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ON AN INTEGRAL OPERATOR IN THE SPACE OF FUNCTIONS WITH BOUNDED VARIATION

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1. BASIC NOTATIONS

Let R^n be the n-dimensional real vector space. Let $\boldsymbol{A} = (a_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ be a $k \times l$-matrix (a_{ij} is the element of $\boldsymbol{A}$ from the i-th row and j-th column). By $\boldsymbol{A}'$ let us denote the transposed matrix of $\boldsymbol{A}$. For a $k \times l$-matrix $\boldsymbol{A}$ we define the number $\|\boldsymbol{A}\| = \max_{i=1,\ldots,k} \sum_{j=1}^{l} |a_{ij}|$. The elements $\boldsymbol{x}$ of R^n are column vectors (i.e. $n \times 1$-matrices). $\|.\|$ is a norm in R^n. The space of all $n \times n$-matrices let be denoted by $L(R^n \to R^n)$. $\|.\|$ is a norm in $L(R^n \to R^n)$ (the obvious operator norm corresponding to the given norm in R^n). We have evidently $\|\boldsymbol{A}'\| \leqq n\|\boldsymbol{A}\|$, $\|\boldsymbol{x}'\| \leqq \leqq n\|\boldsymbol{x}\|$ for $\boldsymbol{A} \in L(R^n \to R^n)$, $\boldsymbol{x} \in R^n$ respectively.

Let $\langle a, b\rangle \subset R = R^1$ be a bounded closed interval, $a < b$. For a given vector function $\boldsymbol{x}(t)$, $\boldsymbol{x} : \langle a, b\rangle \to R^n$ we define the (total) variation of $\boldsymbol{x}$ on $\langle a, b\rangle$ as usual:

$$\operatorname{var}_a^b \boldsymbol{x} = \sup_D \sum_{i=1}^{m} \|\boldsymbol{x}(t_i) - \boldsymbol{x}(t_{i-1})\|$$

where the supremum is taken over all finite decompositions $D : a = t_0 < t_1 < \ldots \ldots < t_m = b$ of $\langle a, b\rangle$.

We denote

$$V_n(a, b) = \{\boldsymbol{x} : \langle a, b\rangle \to R^n;\ \operatorname{var}_a^b \boldsymbol{x} < +\infty\}.$$

If no misunderstanding may occur, we write simply V_n instead of $V_n(a, b)$. If $n = 1$ we write $V(a, b)$ or V instead of $V_1(a, b)$ or V_1.

The following statement is obvious: $\boldsymbol{x} \in V_n(a, b)$ if and only if $x_i \in V(a, b)$ for all $i = 1, \ldots, n$, $\boldsymbol{x}' = (x_1, \ldots, x_n)$. The inequality

$$\operatorname{var}_a^b x_i \leqq \operatorname{var}_a^b \boldsymbol{x} \tag{1,1}$$

is satisfied for all $i = 1, 2, \ldots, n$.

For $\mathbf{x} \in V_n(a, b)$ the limits $\lim_{\tau \to t+} \mathbf{x}(\tau) = \mathbf{x}(t+)$, $\lim_{\tau \to t-} \mathbf{x}(\tau) = \mathbf{x}(t-)$ exist for all $t \in \langle a, b \rangle$. We use the notations

$$\Delta^+ \mathbf{x}(t) = \mathbf{x}(t+) - \mathbf{x}(t), \quad \Delta^- \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-),$$
$$\Delta \mathbf{x}(t) = \mathbf{x}(t+) - \mathbf{x}(t-) = \Delta^+ \mathbf{x}(t) + \Delta^- \mathbf{x}(t).$$

2. THE INTEGRAL

For our purpose we use the concept of the generalized Perron-Stieltjes integral introduced by J. Kurzweil in [1].

Let $f : \langle a, b \rangle \to R$, $g : \langle a, b \rangle \to R$ be given. If $g(t)$ is not defined for $t < a$ and $t > b$ then we suppose that $g(t) = g(a)$ for $t < a$ and $g(t) = g(b)$ for $t > b$. For $\langle c, d \rangle \subset \langle a, b \rangle$ we denote

$$\int_c^d f(t)\, \mathrm{d}g(t) = \int_c^d \mathrm{D} f(\tau)\, g(t)$$

where the right hand side is the Kurzweil integral ([1]) of the function $U(\tau, t) = f(\tau)\, g(t)$. In [1] the following is shown:

If $f : \langle a, b \rangle \to R$ is finite and $g \in V(c, d)$ then the integral $\int_c^d f(t)\, \mathrm{d}g(t)$ exists if and only if the Perron-Stieltjes integral (P.S.) $\int_c^d f(t)\, \mathrm{d}g(t)$ exists (in the usual sense) and both integrals are equal.

Further we have: If $f, g : \langle a, b \rangle \to R$, $\langle c, d \rangle \subset \langle a, b \rangle$, $f \in V(c, d)$, $g \in V(c, d)$ then the integral $\int_c^d f(t)\, \mathrm{d}g(t)$ exists.

This follows essentially from the same statement which holds for the Perron-Stieltjes integral and the above quoted equivalence of both concepts of integral.

If $|f(t)| \leqq M$ for $t \in \langle a, b \rangle$ and $g \in V(c, d)$, $\langle c, d \rangle \subset \langle a, b \rangle$ then

$$|f(\tau)\,(g(t_2) - g(t_1))| \leqq M \operatorname{var}_{t_1}^{t_2} g \leqq M |\operatorname{var}_c^{t_2} g - \operatorname{var}_c^{t_1} g|$$

for all $t_1, t_2, \tau \in \langle c, d \rangle$. From this inequality and from Lemma 2,1 in [3] the following proposition immediately follows:

If $f : \langle a, b \rangle \to R$, $|f(t)| \leqq M$ for all $t \in \langle a, b \rangle$, $g \in V(c, d)$ and if $\int_c^d f(t)\, \mathrm{d}g(t)$ exists, then

$$\left| \int_c^d f(t)\, \mathrm{d}g(t) \right| \leqq M \int_c^d \mathrm{d}(\operatorname{var}_c^t g) = M \operatorname{var}_c^d g. \tag{2,1}$$

If $f \in V(a, b)$ then obviously the inequality $|f(t)| \leqq |f(a)| + \operatorname{var}_a^b f$ holds for all $t \in \langle a, b \rangle$. Let $g \in V(c, d)$, $\langle c, d \rangle \subset \langle a, b \rangle$. By (2,1) we obtain

$$\left| \int_c^d f(t)\, \mathrm{d}g(t) \right| \leqq (|f(a)| + \operatorname{var}_a^b f) \operatorname{var}_c^d g. \tag{2,2}$$

From (2,2) we can easily obtain that for $f, g \in V(a, b)$ the function $\int_a^t f(\tau)\, dg(\tau)$: $: \langle a, b\rangle \to R$ belongs to $V(a, b)$, namely, the inequality

$$\operatorname{var}_a^b\left(\int_a^t f(\tau)\, dg(\tau)\right) \leqq (|f(a)| + \operatorname{var}_a^b f) \operatorname{var}_a^b g < +\infty$$

holds.

Proposition 2,1. *Let $f, g \in V(a, b)$. For $\alpha \in \langle a, b\rangle$, $t \in \langle a, b\rangle$ we define $\psi_\alpha^+(t) = 0$ for $t \leqq \alpha$, $\psi_\alpha^+(t) = 1$ for $\alpha < t$ if $\alpha < b$ and $\psi_\alpha^-(t) = 0$ for $t < \alpha$ if $a < \alpha$, $\psi_\alpha^-(t) = 1$ for $\alpha \leqq t$.*

Then we have

$$\int_a^b \psi_\alpha^+(t)\, dg(t) = \begin{cases} g(b) - g(\alpha+) & \text{if } \alpha < b \\ 0 & \text{if } \alpha = b \end{cases} \tag{2,3}$$

$$\int_a^b \psi_\alpha^-(t)\, dg(t) = \begin{cases} g(b) - g(\alpha-) & \text{if } a < \alpha \\ g(b) - g(a) & \text{if } a = \alpha \end{cases}$$

and

$$\int_a^b f(t)\, d\psi_\alpha^+(t) = \begin{cases} f(\alpha) & \text{if } \alpha < b \\ 0 & \text{if } \alpha = b \end{cases} \tag{2,4}$$

$$\int_a^b f(t)\, d\psi_\alpha^-(t) = \begin{cases} f(\alpha) & \text{if } \alpha > a \\ 0 & \text{if } \alpha = a\,. \end{cases}$$

Proof. If $\alpha = b$ then $\int_a^b \psi_\alpha^+(t)\, dg(t) = \int_a^b 0\,.\, dg(t) = 0$. Let $\alpha < b$. Then we have

$$\int_a^b \psi_\alpha^+(t)\, dg(t) = \int_a^{\alpha+\delta} \psi_\alpha^+(t)\, dg(t) + g(b) - g(\alpha + \delta)$$

for all $0 < \delta < b - \alpha$ because

$$\int_{\alpha+\delta}^b \psi_\alpha^+(t)\, dg(t) = \int_{\alpha+\delta}^b dg(t) = g(b) - g(\alpha + \delta)\,.$$

By Theorem 1, 3, 6 [1] there is

$$\lim_{\delta\to 0+} \int_a^{\alpha+\delta} \psi_\alpha^+(t)\, dg(t) = \int_a^\alpha \psi_\alpha^+(t)\, dg(t) + \lim_{\delta\to 0+} \psi_\alpha^+(\alpha)\,[g(\alpha + \delta) - g(\alpha)] = 0$$

since $\int_a^\alpha \psi_\alpha^+(t)\, dg(t) = \int_a^\alpha 0\, dg(t) = 0$ and $\psi_\alpha^+(\alpha) = 0$. For $\delta \to 0+$ so we obtain the first equation from (2,3). The second one can be proved similarly.

We verify for example the second formula from (2,4). The first one can be verified in a similar manner. If $\alpha = a$ then $\psi_\alpha^-(t) = 1$ in $\langle a, b\rangle$. For every $0 < \delta < b - a$ we have $f(\tau_0)(\psi_\alpha^-(\tau) - \psi_\alpha^-(\tau_0)) = 0$ for each $\tau_0 \in \langle a + \delta, b\rangle$ and $\tau_0 - \delta < \tau < < +\infty$. By Lemma 1, 3, 1 [1] we obtain therefore $\int_{a+\delta}^b f(t)\, d\psi_\alpha^-(t) = 0$. This implies

$$\int_a^b f(t)\,\mathrm{d}\psi_\alpha^-(t) = \int_a^{a+\delta} f(t)\,\mathrm{d}\psi_\alpha^-(t)$$

for each $0 < \delta < b - a$ and by Theorem 1, 3, 6 [1] we have

$$\int_a^b f(t)\,\mathrm{d}\psi_\alpha^-(t) = f(a)\,\Delta^+\psi_\alpha^-(a) = 0\,.$$

Let $a < \alpha \leqq b$. By the same reason as above we have $\int_a^b f(t)\,\mathrm{d}\psi_\alpha^-(t) = \int_{\alpha-\delta}^{\alpha+\delta} f(t)\,.\,\mathrm{d}\psi_\alpha^-(t)$ if $\alpha < b$ and $\int_a^b f(t)\,\mathrm{d}\psi_\alpha^-(t) = \int_{\alpha-\delta}^b f(t)\,\mathrm{d}\psi_\alpha^-(t)$ if $\alpha = b$ for all sufficiently small $\delta > 0$. Using Theorem 1, 3, 6 [1] we can evaluate

$$\lim_{\delta\to 0+}\int_{\alpha-\delta}^{\alpha+\delta} f(t)\,\mathrm{d}\psi_\alpha^-(t) = \lim_{\delta\to 0+}\left(\int_{\alpha-\delta}^{\alpha} + \int_\alpha^{\alpha+\delta}\right) f(t)\,\mathrm{d}\psi_\alpha^-(t) =$$
$$= f(\alpha)\,\Delta^-\psi_\alpha^-(\alpha) + f(\alpha)\,\Delta^+\psi_\alpha^-(\alpha) = f(\alpha)\,\Delta\psi_\alpha^-(\alpha) = f(\alpha)$$

if $\alpha < b$ and similarly

$$\lim_{\delta\to 0+}\int_{\alpha-\delta}^{b} f(t)\,\mathrm{d}\psi_\alpha^-(t) = f(b)\,\Delta^-\psi_b^-(b) = f(b)$$

if $\alpha = b$. Therefore the second equation in (2,4) holds.

