and Theorem 1. ∎

Remark. There are simple examples of "proper" weak epimorphisms of nets in which necessarily the image net is of order 2.

Probably the simplest is the following (which is easily seen from the figure 11).

Reference

[1] *Belousov, Valentin Danilovič*: Algebraic Nets and Quasigroups (in Russian), Kišinev 1971.

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