Corollary 2,1. *If $\alpha \in \langle a, b\rangle$, $\psi_\alpha(t) = 0$ for $t \in \langle a, b\rangle$, $t \neq \alpha$, $\psi_\alpha(\alpha) = 1$, $g \in V(a, b)$ then we have by* (2,3)

$$\text{(2,5)} \qquad \int_a^b \psi_\alpha(t)\,\mathrm{d}g(t) = g(\alpha+) - g(\alpha-) = \Delta g(\alpha)$$

since $\psi_\alpha(t) = \psi_\alpha^-(t) - \psi_\alpha^+(t)$ for $t \in \langle a, b\rangle$.

Let a countable set $(t_1, t_2, \ldots)$ of points in $\langle a, b\rangle$ be given, $t_i \neq t_j$ for $i \neq j$ and let us have two sequences c_i^+, c_i^-, $i = 1, 2, \ldots$ of real numbers such that the series $\sum_{a\leqq t_i<b} c_i^+$, $\sum_{a<t_i\leqq b} c_i^-$ converge absolutely. The function

$$g_B(t) = \sum_{a\leqq t_i<t} c_i^+ + \sum_{a<t_i\leqq t} c_i^-$$

will be called a break function in $\langle a, b\rangle$. For this break function $g_B(t)$ we have evidently

$$\operatorname{var}_a^b g_B = \sum_{a<t_i\leqq b} |c_i^-| + \sum_{a\leqq t_i<b} |c_i^+| < +\infty\,,$$

$\Delta^+ g_B(t) = \Delta^- g_B(t) = 0$ if $t \neq t_i$, $i = 1, 2, \ldots$ and $\Delta^+ g_B(t_i) = c_i^+$, $\Delta^- g_B(t_i) = c_i^-$, $i = 1, 2, \ldots$

Using the functions ψ_a^+ and ψ_a^- introduced in Proposition 2,1 we can evidently express the break function in the form

$$g_B(t) = \sum_{i=1}^{\infty} [c_i^+ \psi_{t_i}^+(t) + c_i^- \psi_{t_i}^-(t)] =$$

$$= \sum_{i=1}^{\infty} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

Remark 2,1. The notion of a break function can be similarly introduced for n-vector functions too; for this case it is sufficient to take $c_i^+ \in R^n$, $c_i^- \in R^n$ and repeat the above procedure where instead of $|.|$ should be written $\|.\|$.

Proposition 2,2. *Let $g_B \in V(a, b)$ be a break function. If $f \in V(a, b)$ then*

$$(2,6) \qquad \int_a^b f(t) \, \mathrm{d}g_B(t) = f(a) \Delta^+ g_B(a) + \sum_{a<\tau<b} f(\tau) \Delta g_B(\tau) + f(b) \Delta^- g_B(b) .$$

Proof. Since g_B is a break function there exists a sequence $\{t_i\}$, $t_i \in \langle a, b \rangle$, $t_i \neq t_j$ for $i \neq j$ such that

$$g_B(t) = \sum_{i=1}^{\infty} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

We put

$$g_B^N(t) = \sum_{i=1}^{N} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

We have

$$\mathrm{var}_a^b (g_B - g_B^N) = \mathrm{var}_a^b \Big(\sum_{i=N+1}^{\infty} [\Delta^+ g(t_i) \psi_{t_i}^+(t) + \Delta^- g(t_i) \psi_{t_i}^-(t)] =$$

$$= \sum_{i=N+1}^{\infty} [|\Delta^+ g(t_i)| + |\Delta^- g(t_i)|] .$$

The relation $g \in V(a, b)$ implies the convergence of the series $\sum_{i=1}^{\infty} [|\Delta^+ g(t_i)| + |\Delta^- g(t_i)|]$ and therefore we have

$$\lim_{N \to \infty} \mathrm{var}_a^b (g_B - g_B^N) = 0 .$$

Hence by (2,2) we obtain

$$\lim_{N \to \infty} \left| \int_a^b f(t) \, \mathrm{d}g_B(t) - \int_a^b f(t) \, \mathrm{d}g_B^N(t) \right| \leq \lim_{N \to \infty} [|f(a)| + \mathrm{var}_a^b f] \, \mathrm{var}_a^b (g_B - g_B^N) = 0$$

i.e.

$$\int_a^b f(t)\,\mathrm{d}g_B(t) = \lim_{N\to\infty} \int_a^b f(t)\,\mathrm{d}g_B^N(t)\,. \tag{2,7}$$

Using (2,4) we have

$$\int_a^b f(t)\,\mathrm{d}g_B^N(t) = \sum_{i=1}^N \int_a^b f(t)\,\mathrm{d}[\Delta^+ g_B(t_i)\,\psi_{t_i}^+(t) + \Delta^- g_B(t_i)\,\psi_{t_i}^-(t)] =$$
$$= \sum_{i=1}^N \left[\Delta^+ g_B(t_i) \int_a^b f(t)\,\mathrm{d}\psi_{t_i}^+(t) + \Delta^- g_B(t_i) \int_a^b f(t)\,\mathrm{d}\psi_{t_i}^-(t)\right] =$$
$$= \sum_{i=1}^N [\Delta^+ g_B(t_i) f(t_i) + \Delta^- g_B(t_i) f(t_i)] =$$
$$= \sum_{i=1}^N f(t_i)\,[\Delta^+ g_B(t_i) + \Delta^- g_B(t_i)]\,.$$

This and (2,7) give (2,6) and Proposition 2,2 is proved.

Corollary 2,2. *If $g \in V(a, b)$ is a break function such that $\Delta g(t) = 0$ for all $t \in (a, b)$, $\Delta^+ g(a) = \Delta^- g(b) = 0$ then $\int_a^b f(t)\,\mathrm{d}g(t) = 0$ for all $f \in V(a, b)$.*

The proof follows immediately from (2,6).

In [2] the following theorem on integration by parts is proved:

Let $f, g \in V(a, b)$ then for any interval $\langle c, d\rangle \subset \langle a, b\rangle$ we have

$$\int_c^d f(t)\,\mathrm{d}g(t) + \int_c^d g(t)\,\mathrm{d}f(t) = \tag{2,8}$$
$$= f(d)\,g(d) - f(c)\,g(c) - \sum_{c\le\tau<d} \Delta^+ f(\tau)\,\Delta^+ g(\tau) + \sum_{c<\tau\le d} \Delta^- f(\tau)\,\Delta^- g(\tau)\,.$$

Let $\mathbf{z}, \mathbf{w} \in V_n(a, b)$; we denote for $\langle c, d\rangle \subset \langle a, b\rangle$

$$\int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) = \int_c^d \mathrm{d}[\mathbf{w}'(t)]\,\mathbf{z}(t) = \sum_{i=1}^n \int_c^d z_i(t)\,\mathrm{d}w_i(t)\,.$$

Using this notation and the integration by parts formula (2,8) we can easily derive the integration by parts formula for n-vector functions $\mathbf{z}, \mathbf{w} \in V_n(a, b)$, $\langle c, d\rangle \subset \langle a, b\rangle$ in the form

$$\int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) + \int_c^d \mathbf{w}'(t)\,\mathrm{d}\mathbf{z}(t) = \int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) + \int_c^d \mathrm{d}[\mathbf{z}'(t)]\,\mathbf{w}(t) = \tag{2,9}$$
$$= \mathbf{z}'(d)\,\mathbf{w}(d) - \mathbf{z}'(c)\,\mathbf{w}(c) - \sum_{c\le\tau<d} \Delta^+ \mathbf{z}'(\tau)\,\Delta^+ \mathbf{w}(\tau) + \sum_{c<\tau\le d} \Delta^- \mathbf{z}'(\tau)\,\Delta^- \mathbf{w}(\tau)\,.$$

Remark 2,2. In a similar manner can be obtained the result of Corollary 2,2 for n-vector functions: If $\mathbf{w} \in V_n(a, b)$ is a break function (cf. Remark 2,1) such that $\Delta \mathbf{w}(t) = 0$ for all $t \in (a, b)$, $\Delta^+ \mathbf{w}(a) = \Delta^- \mathbf{w}(b) = 0$ then $\int_a^b \mathbf{z}'(t)\, d\mathbf{w}(t) = 0$ for all $\mathbf{z} \in V_n(a, b)$.

Now let a nondegenerate interval $I = \langle a, b\rangle \times \langle c, d\rangle$ in R^2 be given; $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ let be a matrix function defined on the interval I. The elements of the matrix $\mathbf{K}(s, t)$ are denoted by $k_{ij}(s, t)$, i.e. $\mathbf{K}(s, t) = (k_{ij}(s, t))$, $i, j = 1, 2, \ldots, n$.

For a given subinterval $J = \langle \bar{a}, \bar{b}\rangle \times \langle \bar{c}, \bar{d}\rangle \subset I$ we set $m_{\mathbf{K}}(J) = \mathbf{K}(\bar{b}, \bar{d}) - \mathbf{K}(\bar{b}, \bar{c}) - \mathbf{K}(\bar{a}, \bar{d}) + \mathbf{K}(\bar{a}, \bar{c}) \in L(R^n \to R^n)$ and define

$$v_I(\mathbf{K}) = \sup \sum_i \|m_{\mathbf{K}}(J_i)\| \tag{2,10}$$

where the supremum is taken over all finite systems of subintervals $J_i \subset I$ such that for the interiors J_i^0 of J_i (in the topology of R^2) we have $J_i^0 \cap J_j^0 = \emptyset$ when $i \neq j$. The norm $\|.\|$ used in (2,10) is the operator norm in $L(R^n \to R^n)$ (see Sec. 1.).

The number $v_I(\mathbf{K})$ established in (2,10) is a kind of twodimensional variation of the matrix function $\mathbf{K}(s, t)$ in the interval I. This notion of a twodimensional variation is considered in the book of T. H. Hildebrandt [5] (for the case $n = 1$).

For a real function $k(s, t) : I \to R$ we can define the number $v_I(k)$ as above if we take $n = 1$. The properties of our operator norm imply

$$v_I(k_{ij}) \leqq v_I(\mathbf{K}) \tag{2,11}$$

for all $i, j = 1, 2, \ldots, n$.

If $I_j \subset I$ is a rectangle for each $j = 1, 2, \ldots, m$ and $I_i^0 \cap I_k^0 = \emptyset$ for $j \neq k$, then we can define the number $v_{I_j}(\mathbf{K})$ for each $j = 1, 2, \ldots, m$ as above and by definition we easily obtain

$$\sum_{j=1}^m v_{I_j}(\mathbf{K}) \leqq v_I(\mathbf{K}) . \tag{2,12}$$

We define as usual

$$\operatorname{var}_c^d \mathbf{K}(s, .) = \sup \sum_i \|\mathbf{K}(s, t_i) - \mathbf{K}(s, t_{i-1})\|$$

for fixed $s \in \langle a, b\rangle$ and

$$\operatorname{var}_a^b \mathbf{K}(., t) = \sup \sum_j \|\mathbf{K}(s_j, t) - \mathbf{K}(s_{j-1}, t)\|$$

for fixed $t \in \langle c, d\rangle$ where the supremums are taken over all finite decompositions of the interval $\langle c, d\rangle$, $\langle a, b\rangle$ respectively.

The properties of the used operator norm imply

$$\operatorname{var}_c^d k_{ij}(s, .) \leqq \operatorname{var}_c^d \mathbf{K}(s, .) , \tag{2,13a}$$

$$\operatorname{var}_a^b k_{ij}(., t) \leqq \operatorname{var}_a^b \mathbf{K}(., t) \tag{2,13b}$$

for any $i, j = 1, 2, \ldots, n$, $s \in \langle a, b\rangle$, $t \in \langle c, d\rangle$.

For any $s, s_0 \in \langle a, b\rangle$, $t_{j-1}, t_j \in \langle c, d\rangle$ we have

$$\|\mathbf{K}(s, t_j) - \mathbf{K}(s, t_{j-1})\| \leqq \|m_{\mathbf{K}}(J_j)\| + \|\mathbf{K}(s_0, t_j) - \mathbf{K}(s_0, t_{j-1})\|$$

where $J_i = \langle s_0, s\rangle \times \langle t_{j-1}, t_j\rangle$. Hence for each decomposition $D : c = t_0 < < t_1 < \ldots < t_m = d$ of the interval $\langle c, d\rangle$ the inequality

$$\sum_{j=1}^{m} \|\mathbf{K}(s, t_j) - \mathbf{K}(s, t_{j-1})\| \leqq \sum_{j=1}^{m} \|m_{\mathbf{K}}(J_j)\| + \\ + \sum_{j=1}^{m} \|\mathbf{K}(s_0, t_j) - \mathbf{K}(s_0, t_{j-1})\| \leqq v_I(\mathbf{K}) + \operatorname{var}_c^d \mathbf{K}(s_0, .)$$

holds; therefore

(2,14a) $$\operatorname{var}_c^d \mathbf{K}(s, .) \leqq v_I(\mathbf{K}) + \operatorname{var}_c^d \mathbf{K}(s_0, .)$$

for each $s \in \langle a, b\rangle$. Similarly can be proved for $t_0 \in \langle c, d\rangle$ the inequality

(2,14b) $$\operatorname{var}_a^b \mathbf{K}(., t) \leqq v_I(\mathbf{K}) + \operatorname{var}_a^b \mathbf{K}(., t_0)$$

which holds for each $t \in \langle c, d\rangle$.

Therefore, if we suppose that $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_0, .) < +\infty$ for some $s_0 \in \in \langle a, b\rangle$ then we have $\operatorname{var}_c^d \mathbf{K}(s, .) < +\infty$ for all $s \in \langle a, b\rangle$ and symmetrically if $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_a^b \mathbf{K}(., t_0) < +\infty$ for some $t_0 \in \langle c, d\rangle$, then $\operatorname{var}_a^b \mathbf{K}(., t) < +\infty$ for all $t \in \langle c, d\rangle$.

Let us put

(2,15a) $$\varphi(\sigma) = v_{\langle a,\sigma\rangle \times \langle c,d\rangle}(\mathbf{K})$$

for $\sigma \in \langle a, b\rangle$; $\varphi(\sigma) : \langle a, b\rangle \to R$ is evidently a nondecreasing function in $\langle a, b\rangle$, $\varphi(a) = 0$, $\varphi(b) = v_I(\mathbf{K})$. In the same way we can define

(2,15b) $$\psi(\tau) = v_{\langle a,b\rangle \times \langle c,\tau\rangle}(\mathbf{K})$$

for $\tau \in \langle c, d\rangle$; $\psi(\tau) : \langle c, d\rangle \to R$ is nondecreasing, $\psi(c) = 0$, $\psi(d) = v_I(\mathbf{K})$.

Note that for an arbitrary decomposition of the interval $\langle a, b\rangle : a = s_0 < s_1 < \ldots \ldots < s_1 = b$ and any two points $t_1, t_2 \in \langle c, d\rangle$ we have

$$|\operatorname{var}_c^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)) - \operatorname{var}_c^{t_1} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))| = \\ = |\operatorname{var}_{t_1}^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))| \leqq v_{\langle s_{i-1},s_i\rangle \times \langle t_1,t_2\rangle}(\mathbf{K})$$

for $i = 1, 2, \ldots, l$, i.e.

(2,16a) $$|\sum_{i=1}^{l} [\operatorname{var}_c^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)) - \operatorname{var}_c^{t_1} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))]| \leqq \\ \leqq \sum_{i=1}^{l} v_{\langle s_{i-1},s_i\rangle \times \langle t_1,t_2\rangle}(\mathbf{K}) \leqq v_{\langle a,b\rangle \times \langle t_1,t_2\rangle}(\mathbf{K}) \leqq |\psi(t_2) - \psi(t_1)|\,.$$

Symmetrically for an arbitrary decomposition $c = t_0 < t_1 < \ldots < t_m = d$ and any two points $s_1, s_2 \in \langle a, b \rangle$ it is

$$(2{,}16\text{b}) \qquad \left|\sum_{j=1}^{m} [\operatorname{var}_a^{s_2} (\mathbf{K}(., t_j) - \mathbf{K}(., t_{j-1})) - \operatorname{var}_a^{s_1} (\mathbf{K}(., t_j) - \mathbf{K}(., t_{j-1}))]\right| \leqq \leqq |\varphi(s_2) - \varphi(s_1)| \, .$$

Lemma 2,1. *Let* $\mathbf{K}(s, t) : I = \langle a, b \rangle \times \langle c, d \rangle \to L(R^n \to R^n)$ *satisfy* $v_I(\mathbf{K}) < +\infty$ *and* $\operatorname{var}_c^d \mathbf{K}(s_*, .) < +\infty$ *for some* $s_* \in \langle a, b \rangle$. *Let further for some* $s_0 \in \langle a, b \rangle$

$$(2{,}17) \qquad \lim_{s \to s_0+} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

or

$$\lim_{s \to s_0-} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

for all $t \in \langle c, d \rangle$ *then*

$$(2{,}18) \qquad \lim_{s \to s_0+} \varphi(s) = \varphi(s_0) \, , \quad \lim_{s \to s_0-} \varphi(s) = \varphi(s_0)$$

respectively where $\varphi : \langle a, b \rangle \to R$ *is the function defined in* (2,15a). (*If* $s_0 = a$ ($s_0 = b$) *then we consider the first* (*second*) *case only.*)

Proof. We prove the first case only, the other one is symmetric. The function φ is nondecreasing, i.e. $\varphi(s) - \varphi(s_0) \geqq 0$ for $s \geqq s_0$. Let us suppose that our Lemma is not valid and that there is a number $\eta > 0$ such that for all $s > s_0$ we have

$$(2{,}19) \qquad \varphi(s) - \varphi(s_0) \geqq \eta > 0 \, .$$

By definition of $\varphi(s)$ there exists a finite system of intervals $J_l \subset \langle a, s \rangle \times \langle c, d \rangle$, $J_l^0 \cap J_j^0 = \emptyset$, $l \neq j$ such that

$$\sum_l \|m_{\mathbf{K}}(J_l)\| > \varphi(s) - \frac{\eta}{8}$$

We put $I_l^* = J_l \cap \langle a, s_0 \rangle \times \langle c, d \rangle$ and $I_l = J_l \cap \langle s_0, s \rangle \times \langle c, d \rangle$. Obviously

$$\|m_{\mathbf{K}}(J_l)\| \leqq \|m_{\mathbf{K}}(I_l^*)\| + \|m_{\mathbf{K}}(I_l)\|$$

and

$$\sum_l \|m_{\mathbf{K}}(I_l)\| \leqq \varphi(s_0) \, .$$

Hence

$$\sum_l \|m_{\mathbf{K}}(I_l)\| + \sum_l \|m_{\mathbf{K}}(I_l^*)\| - \varphi(s_0) > \varphi(s) - \varphi(s_0) - \frac{\eta}{8}$$

and we obtain

$$(2{,}20) \qquad \sum_l \|m_{\mathbf{K}}(I_l)\| > \varphi(s) - \varphi(s_0) - \frac{\eta}{8} \, .$$

At the same time I_l is a finite system of intervals in $\langle s_0, s \rangle \times \langle c, d \rangle$ and $I_l^0 \cap I_j^0 = \emptyset$ for $j \neq l$.

Let $0 < \delta < s - s_0$ and put $\tilde{I}_l = I_l \cap \langle s_0, s_0 + \delta\rangle \times \langle c, d\rangle$, $\hat{I}_l = I_l \cap \langle s_0 + \delta, s\rangle \times \langle c, d\rangle$. If for example $I_l = \langle s_0, s_0 + \delta\rangle \times \langle t', t''\rangle$ then we have

$$\|m_K(\hat{I}_l)\| = \|K(s_0 + \delta, t'') - K(s_0 + \delta, t') - K(s_0, t'') + K(s_0, t')\| \leqq$$
$$\leqq \|K(s_0 + \delta, t'') - K(s_0, t'')\| + \|K(s_0 + \delta, t') - K(s_0, t')\|$$

and we see that a choice of sufficiently small $\delta > 0$ implies by (2,17) that $\|m_K(\hat{I}_l)\|$ is arbitrarily small. This procedure can be repeated for all possible forms of $\hat{I}_l$ in a similar manner. Hence for sufficiently small $\delta > 0$ we can obtain

$$\sum_l \|m_K(\hat{I}_l)\| < \frac{\eta}{8}.$$

This inequality together with $\sum_l \|m_K(I_l)\| \leqq \sum_l \|m_K(\hat{I}_l)\| + \sum_l \|m_K(\tilde{I}_l)\|$ and (2,20) implies

$$\sum_l \|m_K(\tilde{I}_l)\| > \varphi(s) - \varphi(s_0) - \frac{\eta}{4}. \tag{2,21}$$

As (2,19) is assumed there is a finite system of intervals $I_k^{**} \subset \langle a, s_0 + \delta\rangle \times \langle c, d\rangle$, $(I_k^{**})^0 \cap (I_j^{**})^0 = \emptyset$ for $j \neq k$ such that

$$\sum_k \|m_K(I_k)\| - \varphi(s) > \frac{\eta}{2}.$$

From this and from (2,21) we have

$$\sum_l \|m_K(\tilde{I}_l)\| + \sum_k \|m_K(I_k^{**})\| - \varphi(s_0) > \varphi(s) - \varphi(s_0) + \frac{\eta}{4}. \tag{2,22}$$

Since the union of intervals $\tilde{I}_l$ and I_k^{**} forms a finite system of intervals in $\langle a, s\rangle \times \langle c, d\rangle$ with mutually disjoint interiors, we obtain from (2,22) by definition of $\varphi(s)$ the contradictory inequality $\varphi(s) - \varphi(s_0) > \varphi(s) - \varphi(s_0) + \eta/4$. Thus our Lemma is proved.

Remark 2,3. By definition we have for $s_0 \in \langle a, b\rangle$, $0 < \delta < b - s_0$

$$\operatorname{var}_c^d (K(s_0 + \delta, .) - K(s_0, .)) \leqq v_{\langle s_0, s_0+\delta\rangle \times \langle c,d\rangle}(K) \leqq \varphi(s_0 + \delta) - \varphi(s_0) \tag{i}$$

where φ is given in (2,15a). Hence if the first equation in (2,17) holds, we have by (2,18)

$$\lim_{\delta \to 0+} \operatorname{var}_c^d (K(s_0 + \delta, .) - K(s_0, .)) = 0.$$

If an arbitrary $K(s, t) : \langle a, b\rangle \times \langle c, d\rangle \to L(R^n \to R^n)$ is given with $v_I(K) < +\infty$, $\operatorname{var}_c^d K(s_*, .) < +\infty$ for some $s_* \in \langle a, b\rangle$ and $\lim_{s \to s_0+} K(s, t) = K(s_0+, t)$ exists for every $t \in \langle c, d\rangle$ then we can define $K^\circ(s, t) = K(s, t)$ if $(s, t) \in I = \langle a, b\rangle \times$

$\times \langle c, d\rangle$, $s \neq s_0$, $\boldsymbol{K}^\circ(s_0, t) = \boldsymbol{K}(s_0+, t)$. Easily can be obtained $v_I(\boldsymbol{K}^\circ) < +\infty$ and $\operatorname{var}_c^d \boldsymbol{K}^\circ(s_{**}, .) < +\infty$ for some $s_{**} \in \langle a, b\rangle$. We have further evidently $\lim_{s \to s_0+} \|\boldsymbol{K}^\circ(s, t) - \boldsymbol{K}^\circ(s_0, t)\| = 0$ for every $t \in \langle c, d\rangle$ and in the same way as above we obtain

$$\text{(ii)} \qquad \lim_{\delta \to 0+} \operatorname{var}_c^d (\boldsymbol{K}^\circ(s_0 + \delta, .) - \boldsymbol{K}^\circ(s_0, .)) =$$
$$= \lim_{\delta \to 0+} \operatorname{var}_c^d (\boldsymbol{K}(s_0 + \delta, .) - \boldsymbol{K}(s_0+, .)) = 0\,.$$

A similar appointment gives the same result for left hand side limits. This implies the following

Corollary 2,3. *If for $\boldsymbol{K}(s, t) : I \to L(R^n \to R^n)$, $v_I(\boldsymbol{K}) < +\infty$, $\operatorname{var}_c^d \boldsymbol{K}(s_*, .) < +\infty$ for some $s_* \in \langle a, b\rangle$ and if $\lim_{s \to s_0+} \boldsymbol{K}(s, t) = \boldsymbol{K}(s_0+, t)$ exists for any $t \in \langle c, d\rangle$ then we have*

$$\operatorname{var}_c^d (\boldsymbol{K}(s_0+, .) - \boldsymbol{K}(s_0, .)) = \operatorname{var}_c^d \Delta^+ \boldsymbol{K}(s_0, .) \leqq \varphi(s_0+) - \varphi(s_0) = \Delta^+ \varphi(s_0)\,.$$

(*A similar statement for left hand side limits holds.*)

Proof. For any $\delta > 0$ we have

$$\operatorname{var}_c^d (\boldsymbol{K}(s_0+, .) - \boldsymbol{K}(s_0, .)) =$$
$$= \operatorname{var}_c^d (\boldsymbol{K}(s_0 + \delta, .) - \boldsymbol{K}(s_0, .) + \boldsymbol{K}(s_0+, .) - \boldsymbol{K}(s_0 + \delta, .) \leqq$$
$$\leqq \operatorname{var}_c^d (\boldsymbol{K}(s_0 + \delta, .) - \boldsymbol{K}(s_0, .)) + \operatorname{var}_c^d (\boldsymbol{K}(s_0 + \delta, .) - \boldsymbol{K}(s_0+, .)) \leqq$$
$$\leqq \varphi(s_0 + \delta) - \varphi(s_0) + \operatorname{var}_c^d (\boldsymbol{K}(s_0 + \delta, .) - \boldsymbol{K}(s_0+, .))\,.$$

Hence by the limiting process $\delta \to 0+$ we obtain our inequality by means of (i) and (ii).

Now we define integrals of vector functions. If an n-vector $\boldsymbol{U}(\tau, t) = (U_1(\tau, t), \ldots, \ldots, U_n(\tau, t))$ is given, $\boldsymbol{U}(\tau, t) : S \to R^n$, $S = \{(\tau, t) \in R^2;\ c \leqq \tau \leqq d,\ \tau - \delta(\tau) \leqq t \leqq \leqq \tau + \delta(\tau),\ \delta(\tau) > 0$ for every $\tau \in \langle c, d\rangle\}$ then by definition (cf. [1])

$$\int_c^d \mathrm{D}\boldsymbol{U}(\tau, t) = \left(\int_c^d \mathrm{D}U_1(\tau, t), \ldots, \int_c^d \mathrm{D}U_n(\tau, t)\right).$$

Given $\boldsymbol{x} : \langle c, d\rangle \to R^n$, we put for any $s \in \langle a, b\rangle$

$$\boldsymbol{U}(\tau, t) = \boldsymbol{K}(s, t)\, \boldsymbol{x}(\tau) = \Big(\sum_{j=1}^n k_{1j}(s, t)\, x_j(\tau), \ldots, \sum_{j=1}^n k_{nj}(s, t)\, x_j(\tau)\Big)'$$

and denote

$$\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t) = \int_c^d \mathrm{D}\boldsymbol{U}(\tau,t) =$$

$$= \Big(\int_c^d \mathrm{D}(\sum_{j=1}^n k_{1j}(s,t)\,x_j(\tau)), \ldots, \int_c^d \mathrm{D}(\sum_{j=1}^n k_{nj}(s,t)\,x_j(\tau)\Big)' =$$

$$= \Big(\sum_{j=1}^n \int_c^d x_j(t)\,\mathrm{d}_t k_{1j}(s,t), \ldots, \sum_{j=1}^n \int_c^d x_j(t)\,\mathrm{d}_t k_{nj}(s,t)\Big)'.$$

Remark 2,4. For this definition we need to know the values $\boldsymbol{K}(s,t)$ for $t < c$, $t > d$. We suppose therefore $\boldsymbol{K}(s,t) = \boldsymbol{K}(s,c)$ for $t < c$ and $\boldsymbol{K}(s,t) = \boldsymbol{K}(s,d)$ for $t > d$.

For $\boldsymbol{y} : \langle a, b\rangle \to R^n$ we define in a similar manner the integral $\int_a^b \mathrm{d}_s[\boldsymbol{K}(s,t)]\,\boldsymbol{y}(s)$ and as above we suppose $\boldsymbol{K}(s,t) = \boldsymbol{K}(a,t)$ for $s < a$ and $\boldsymbol{K}(s,t) = \boldsymbol{K}(b,t)$ for $s > b$.

Proposition 2,3. *Let $\boldsymbol{K}(s,t) : I = \langle a, b\rangle \times \langle c, d\rangle \to L(R^n \to R^n)$ be given and let $\boldsymbol{K}(s,t) = \boldsymbol{K}(s,c)$ for $t < c$, $\boldsymbol{K}(s,t) = \boldsymbol{K}(s,d)$ for $t > d$. Let us suppose that $v_I(\boldsymbol{K}) < +\infty$ and $\operatorname{var}_c^d \boldsymbol{K}(s_0, .) < +\infty$ for some $s_0 \in \langle a, b\rangle$.*

If $\boldsymbol{x}(t) \in V_n(c,d)$ then the integral $\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t)$ exists for any $s \in \langle a, b\rangle$. The inequality

$$(2,23)\qquad \Big\|\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t)\Big\| \leqq \sup_{t\in\langle c,d\rangle} \|\boldsymbol{x}(t)\| \,.\, \operatorname{var}_c^d \boldsymbol{K}(s, .)$$

holds for any $s \in \langle a, b\rangle$. Further we have

$$(2,24)\qquad \operatorname{var}_a^b\Big(\int_c^d \mathrm{d}_t \boldsymbol{K}(s,t)\,\boldsymbol{x}(t)\Big) \leqq \int_c^d \|\boldsymbol{x}(t)\|\,\mathrm{d}\psi(t) \leqq \sup_{t\in\langle c,d\rangle} \|\boldsymbol{x}(t)\| \,.\, v_I(\boldsymbol{K})$$

where the function ψ is defined in (2,15b). Thus the integral $\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t)$ as a function of the variable s belongs to $V_n(a,b)$.

Proof. By the assumption and by (2,14a) we obtain $\operatorname{var}_c^d \boldsymbol{K}(s, .) < +\infty$ for all $s \in \langle a, b\rangle$; this implies the existence of the integral $\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t)$ for any $s \in \langle a, b\rangle$. Further we have for each $s \in \langle a, b\rangle$

$$\|\boldsymbol{K}(s,t_2)\,\boldsymbol{x}(\tau) - \boldsymbol{K}(s,t_1)\,\boldsymbol{x}(\tau)\| \leqq \|\boldsymbol{x}(\tau)\| \,.\, \operatorname{var}_{t_1}^{t_2} \boldsymbol{K}(s, .) \leqq$$
$$\leqq \|\boldsymbol{x}(\tau)\| \,.\, |\operatorname{var}_c^{t_2} \boldsymbol{K}(s, .) - \operatorname{var}_c^{t_1} \boldsymbol{K}(s, .)|$$

for any t_1, t_2 and for $\tau \in \langle c, d\rangle$. Hence Lemma 2,1 [3] implies

$$\Big\|\int_c^d \mathrm{d}_t[\boldsymbol{K}(s,t)]\,\boldsymbol{x}(t)\Big\| \leqq \int_c^d \|\boldsymbol{x}(t)\|\,\mathrm{d}(\operatorname{var}_c^t \boldsymbol{K}(s, .))$$

and by (2,1) we obtain (2,23).

Let an arbitrary finite decomposition $a = s_0 < s_1 < \ldots < s_l = b$ of the interval $\langle a, b\rangle$ be given. We have

$$\sum_{i=1}^{l}\left\|\int_c^d \mathrm{d}_t[\mathbf{K}(s_i, t) - \mathbf{K}(s_{i-1}, t)]\, \mathbf{x}(t)\right\| \leqq$$
$$\leqq \sum_{i=1}^{l}\int_c^d \|\mathbf{x}(t)\|\, \mathrm{d}(\mathrm{var}_c^t\,[\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)]) \leqq$$
$$\leqq \int_c^d \|\mathbf{x}(t)\|\, \mathrm{d}(\sum_{i=1}^{l}\mathrm{var}_c^t\,[\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)])\,.$$

From this inequality we obtain using (2,16a) and Lemma 2,1 [3] the inequality

$$\sum_{i=1}^{l}\left\|\int_c^d \mathrm{d}_t[\mathbf{K}(s_i, t) - \mathbf{K}(s_{i-1}, t)]\, \mathbf{x}(t)\right\| \leqq \int_c^d \|\mathbf{x}(t)\|\, \mathrm{d}\psi(t)\,.$$

Passing to the supremum on the left hand side in this inequality we obtain immediately the first inequality in (2,24). The other one follows from the relation $\int_c^d d\psi(t) = \psi(d) - \psi(c) = v_I(\mathbf{K})$ and from (2,1). Therefore we have $\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) \in V_n(a, b)$.

Remark 2,5. A similar proposition can also be proved for $\mathbf{y} \in V_n(a, b)$ and the integral $\int_a^b \mathrm{d}_s[\mathbf{K}(s, t)]\, \mathbf{y}(s)$.

Lemma 2,2. *Let $k(s, t) : I = \langle a, b\rangle \times \langle c, d\rangle \to R$ be given. Suppose that $k(s, t) = k(a, t)$ for $s < a$, $k(s, t) = k(b, t)$ for $s > b$, $k(s, t) = k(s, c)$ for $t < c$ and $k(s, t) = k(s, d)$ for $t > d$. Further let $v_I(k) < +\infty$, $\mathrm{var}_c^d\, k(s_0, .) < +\infty$ for some $s_0 \in \langle a, b\rangle$ and $\mathrm{var}_a^b\, k(., t_0) < +\infty$ for some $t_0 \in \langle c, d\rangle$. If $f(s) \in V(a, b)$, $g \in V(c, d)$ then*

$$\int_c^d g(t)\, \mathrm{d}_t\left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)]\right) = \int_a^b f(s)\, \mathrm{d}_s\left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)]\right) \tag{2,25}$$

and

$$\int_c^d g(t)\, \mathrm{d}_t\left(\int_a^b k(s, t)\, \mathrm{d}f(s)\right) = \int_a^b\left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)]\right)\mathrm{d}f(s) \tag{2,26}$$

hold and the integrals on both sides of (2,25) and (2,26) exist.

Proof. By Proposition 2,3 we have $\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \in V(c, d)$, $\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \in V(a, b)$ and the existence of the integrals on both sides of (2,25) follows from this fact immediately.

Let us put $f(s) = \psi_\alpha^+(s)$, $g(t) = \psi_\beta^-(t)$ for $\alpha \in \langle a, b\rangle$, $\beta \in \langle c, d\rangle$ (cf. Proposition 2,1). Using (2,3) from Proposition 2,1 we obtain by easy computation

$$\left(\int_c^d g(t)\, \mathrm{d}_t \int_a^b f(s)\, \mathrm{d}_s[k(s, t)]\right) = k(b, d) - k(b, \beta-) - k(\alpha+, d) + k(\alpha+, \beta-) =$$
$$= \int_a^b f(s)\, \mathrm{d}_s\left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)]\right)$$

i.e. (2,25) holds for this choice of f and g. We note that the term $k(\alpha+, \beta-)$ in the above computation is in one case obtained as $\lim_{t\to\beta-} \lim_{s\to\alpha+} k(s, t)$ and as $\lim_{s\to\alpha+} \lim_{t\to\beta-} k(s, t)$ in the other one. By Theorem III. 5.3 in [5] both iterated limits are equal since by assumption the existence of all quadrantal limits in any point of I is quaranted. Similarly it can be proved by direct computation that (2,25) is true if $f(s)$ equals $\psi_\alpha^-(s)$ or $\psi_\alpha^+(s)$ and $g(t)$ equals $\psi_\beta^+(t)$ or $\psi_\beta^-(t)$ for some $\alpha \in \langle a, b\rangle$, $\beta \in \langle c, d\rangle$ (c.f. Proposition 2,1). Hence from the linearity of the integral we obtain that (2,25) holds if $f(s)$, $g(t)$ are step functions because it is clear that every step function can be expressed as a finite linear combination of functions of the type ψ_α^+ and ψ_α^-. It is known that if $f \in V(a, b)$ and $g \in V(c, d)$ then there exist sequences $\{f_l(s)\}$ $f_l : \langle a, b\rangle \to R$ and $\{g_l(t)\}$, $g_l : \langle c, d\rangle \to R$, $l = 1, 2, \ldots, f_l, g_l$ are step functions such that $\lim_{l\to\infty} f_l(s) = f(s)$, $\lim_{l\to\infty} g_l(t) = g(t)$ uniformly in $\langle a, b\rangle$, $\langle c, d\rangle$ respectively. We denote

$$I_{1,l} = \int_c^d g_l(t)\, \mathrm{d}_t \left(\int_a^b f_l(s)\, \mathrm{d}_s[k(s, t)] \right),$$

$$I_{2,l} = \int_a^b f_l(s)\, d_s \left(\int_c^d g_l(t)\, \mathrm{d}_t[k(s, t)] \right);$$

since (2,25) holds for step functions we have

$$I_{1,l} = I_{2,l} \quad \text{for every} \quad l = 1, 2, \ldots \tag{2,27}$$

Further by (2,2) and (2,24) we have

$$\left| \int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \right) - I_{1,l} \right| \leqq \left| \int_c^d (g(t) - g_l(t))\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \right) \right| +$$

$$+ \left| \int_c^d g_l(t)\, \mathrm{d}_t \left(\int_a^b (f(s) - f_l(s))\, \mathrm{d}_s[k(s, t)] \right) \right| \leqq$$

$$\leqq \sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)|\, \mathrm{var}_c^d \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, .)] \right) +$$

$$+ \sup_{t\in\langle c,d\rangle} |g_l(t)|\, \mathrm{var}_c^d \left(\int_a^b (f(s) - f_l(s))\, \mathrm{d}_s[k(s, .)] \right) \leqq$$

$$\leqq [\sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)|\, (|f(a)| + \mathrm{var}_a^b f) + \sup_{t\in\langle c,d\rangle} |g_l(t)| \sup_{s\in\langle a,b\rangle} |f(s) - f_l(s)|]\, v_I(k)$$

and therefore

$$\lim_{l\to\infty} I_{1,l} = \int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \right).$$

Similarly it also holds

$$\lim_{l\to\infty} I_{2,l} = \int_a^b f(s)\, \mathrm{d}_s \left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \right).$$

In this way (2,27) implies (2,25) for arbitrary $f \in V(a, b)$, $g \in V(c, d)$.

The integral on the right hand side of (2,26) exists evidently $\left(\int_c^d g(t)\,\mathrm{d}_t[k(s,t)] \in \in V(a,b)\right)$. It can be easily proved that the inequality

$$\mathrm{var}_c^d\left(\int_a^b k(s,.)\,\mathrm{d}f(s)\right) \leqq [v_I(k) + \mathrm{var}_c^d\, k(s_0,.)]\,\mathrm{var}_a^b f < +\infty$$

holds. Thus $\int_a^b k(s,t)\,\mathrm{d}f(s) \in V(c,d)$ and the integral on the left hand side of (2,26) also exists.

The equality (2,26) holds if we set $g(t) = \psi_\alpha^+(t)$ (cf. Proposition 2,1). In fact we have by (2,3)

$$\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)] = k(s,d) - k(s,\alpha+)\,.$$

Hence

$$\int_a^b\left(\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s) = \int_a^b (k(s,d) - k(s,\alpha+))\,\mathrm{d}f(s)\,.$$

By (2,3) we have also

$$\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right) = \int_a^b (k(s,d) - k(s,\alpha+))\,\mathrm{d}f(s)\,.$$

Therefore

$$\int_a^b\left(\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s) = \int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right).$$

Similarly can be proved that (2,26) holds if $g(t) = \psi_\alpha^-(t)$ and we obtain that (2,26) holds for every step function $g(t)$. Let for $g \in V(c,d)$ a sequence of step functions $g_l : \langle c,d\rangle \to R$ be given, $\lim_{l\to\infty} g_l(t) = g(t)$ uniformly in $\langle c,d\rangle$. Then we have

$$\left|\int_c^d (g(t) - g_l(t))\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right)\right| \leqq \sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)|\,(v_I(k) + \mathrm{var}_c^d\, k(s_0,.))\,\mathrm{var}_a^b f$$

and

$$\left|\int_a^b\left(\int_c^d (g(t) - g_l(t))\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s)\right| \leqq$$
$$\leqq \sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)|\,(v_I(k) + \mathrm{var}_c^d\, k(s_0,.))\,\mathrm{var}_a^b f\,.$$

Thus in the same way as in the case of (2,25) we obtain that (2,26) holds for any $g \in V(c,d)$.

Let us now denote

$$\langle \mathbf{z}, \mathbf{w}\rangle_{(c,d)} = \int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) = \int_c^d \mathrm{d}[\mathbf{w}'(t)]\,\mathbf{z}(t) = \sum_{j=1}^n \int_c^d z_j(t)\,\mathrm{d}w_j(t)$$

if $\mathbf{z}, \mathbf{w} \in V_n(c, d)$ and

$$\langle \mathbf{z}, \mathbf{w} \rangle_{(a,b)} = \int_a^b \mathbf{z}'(s)\, \mathrm{d}\mathbf{w}(s) = \int_a^b \mathrm{d}[\mathbf{w}'(s)]\, \mathbf{z}(s) = \sum_{j=1}^n \int_a^b z_j(s)\, \mathrm{d}w_j(s)$$

if $\mathbf{z}, \mathbf{w} \in V_n(a, b)$.

Proposition 2,4. *Let* $\mathbf{K}(s, t) : I = \langle a, b \rangle \times \langle c, d \rangle \to L(R^n \to R^n)$ *be given and let* $\mathbf{K}(s, t)$ *be extended for* $s < a$, $s > b$, $t < c$, $t > d$ *as in Remark* 2,4. *Let us suppose that* $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_0, .) < +\infty$ *for some* $s_0 \in \langle a, b \rangle$, $\operatorname{var}_a^b \mathbf{K}(., t_0) < < +\infty$ *for some* $t_0 \in \langle c, d \rangle$. *Let* $\mathbf{x} \in V_n(c, d)$, $\mathbf{y} \in V_n(a, b)$. *Then*

$$\left\langle \mathbf{y}, \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t) \right\rangle_{(a,b)} = \left\langle \mathbf{x}, \int_a^b \mathrm{d}_s[\mathbf{K}'(s, .)]\, \mathbf{y}(s) \right\rangle_{(c,d)} \tag{2,28}$$

and

$$\left\langle \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t), \mathbf{y} \right\rangle_{(a,b)} = \left\langle \mathbf{x}, \int_a^b \mathbf{K}'(s, .)\, \mathrm{d}\mathbf{y}(s) \right\rangle_{(c,d)}. \tag{2,29}$$

Proof. By (1.1), (2,11), (2,13a), (2,13b) all assumptions of Lemma 2,2 are satisfied for $k_{ij}(s, t)$, $x_l(t)$, $y_m(s)$, $i, j, l, m = 1, \dots, n$. Therefore from (2,25) we obtain

$$\int_a^b y_m(s)\, \mathrm{d}_s \left(\int_c^d x_l(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \int_c^d x_l(t)\, \mathrm{d}_t \left(\int_a^b y_m(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) \tag{2,30}$$

for every $i, j, l, m = 1, 2, \dots, n$. Hence

$$\begin{aligned}
\left\langle \mathbf{y}, \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t) \right\rangle_{(a,b)} &= \int_a^b \mathbf{y}'(s)\, \mathrm{d}_s \left(\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) \right) = \\
&= \sum_{i=1}^n \int_a^b y_i(s)\, \mathrm{d}_s \left(\sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_a^b y_i(s)\, \mathrm{d}_s \left(\int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t \left(\int_a^b y_i(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) = \\
&= \sum_{j=1}^n \int_c^d x_j(t)\, d_t \left(\sum_{i=1}^n \int_a^b y_i(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) = \\
&= \int_c^d \mathbf{x}'(t)\, \mathrm{d}_t \left(\int_a^b \mathrm{d}_s[\mathbf{K}'(s, t)]\, \mathbf{y}(s) \right) = \\
&= \left\langle \mathbf{x}, \int_a^b \mathrm{d}_s[\mathbf{K}'(s, .)]\, \mathbf{y}(s) \right\rangle_{(c,d)}.
\end{aligned}$$

Thus the equality (2,28) is proved.

From (2,26) we have

$$\int_c^d x_l(t)\, \mathrm{d}_t \left(\int_a^b k_{ij}(s, t)\, \mathrm{d}y_m(s) \right) = \int_a^b \left(\int_c^d x_l(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) \mathrm{d}y_m(s) \tag{2,31}$$

for every $i, j, l, m = 1, 2, \ldots, n$. Hence

$$\left\langle \int_c^d \mathrm{d}_t[\boldsymbol{K}(., t)]\, \boldsymbol{x}(t), \boldsymbol{y} \right\rangle_{(a,b)} =$$

$$= \sum_{i=1}^n \int_a^b \left(\sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) \mathrm{d}y_i(s) =$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_a^b \left(\int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) \mathrm{d}y_i(s) =$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t \left(\int_a^b k_{ij}(s, t)\, \mathrm{d}y_i(s) \right) =$$

$$= \sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t \left(\sum_{i=1}^n \int_a^b k_{ij}(s, t)\, \mathrm{d}y_i(s) \right) =$$

$$= \int_c^d \boldsymbol{x}'(t)\, \mathrm{d}_t \left(\int_a^b \boldsymbol{K}'(s, t)\, \mathrm{d}\boldsymbol{y}(s) \right) = \left\langle \boldsymbol{x}, \int_a^b \boldsymbol{K}'(s, .)\, \mathrm{d}\boldsymbol{y}(s) \right\rangle_{(c,d)}$$

and (2,29) is also proved.

Remark 2,6. The equality (2,30) (resp. (2,31)) enables us also to derive the following relations which are symmetric to the relations (2,28) (resp. (2,29))

$$\left\langle \boldsymbol{x}, \int_a^b \mathrm{d}_s[\boldsymbol{K}(s, .)]\, \boldsymbol{y}(s) \right\rangle_{(c,d)} = \left\langle \boldsymbol{y}, \int_c^d \mathrm{d}_t \boldsymbol{K}'(., t)\, \boldsymbol{x}(t) \right\rangle_{(a,b)} \tag{2,32}$$

and

$$\left\langle \int_a^b \mathrm{d}_s[\boldsymbol{K}(s, .)]\, \boldsymbol{y}(s), \boldsymbol{x} \right\rangle_{(c,d)} = \left\langle \boldsymbol{y}, \int_c^d \boldsymbol{K}'(., t)\, \mathrm{d}\boldsymbol{x}(t) \right\rangle_{(a,b)}. \tag{2,33}$$

We note that (2,29) can be written in the form

$$\int_a^b \mathrm{d}\boldsymbol{y}'(s) \left(\int_c^d \mathrm{d}_t \boldsymbol{K}(s, t)\, \boldsymbol{x}(t) \right) = \int_c^d \mathrm{d}_t \left[\int_a^b \mathrm{d}\boldsymbol{y}'(s)\, \boldsymbol{K}(s, t) \right] \boldsymbol{x}(t).$$

3. OPERATORS $\int_0^1 d_t[K(s, t)]\, x(t)$ AND $\int_0^1 K(s, t)\, dx(t)$ IN THE SPACE V_n

In the sequel we denote $V_n = V_n(0, 1)$. For $x \in V_n$ we denote

$$(3,1) \qquad \|x\|_{V_n} = \|x(0)\| + \operatorname{var}_0^1 x .$$

$\|\cdot\|_{V_n}$ is the usual norm in V_n, V_n with this norm forms a Banach space.

Let $K(s, t) : I = \langle 0, 1\rangle \times \langle 0, 1\rangle \to L(R^n \to R^n)$ be given. Let us suppose that $K(s, t) = K(s, 0)$ for $t < 0$ and $K(s, t) = K(s, 1)$ for $t > 1$.

Further we assume in this section that

$$(3,2) \qquad v(K) = v_I(K) < +\infty$$

and

$$(3,3) \qquad \operatorname{var}_0^1 K(0, .) < +\infty .$$

Proposition 2,3 quarantees for every $x \in V_n$, $x'(t) = (x_1(t), \ldots, x_n(t))$ the existence of the integral

$$(3,4) \qquad \int_0^1 d_t[K(s, t)]\, x(t) = y(s) , \quad s \in \langle 0, 1\rangle .$$

By (2,24) from the same Proposition we obtain the inequality

$$(3,5) \qquad \operatorname{var}_0^1 y = \operatorname{var}_0^1 \left(\int_0^1 d_t[K(s, t)]\, x(t) \right) \leqq \|x\|_{V_n} v(K) .$$

The map

$$(3,6) \qquad Kx = y$$

(y is determined by (3,4)) is evidently a linear operator on the space V_n because (3,5) implies $y \in V_n$.

Further it is

$$\|Kx\|_{V_n} = \|y(0)\| + \operatorname{var}_0^1 y \leqq \left\| \int_0^1 d_t[K(0, t)]\, x(t) \right\| + \|x\|_{V_n} v(K) \leqq$$
$$\leqq \sup_{t \in \langle 0,1\rangle} \|x(t)\| \operatorname{var}_0^1 K(0, .) + \|x\|_{V_n} v(K)$$

and therefore

$$(3,7) \qquad \|Kx\|_{V_n} \leqq (\operatorname{var}_0^1 K(0, .) + v(K))\, \|x\|_{V_n}$$

i.e. $K : V_n \to V_n$ is a continuous linear operator on V_n.

Theorem 3,1. *The linear operator* $\boldsymbol{K} : V_n \to V_n$ *from* (3,6) *is completely continuous if* (3,2) *and* (3,3) *is satisfied.*

Proof. We denote by $B = \{\boldsymbol{x} \in V_n;\ \|\boldsymbol{x}\|_{V_n} < 1\}$ the unit ball in V_n. Let a sequence $\boldsymbol{x}^l$, $l = 1, 2, \ldots$ be given such that $\boldsymbol{x}^l \in B$ for $l = 1, 2, \ldots$. By Helly's Choice Theorem it is possible to select from the sequence $\boldsymbol{x}^l$ a subsequence $\boldsymbol{x}^{l_k}$ such that

$$\lim_{k \to \infty} \boldsymbol{x}^{l_k}(t) = \boldsymbol{x}^*(t)$$

for any $t \in \langle 0, 1 \rangle$ so that at the same time $\operatorname{var}_0^1 \boldsymbol{x}^* \leqq 1$ (i.e. $\boldsymbol{x}^* \in V_n$). We put $\boldsymbol{z}^k(t) = \boldsymbol{x}^{l_k}(t) - \boldsymbol{x}^*(t)$ for $t \in \langle 0, 1 \rangle$. Evidently $\boldsymbol{z}^k \in V_n$ for all $k = 1, 2, \ldots$, $\|\boldsymbol{z}^k\|_{V_n} \leqq \|\boldsymbol{x}^{l_k}\|_{V_n} + \|\boldsymbol{x}^*(0)\| + \operatorname{var}_0^1 \boldsymbol{x} < 3$ and

$$\lim_{k \to \infty} \boldsymbol{z}^k(t) = 0 \quad \text{for all} \quad t \in \langle 0, 1 \rangle\,. \tag{3,8}$$

(2,24) implies

$$\operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{z}^k(t) \right) \leqq \int_0^1 \|\boldsymbol{z}^k(t)\| \,\mathrm{d}\psi(t)$$

where $\psi : \langle 0, 1 \rangle \to R$ is a nondecreasing function $\psi(0) = 0$, $\psi(1) = v(\boldsymbol{K})$ (cf. (2,15b) where $\langle a, b \rangle = \langle c, d \rangle = \langle 0, 1 \rangle$). Clearly $0 \leqq \|\boldsymbol{z}^k(t)\| \leqq 3$ for $t \in \langle 0, 1 \rangle$ and the function $\|\boldsymbol{z}^k(t)\|$ belongs to $V(0, 1)$ for all $k = 1, 2, \ldots$, hence the integral $\int_0^1 \|\boldsymbol{z}^k(t)\| \,\mathrm{d}\psi(t)$ exists (as Kurzweil integral or equivalently as Perron-Stieltjes integral). The Dominated Convergence Theorem for the Perron-Stieltjes integral implies

$$\lim_{k \to \infty} \int_0^1 \|\boldsymbol{z}^k(t)\| \,\mathrm{d}\psi(t) = 0\,.$$

Hence we have

$$\lim_{k \to \infty} \operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}^{l_k}(t) - \int_0^1 \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}^*(t) \right) = 0\,. \tag{3,9}$$

Similarly we can show that

$$\lim_{k \to \infty} \left\| \int_0^1 \mathrm{d}_t[\boldsymbol{K}(0, t)]\, \boldsymbol{x}^{l_k}(t) - \int_0^1 \mathrm{d}_t[\boldsymbol{K}(0, t)]\, \boldsymbol{x}^*(t) \right\| = 0\,. \tag{3,10}$$

From (3,9) and (3,10) we have

$$\lim_{k \to \infty} \left\| \int_0^1 \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}^{l_k}(t) - \boldsymbol{y}^*(s) \right\|_{V_n} = 0$$

where $\boldsymbol{y}^*(s) = \int_0^1 \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}^*(t) \in V_n$ since $\boldsymbol{x}^* \in V_n$. Hence we conclude that the image of the unit ball B is precompact in V_n and consequently the operator $\boldsymbol{K}$ is completely continuous.

We shall derive now some special analytic properties of the operator K if $\mathbf{K}(s, t) : : I \to L(R^n \to R^n)$ satisfies some additional assumptions.

Proposition 3,1. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2) and (3,3), $\mathbf{x} \in V_n$. Let further for some $s_0 \in \langle 0, 1 \rangle$*

$$\lim_{s \to s_0+} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0 \quad \text{or} \lim_{s \to s_0-} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

for all $t \in \langle 0, 1 \rangle$. Then

$$\lim_{s \to s_0+} \mathbf{y}(s) = \mathbf{y}(s_0)\,, \quad \lim_{s \to s_0-} \mathbf{y}(s) = \mathbf{y}(s_0)$$

respectively where $\mathbf{y} \in V_n$ is given in (3,4).

Proof. The statement follows in an easy way from the inequality (see (2,23) from Proposition 2,3 and (2,15a))

$$\|\mathbf{y}(s) - \mathbf{y}(s_0)\| = \left\| \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t) - \mathbf{K}(s_0, t)]\, \mathbf{x}(t) \right\| \leqq$$

$$\leqq \|\mathbf{x}\|_{V_n} \operatorname{var}_0^1 (\mathbf{K}(s, .) - \mathbf{K}(s_0, .)) \leqq \|\mathbf{x}\|_{V_n} |\varphi(s) - \varphi(s_0)|$$

and from Lemma 2,1.

Corollary 3,1. *If $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfies (3,2), (3,3) and $\mathbf{K}(s, t)$ is continuous in the variable s for any $t \in \langle 0, 1 \rangle$ then the vector function $\mathbf{y}(s) : \langle 0, 1 \rangle \to R^n$ from (3,4) is continuous for any $\mathbf{x} \in V_n$, i.e. K maps V_n into CV_n (the space of continuous n-vector functions with bounded variation).*

Lemma 3,1. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2) and (3,3), $\mathbf{x} \in V_n$. Let $s_0 \in \in \langle 0, 1 \rangle$. If the limit*

$$\lim_{s \to s_0+} \mathbf{K}(s, t) = \mathbf{K}(s_0+, t) \quad \text{or} \lim_{s \to s_0-} \mathbf{K}(s, t) = \mathbf{K}(s_0-, t) \tag{3,11}$$

exists for all $t \in \langle 0, 1 \rangle$ then we have

$$\mathbf{y}(s_0+) = \lim_{s \to s_0+} \mathbf{y}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0+, t)]\, \mathbf{x}(t)\,, \tag{3,12}$$

$$\mathbf{y}(s_0-) = \lim_{s \to s_0-} \mathbf{y}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0-, t)]\, \mathbf{x}(t)$$

respectively.

Proof. We have

$$\mathbf{y}(s_0+) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0+, t)]\, \mathbf{x}(t) = \lim_{s \to s_0+} \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t) - \mathbf{K}(s_0+, t)]\, \mathbf{x}(t) = 0$$

because for $\delta > 0$

$$\left\| \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0 + \delta, t) - \mathbf{K}(s_0+, t)]\, \mathbf{x}(t) \right\| \leqq \|\mathbf{x}\|_{V_n} \operatorname{var}_0^1 (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .))$$

and by Corollary 2, 3 it is

$$\lim_{\delta \to 0+} \operatorname{var}_0^1 (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .)) = 0\,.$$

The second statement can be proved similarly.

Remark 3,1. If we suppose that $\operatorname{var}_0^1 \mathbf{K}(., t_*) < +\infty$ for some $t_* \in \langle 0, 1 \rangle$ for $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ in addition to the conditions (3,2), (3,3) then $\operatorname{var}_0^1 \mathbf{K}(., t) < < +\infty$ for all $t \in \langle 0, 1 \rangle$ and the limits (3,11) exist for all $s_0 \in \langle 0, 1 \rangle$ and all $t \in \in \langle 0, 1 \rangle$, and (3,12) holds for all $s_0 \in \langle 0, 1 \rangle$.

Proposition 3,2. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2) and (3,3). If $\mathbf{K}(s, t)$ is for every $t \in \langle 0, 1 \rangle$ regular at $s_0 \in \langle 0, 1 \rangle$, i.e. the limits (3,11) exist for every $t \in \langle 0, 1 \rangle$ and*

$$\mathbf{K}(s_0+, t) + \mathbf{K}(s_0-, t) - 2\mathbf{K}(s_0, t) = 0 \tag{3,13}$$

for all $t \in \langle 0, 1 \rangle$ then

$$\mathbf{y}(s_0+) + \mathbf{y}(s_0-) - 2\mathbf{y}(s_0) = 0\,, \tag{3,14}$$

i.e. $\mathbf{y}(s)$ is a regular function at $s_0 \in \langle 0, 1 \rangle$.

Proof follows immediately from Lemma 3,1.

Corollary 3,2. *If $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfies (3,2), (3,3), the limits (3,11) exist for every $s_0 \in \langle 0, 1 \rangle$ and $t \in \langle 0, 1 \rangle$ and (3,13) holds for every $t \in \langle 0, 1 \rangle$ then the n-vector function $\mathbf{y}(s) : \langle 0, 1 \rangle \to R^n$ from (3,4) is regular for any $\mathbf{x} \in V_n$ (this means that (3,22) holds for all $s_0 \in \langle 0, 1 \rangle$), i.e. the operator $\boldsymbol{K}$ maps V_n into the space RV_n of all regular n-vector functions of bounded variation.*

Proposition 3,3. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2), (3,3) and*

$$\operatorname{var}_0^1 \mathbf{K}(., 0) < +\infty\,. \tag{3,15}$$

If $\alpha, \beta \in \langle 0, 1\rangle$ then for $\mathbf{x} \in V_n$ the expressions

$$
(3,16) \qquad \begin{aligned}
K_1 \mathbf{x} &= \mathbf{K}(s, \alpha)\, \mathbf{x}(\beta) = \mathbf{y}_1(s)\,, \\
K_2 \mathbf{x} &= \Delta_t^+ \mathbf{K}(s, \alpha)\, \Delta^+ \mathbf{x}(\beta) = \mathbf{y}_2(s)\,, \\
K_3 \mathbf{x} &= \Delta_t^- \mathbf{K}(s, \alpha)\, \Delta^- \mathbf{x}(\beta) = \mathbf{y}_3(s)
\end{aligned}
$$

define completely continuous operators on V_n.

Proof. We have (by (2,14b))

$$
\begin{aligned}
\|K_1 \mathbf{x}\|_{V_n} &= \|\mathbf{K}(0, \alpha)\, \mathbf{x}(\beta)\| + \mathrm{var}_0^1 (\mathbf{K}(., \alpha)\, \mathbf{x}(\beta)) \leqq \\
&\leqq \{\|\mathbf{K}(0, \alpha)\| + \mathrm{var}_0^1 \mathbf{K}(., \alpha)\}\, \|\mathbf{x}(\beta)\| \leqq \\
&\leqq \{\|\mathbf{K}(0, \alpha)\| + \mathrm{var}_0^1 \mathbf{K}(., 0) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n}\,.
\end{aligned}
$$

Further it is (cf. Corollary 2,3)

$$
\begin{aligned}
\|K_2 \mathbf{x}\|_{V_n} &\leqq \|\Delta^+ \mathbf{K}(0, \alpha)\|\, \|\Delta^+ \mathbf{x}(\beta)\| + \mathrm{var}_0^1 \Delta_t^+ \mathbf{K}(., \alpha)\, \|\Delta^+ \mathbf{x}(\beta)\| = \\
&= \{\mathrm{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\Delta^+ \mathbf{x}(\beta)\| \leqq \\
&\leqq \{\mathrm{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n}
\end{aligned}
$$

and similarly

$$
\|K_3 \mathbf{x}\|_{V_n} \leqq \{\mathrm{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\Delta^- \mathbf{x}(\beta)\| \leqq \{\mathrm{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n}\,.
$$

Let B be the unit ball in the space V_n and let $\mathbf{x}_l \in B$, $l = 1, 2, \ldots$ be given. The sequence $\mathbf{x}_l(\beta)$ $(\Delta^+ \mathbf{x}_l(\beta), \Delta^- \mathbf{x}_l(\beta))$ is bounded in R^n. Hence there is a point $\mathbf{z}(\beta)$ $(\mathbf{z}^+(\beta), \mathbf{z}^-(\beta))$ in R^n and a subsequence $\mathbf{x}_{l_j}(\beta)$ $(\Delta^+ \mathbf{x}_{l_j}(\beta), \Delta^- \mathbf{x}_{l_j}(\beta))$ such that $\lim_{j\to\infty} \mathbf{x}_{l_j}(\beta) = \mathbf{z}(\beta)$ $(\lim_{j\to\infty} \Delta^+ \mathbf{x}_{l_j}(\mathrm{b}) = \mathbf{z}^+(\beta),\ \lim_{j\to\infty} \Delta^- \mathbf{x}_{l_j}(\beta) = \mathbf{z}^-(\beta))$.

Let us set $\mathbf{y}_1^*(s) = \mathbf{K}(s, \alpha)\, \mathbf{z}(\beta) \in V_n$, then we have
$\|K_1 \mathbf{x}_{l_j} - \mathbf{y}_1^*\|_{v_n} \leqq \{\mathbf{K}(0, 0) + \mathrm{var}_0^1 \mathbf{K}(0, .) + \mathrm{var}_0^1 \mathbf{K}(., 0) + v(\mathbf{K})\}\| \mathbf{x}_{l_j}(\beta) - \mathbf{z}(\beta)\|$, hence $\lim_{j\to\infty} \|K_1 \mathbf{x}_{l_j} - \mathbf{y}_1^*\|_{V_n} = 0$, i.e. the operator K_1 maps B into precompact set and therefore K_1 is completely continuous.

By setting $\mathbf{y}_2^*(s) = \Delta_t^+ \mathbf{K}(s, \alpha)\, \mathbf{z}^+(\beta)$ we obtain

$$
\|K_2 \mathbf{x}_{l_j} - \mathbf{y}_2^*\|_{V_n} \leqq \{\Delta_t^+ \mathbf{K}(0, \alpha) + \mathrm{var}_0^1 \Delta_t^+ \mathbf{K}(., \alpha)\}\, \Delta^+ \mathbf{x}_{l_j}(\beta) - \mathbf{z}^+(\mathrm{b})\|\,,
$$

hence $\lim_{j\to\infty} \|K_2 \mathbf{x}_{l_j} - \mathbf{y}_2^*\|_{V_n} = 0$ and K_2 is a completely continuous operator. Similarly the complete continuity of K_3 can be obtained.

Proposition 3,4. *If* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfies* (3,2), (3,3) *and* (3,15) *then the series*

$$\sum_{0 \leqq \tau < 1} \Delta_t^+ \mathbf{K}(s, \tau) \, \Delta^+ \mathbf{x}(\tau) \, , \quad \sum_{0 < \tau \leqq 1} \Delta_t^- \mathbf{K}(s, \tau) \, \Delta^- \mathbf{x}(\tau) \tag{3,17}$$

define completely continuous operators on V_n.

Proof. For any $t', t'' \in \langle 0, 1 \rangle$ and all $s \in \langle 0, 1 \rangle$ we have

$$\begin{aligned} \|\mathbf{K}(s, t') - \mathbf{K}(s, t'')\| &\leqq \|\mathrm{K}(0, t') - \mathbf{K}(0, t'')\| + \\ &+ \|\mathbf{K}(s, t') - \mathbf{K}(0, t') - \mathbf{K}(s, t'') + \mathbf{K}(0, t'')\| \leqq \\ &\leqq |\mathrm{var}_0^{t'} \mathbf{K}(0, .) - \mathrm{var}_0^{t''} \mathbf{K}(0, .)| + v_{\langle 0,1\rangle \times \langle t', t'' \rangle}(\mathbf{K}) \leqq \\ &\leqq |\mathrm{var}_0^{t'} \mathbf{K}(0, .) - \mathrm{var}_0^{t''} \mathbf{K}(0, .)| + |\psi(t') - \psi(t'')| \end{aligned}$$

(cf. (2,15b) for ψ). This implies that the set of discontinuities of $\mathbf{K}(s, t)$ in the variable t lies on a denumerable family of lines parallel to the s-axis: $t = t_l$, $l = 1, 2, \ldots$ since $\mathrm{var}_0^1 \mathbf{K}(0, .)$ and $\psi(t)$ are functions ($\langle 0, 1 \rangle \to R$) of bounded variation. In this way it is possible to rewrite the first expression from (3,17) in the form

$$\sum_{l=1}^{\infty} \Delta_t^+ \mathbf{K}(s, t_l) \, \Delta^+ \mathbf{x}(t_l) \tag{3,18}$$

and similarly the second one.

The operator (3,18) is defined as the limit of the sequence of operators $U_N : V_n \to V_n$ where

$$U_N \mathbf{x} = \sum_{l=1}^{N} \Delta_t^+ \mathbf{K}(s, t_l) \, \Delta_t^+ \mathbf{x}(t_l) \, , \tag{3,19}$$

i.e.

$$U\mathbf{x} = \sum_{l=2}^{\infty} \Delta_t^+ \mathbf{K}(s, t_l) \, \Delta^+ \mathbf{x}(t_l) = \lim_{N \to \infty} U_N \mathbf{x} \, .$$

By Proposition 3,3 for any integer N the operator U_N from (3,19) is completely continuous because U_N is a finite sum of completely continuous operators.

Let us denote $[V_n \to V_n]$ the space of all linear operators acting on V_n, $[V_n \to V_n]$ is a normed linear space with the norm

$$\|U\|_{[V_n \to V_n]} = \sup_{\|\mathbf{x}\|_{V_n} = 1} \|U\mathbf{x}\|_{V_n} \, .$$

The completeness of V_n implies that the space $[V_n \to V_n]$ is complete. Further we have

$$\begin{aligned} \|U_M \mathbf{x} - U_N \mathbf{x}\|_{V_n} &= \Big\| \sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(s, t_1) \, \Delta^+ \mathbf{x}(t_1) \Big\|_{V_n} = \\ = \Big\| \sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(0, t_l) \, \Delta^+ \mathbf{x}(t_l) \Big\| &+ \mathrm{var}_0^1 \sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(., t_l) \, \Delta^+ \mathbf{x}(t_l) \leqq \\ &\leqq \mathrm{var}_0^1 \mathbf{x} \Big(\sum_{l=N+1}^{M} \|\Delta_t^+ \mathbf{K}(0, t_l)\| + \sum_{l=N+1}^{M} \mathrm{var}_0^1 \Delta_t^+ \mathbf{K}(., t_l) \Big) \, . \end{aligned}$$

Hence

$$(3,20)\quad \|U_M - U_N\|_{[V_n \to V_n]} \leqq \sum_{l=N+1}^{M} \|\Delta_t^+ K(0, t_l)\| + \sum_{l=N+1}^{M} \operatorname{var}_0^1 \Delta_t^+ K(., t_l) .$$

The assumption (3,3) yields the convergence of the series $\sum_{l=1}^{\infty} \|\Delta_t^+ K(0, t_l)\|$. Further we have (cf. Corollary 2,3)

$$\operatorname{var}_0^1 \Delta_t^+ K(., t_l) \leqq \psi(t_l+) - \psi(t_l) = \Delta^+ \psi(t_l)$$

where $\psi : \langle 0, 1 \rangle \to R$ is a non-decreasing function, $\psi(0) = 0$, $\psi(1) = v(K)$. Since the series $\sum_{l=1}^{\infty} \Delta^+ \psi(t_l)$ evidently converges we obtain that the series $\sum_{l=1}^{\infty} \operatorname{var}_0^1 \Delta_t^+ K(., t_l)$ converges as well. This implies by (3,20) that U_N, $N = 1, 2, \ldots$ forms a fundamental sequence in the (complete) Banach space $[V_n \to V_n]$ and that $\lim_{N \to \infty} U_N = U$ exists, hence the operator

$$Ux = \sum_{l=1}^{\infty} \Delta_t^+ K(s, t_l)\, \Delta^+ x(t_l) = \sum_{0 \leqq \tau < 1} \Delta_t^+ K(s, \tau)\, \Delta^+ x(\tau)$$

is completely continuous. The proof of complete continuity of the second operator in (3,17) can be carried out in the same manner.

Theorem 3,2. *If* $K(s, t) : I \to L(R^n \to R^n)$ *satisfies* (3,2), (3,3) *and* (3,15) *then the expression*

$$(3,20)\qquad \int_0^1 K(s, t)\, dx(t) = \hat{K}x\,, \quad x \in V_n$$

defines a completely continuous operator on V_n.

Proof. Using the integration by parts formula (2,9) for row vectors of $K(s, t)$ for any $s \in \langle 0, 1 \rangle$ we can write

$$\int_0^1 K(s, t)\, dx(t) = -\int_0^1 d_t[K(s, t)]\, x(t) + K(s, 1)\, x(1) - K(s, 0)\, x(0) -$$
$$- \sum_{0 \leqq \tau < 1} \Delta_t^+ K(s, \tau)\, \Delta^+ x(\tau) + \sum_{0 < \tau \leqq 1} \Delta_t^- K(s, \tau)\, \Delta^- x(\tau) .$$

By Theorem 3,1, Propositions 3,3 and 3,4 we see from this expression that the operator $\hat{K}$ from (3,20) is a linear combination of completely continuous operators and we obtain in this way our Theorem.

Remark 3,2. We note that for the operator $\hat{K} : V_n \to V_n$ given in (3,20) it is possible to derive further analytic properties (continuity, regularity of the result of the operation) if some additional conditions for $K(s, t) : I \to L(R^n \to R^n)$ are assumed. This can be made if we employ the properties of our integral.

4. AUXILIARY STATEMENT FROM FUNCTIONAL ANALYSIS

In this Section we give a simple general statement based on the well known Riesz theory from functional analysis which will be useful in our considerations about Fredholm-Stieltjes integral equations in Section 5.

Let X, Y be normed spaces with the norms $\|.\|_X$, $\|.\|_Y$ respectively. By X', Y' we denote the dual spaces to X, Y respectively. Let a bilinear form $\langle x, y\rangle$ on $X \times Y$ be given which separates points of X, i.e.

(i) for $x \in X$, $x \neq 0$ there exists $y \in Y$ such that $\langle x, y\rangle \neq 0$

and which separates points of Y, i.e.

(ii) for $y \in Y$, $y \neq 0$ there exists $x \in X$ such that $\langle x, y\rangle \neq 0$.

Further we assume that

$$|\langle x, y\rangle| \leqq C\|x\|_X \|y\|_Y \tag{4,1}$$

for any $x \in X$, $y \in Y$ where $C \geqq 0$ is a constant.

For any fixed $y \in Y$ we denote by $[y]$ the linear functional on X which corresponds to $y \in Y$ in terms of the bilinear form $\langle ., .\rangle$; $[y]$ is defined by the relation

$$[y](x) = \langle x, y\rangle\,. \tag{4,2}$$

The inequality (4,1) quarantees the continuity of $[y]$, i.e. we have $[y] \in X'$.

We denote by $[Y]$ the linear set in X' of all continuous linear functionals of the form (4,2).

Since the bilinear form $\langle x, y\rangle$ separates points of Y we have $[y] = 0 \in X'$ ($[y] \in$ $\in [Y]$) if and only if $y = 0$ ($y \in Y$), i.e. $[y] \neq [\tilde{y}]$ if and only if $y \neq \tilde{y}$, $y, \tilde{y} \in Y$.

In this way a one-to-one correspondence between elements of Y and $[Y]$ is given, in other words we have a one-to-one correspondence between the space Y and its immersion $[Y]$ into X' which is given by the bilinear form $\langle x, y\rangle$.

For a set $M \subset Y$ we denote by $[M]$ the set of all elements in X' which are determined by an element in M, i.e.

$$[M] = \{f \in X';\ f = [y],\ y \in M\}\,.$$

In the same way for any fixed $x \in X$ a continuous linear functional $[x] \in Y'$ is given and we have a one-to-one correspondence between X and $[X] \subset Y'$. This follows from the fact that the bilinear form $\langle x, y\rangle$ separates points of X.

For a given operator $\boldsymbol{K} : X \to X$ we denote by $\boldsymbol{K}^* : X' \to X'$ the adjoint operator which is defined by the obvious relation $f(\boldsymbol{K}x) = \boldsymbol{K}^*f(x)$ for $f \in X'$, $x \in X$ (similarly for operators $\boldsymbol{L} : Y \to Y$). If a linear operation $\boldsymbol{T} : X \to X$ is given then $\boldsymbol{T}^{-1}(0)$ means the null – space of this operator, i.e.

$$\boldsymbol{T}^{-1}(0) = \{x \in X;\ \boldsymbol{T}x = 0\}\,.$$

Proposition 4,1. *Let X, Y be normed spaces, $K: X \to X$, $L: Y \to Y$ completely continuous operators in X, Y respectively. Further let $\langle x, y\rangle$ be a bilinear form on $X \times Y$ which separates points of X and Y such that for any $x \in X$, $y \in Y$ the inequality* (4,1) *holds, and let*

$$\langle Kx, y\rangle = \langle x, Ly\rangle \tag{4,3}$$

for any $x \in X$, $y \in Y$.

We denote $T = I_X - K$, $S = I_Y - L$, $T^* = I_{X'} - K^*$, $S^* = I_{Y'} - L^*$ (I_X, I_Y, $I_{X'}$, $I_{Y'}$ are the identity operators in X, Y, X', Y' respectively). Then we have

$$\dim T^{-1}(0) = \dim T^{*-1}(0) = \dim S^{-1}(0) = \dim S^{*-1}(0) = r \tag{4,4}$$

where r is a nonnegative integer (by dim *the dimension of a linear set is denoted) and*

$$T^{*-1}(0) \subset [Y], \quad S^{*-1}(0) \subset [X]. \tag{4,5}$$

Proof. We have (see VIII.2 in [6]): The equation

$$Tx = x - Kx = \tilde{x}, \quad \tilde{x} \in X \tag{4,6}$$

has a solution $x \in X$ if and only if for any solution $f \in X'$ of the equation

$$T^*f = f - K^*f = 0 \tag{4,7}$$

the relation

$$f(\tilde{x}) = 0 \tag{4,8}$$

holds and the dimension of the linear set

$$T^{-1}(0) = \{x \in X;\ Tx = x - Kx = 0\}$$

is finite and equal to the dimension of the linear set

$$T^{*-1}(0) = \{f \in X';\ T^*f = f - K^*f = 0\},$$

i.e. we have

$$\dim T^{-1}(0) = \dim T^{*-1}(0) = r. \tag{4,9}$$

Observe that (4,3) can be written in the form $[y](Kx) = [Ly](x)$ hence we have

$$K^*[y] = [Ly] \tag{4,10}$$

for any functional $[y] \in [Y] \subset X'$. Further by (4,10) it is

$$T^{*-1}(0) \cap [Y] = \{[y] \in [Y];\ T^*[y] = [y] - K^*[y] = [y] - [Ly] =$$
$$= [y - Ly] = 0\} = [\{y \in Y;\ Sy = y - Ly = 0\}] = [S^{-1}(0)]$$

and evidently

$$(4,11) \qquad \dim [S^{-1}(0)] = \dim (T^{*-1}(0) \cap [Y]) = p \leqq r .$$

With respect to the one-to-one correspondence between Y and $[Y]$ we have evidently

$$(4,12) \qquad \dim [S^{-1}(0)] = \dim S^{-1}(0) = p .$$

Since $L : Y \to Y$ is a completely continuous operator, the Riesz theory yields

$$(4,13) \qquad \dim S^{-1}(0) = \dim S^{*-1}(0) = \mathrm{p} .$$

The equation (4,3) can be also rewritten in the form

$$[x](Ly) = [Kx](y)$$

for all $x \in X$, $y \in Y$, i.e. we have

$$(4,14) \qquad L^*[x] = [Kx]$$

for any functional $[x] \in [X] \subset Y'$.

Analoguously as above we obtain by (4,14)

$$S^{*-1}(0) \cap [X] =$$
$$= \{[x] \in [X];\ [x] - L^*[x] = [x] - [Kx] = [x - Kx] = 0\} = [T^{-1}(0)] .$$

Hence (by (4,13)) we have

$$(4,15) \qquad \dim [T^{-1}(0)] = \dim (S^{*-1}(0) \cap [X]) = q \leqq p .$$

Further we have evidently by (4,9)

$$q = \dim [T^{-1}(0)] = \dim T^{-1}(0) = r .$$

From this equality together with (4,11) and (4,15) we obtain $r = p$; thus (4,9) and (4,11) imply

$$\dim T^{*-1}(0) = \dim (T^{*-1}(0) \cap [Y]) = r$$

and hence we have

$$T^{*-1}(0) \subset [Y] .$$

The second relation from (4,5) can be derived similarly.

Proposition 4,1 enables us to derive the following

Theorem 4,1. *Let the assumptions of Proposition* 4,1 *be satisfied. Then either the equation*

$$(4,16) \qquad Tx = x - Kx = \tilde{x} , \quad \tilde{x} \in X$$

admits a unique solution $x \in X$ *for any* $\tilde{x} \in X$, *in particular* $x = 0$ *for* $\tilde{x} = 0$; *or the homogeneous equation*

$$x - Kx = 0 \tag{4,17}$$

admits r linearly independent solutions $x_1, \ldots, x_r$ *in X.*

In the first case the equation

$$Sy = y - Ly = \tilde{y}, \quad \tilde{y} \in Y \tag{4,18}$$

has also a unique solution $y \in Y$ *for any* $\tilde{y} \in Y$. *In the second case the equation*

$$y - Ly = 0 \tag{4,19}$$

admits r linearly independent solutions $y_1, \ldots, y_r$ *in Y. Moreover in the second case the equation* (4,16) *has a solution in X if and only if*

$$\langle \tilde{x}, y \rangle = 0 \tag{4,20}$$

for any solution $y \in Y$ *of* (4,19) *and symmetrically* (4,18) *has a solution in Y if and only if*

$$\langle x, \tilde{y} \rangle = 0 \tag{4,21}$$

for any solution $x \in X$ *of* (4,17).

Proof. The first part of this theorem corresponds to the case when $r = 0$ in (4,4) from Proposition 4,1. The result of this part is a consequence of the well known Riesz theory of completely continuous operators (cf. 11.3 in [4] or Chapter VIII. in [6]).

For the second part of the theorem we have $r > 0$ in Proposition 4,1. Using the duality theory for completely continuous operators in normed spaces (see [6], VIII. 2) we know that (4,16) has a solution if and only if $f(\tilde{x}) = 0$ for any functional $f \in T^{*-1}(0) = \{f \in X'; f - K^*f = 0\}$ (K^* is the adjoint operator to K). From (4,5) we have $T^{*-1}(0) \subset [Y]$ and (4,3) implies $K^*[y] = [Ly]$ for any $[y] \in [Y]$. Hence (4,16) has a solution if and only if $[y](\tilde{x}) = \langle \tilde{x}, y \rangle = 0$ for any $[y]$ from the set

$$\begin{aligned} \{[y] \in [Y];\ [y] - K^*[y] = [y - Ly] = 0\} &= \\ = [\{y \in Y;\ Sy = y - Ly = 0\}] &= [S^{-1}(0)]. \end{aligned}$$

Further evidently $[y] \in [S^{-1}(0)]$ if and only if y is a solution of (4,19) and we obtain in this way the result of the second part of the Theorem for Eq. (4,16). The result for Eq. (4,18) can be derived similarly.

Remark 4,1. The following statement is an easy consequence of Theorem 4,1: If (4,20) for all solutions of (4,19) is satisfied then the general solution $x \in X$ of (4,16) is written as

$$x = \hat{x} + \sum_{l=1}^{r} c_l x_l$$

where $\hat{x}$ is a particular solution of (4,18), $x_1, \ldots, x_r$ are the linearly independent solutions of (4,17) (the base of $T^{-1}(0)$) and $c_1, \ldots, c_r$ are arbitrary constants. A similar statement for the general solution of (4,18) also holds.

5. ALTERNATIVE FOR FREDHOLM-STIELTJES INTEGRAL EQUATIONS IN V_n

We denote by S_n the set of all break functions w in the Banach space space $V_n = V_n(0, 1)$ such that $\Delta w(t) = w(t+) - w(t-) = 0$ for all $t \in (0, 1)$, $\Delta^+ w(0) = \Delta^- w(1) = 0$. Obviously S_n is a linear set in V_n. The set S_n is closed in V_n. In fact, if $w \in V_n$ is an adherent point of S_n then there exists a sequence $w_l \in S_n$, $l = 1, 2, \ldots$ such that $\lim_{l \to \infty} \|w_l - w\|_{V_n} = 0$. For any $t \in (0, 1)$ and $l = 1, 2, \ldots$ we have

$$\|\Delta w(t)\| = \|\Delta w(t) - \Delta w_l(t)\| \leqq \|w_l - w\|_{V_n}$$

thus $\Delta w(t) = 0$. Similarly $\Delta^+ w(0) = \Delta^- w(1) = 0$.

The convergence in V_n implies that w_l converges uniformly on $\langle 0, 1 \rangle$ to w. Let $A \subset \langle 0, 1 \rangle$ be the union of all discontinuity points of w_l, $l = 1, 2, \ldots$ and w; A is a countable set. Any w_l is a constant function in $\langle 0, 1 \rangle - A$. The uniform convergence implies that w is a constant function in $\langle 0, 1 \rangle - A$ and hence we have $w \in S_n$.

Let us consider the quotient space V_n/S_n. An element of V_n/S_n is a class of functions in V_n such that their difference belongs to S_n. Elements of V_n/S_n let be denoted by capitals. The canonical mapping of V_n onto V_n/S_n let be denoted by $\varkappa$; for $\varphi \in V_n$ we have $\varkappa(\varphi) = \varphi + S_n = \Phi \in V_n/S_n$. Any element $\varphi \in V_n$ for which $\varkappa(\varphi) = \Phi$ will be called a representant of the class $\Phi \in V_n/S_n$.

The space V_n/S_n forms a Banach space with the norm

$$\|\Phi\|_{V_n/S_n} = \inf_{\varphi \in \Phi} \|\varphi\|_{V_n} = \inf_{\varkappa(\varphi) = \Phi} \|\varphi\|_{V_n} = \inf_{\varphi \in \Phi} \operatorname{var}_0^1 \varphi \,. \tag{5,1}$$

We have evidently

$$\|\Phi\|_{V_n/S_n} \leqq \operatorname{var}_0^1 \varphi \tag{5,2}$$

for all $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$.

Theorem 5,1. *If $K(s, t) : I = \langle 0, 1 \rangle \times \langle 0, 1 \rangle \to L(R^n \to R^n)$ satisfies $v(K) < +\infty$, $\operatorname{var}_0^1 K(0, .) < +\infty$, $\operatorname{var}_0^1 K(., 0) < +\infty$ then the expression*

$$L\Phi = \varkappa\left(\int_0^1 K'(s, t) \, d\varphi(s)\right), \quad \varphi \in V_n\,, \quad \varkappa(\varphi) = \Phi \tag{5,3}$$

defines a completely continuous linear operator on the Banach space V_n/S_n.

Proof. Let $B(V_n/S_n) = \{\Phi \in V_n/S_n;\ \|\Phi\|_{V_n/S_n} \leqq 1\}$ be the unit ball in V_n/S_n. Let

$$A = \{\Phi \in V_n/S_n;\ \Phi = \varkappa(\varphi),\ \varphi \in V_n,\ \varphi(0) = 0,\ \operatorname{var}_0^1 \varphi \leqq 1\};$$

according to (5,2) it is $B(V_n/S_n) \subset A$. Let $\Phi_l \in B(V_n/S_n)$, $l = 1, 2, \ldots$ then there exist $\varphi_l \in V_n$ with $\operatorname{var}_0^1 \varphi_l \leqq 1$, $\varphi_l(0) = 0$, $l = 1, 2, \ldots$ such that $\varkappa(\varphi_l) = \Phi_l$. The matrix $\mathbf{K}'(s, t) : I \to L(R^n \to R^n)$ satisfies evidently all conditions of Theorem 3,2 hence the operator $\int_0^1 \mathbf{K}'(s, t)\, d\varphi(s)$ is completely continuous in V_n. This implies that there exist $\mathbf{z} \in V_n$ and a subsequence φ_{l_j}, $j = 1, 2, \ldots$ such that

$$\lim_{j \to \infty} \left\| \int_0^1 \mathbf{K}'(s, t)\, d\varphi_{l_j}(s) - \mathbf{z}(t) \right\|_{V_n} = 0$$

i.e.

$$\lim_{j \to \infty} \operatorname{var}_0^1 \left(\int_0^1 \mathbf{K}'(s, t)\, d\varphi_{l_j}(s) - \mathbf{z}(t) \right) = 0 .$$

From (5,2) we have

$$\lim_{j \to \infty} \|\mathbf{L}\Phi_{l_j} - \varkappa(\mathbf{z})\|_{V_n/S_n} \leqq \lim_{j \to \infty} \operatorname{var}_0^1 \left(\int_0^1 \mathbf{K}'(s, t)\, d\varphi_{l_j}(s) - \mathbf{z}(t) \right) = 0$$

and in this way we obtain the complete continuity of $\mathbf{L} : V_n/S_n \to V_n/S_n$.

Let further $\mathbf{x} \in V_n$, $\Phi \in V_n/S_n$. We denote

$$(5,4) \qquad \langle \mathbf{x}, \Phi \rangle = \langle \mathbf{x}, \varphi \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, d\varphi(t)$$

where $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$. The expression $\langle \mathbf{x}, \Phi \rangle$ is independent of the choice of the representant φ of the class Φ. Indeed, if we have $\varphi^\circ \in V_n$, $\varkappa(\varphi^\circ) = \Phi$ then (cf. Remark 2,2)

$$\langle \mathbf{x}, \varphi - \varphi^\circ \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, d(\varphi(t) - \varphi^\circ(t)) = 0$$

because $\varphi - \varphi^\circ \in S_n$. The expression $\langle .,. \rangle$ from (5,4) is a bilinear form on $V_n \times \times V_n/S_n$ and the following lemma holds:

Lemma 5,1. *The bilinear form $\langle .,. \rangle$ from (5,4) separates points of V_n and V_n/S_n (cf. (i) and (ii) in Section 4.).*

Proof. (i) Let $\mathbf{x} \in V_n$, $\mathbf{x} \neq 0$. Then there exists $\alpha \in \langle 0, 1 \rangle$ such that $\mathbf{x}(\alpha) \neq 0$, i.e. there is an index $i = 1, \ldots, n$ such that $x_i(\alpha) \neq 0$. We define $\varphi(t) \in V_n$ as follows: $\varphi_k(t) = 0$ for $t \in \langle 0, 1 \rangle$, $k \neq i$, $\varphi_i(t) = 0$ for $0 \leqq t < \alpha$, $\varphi_i(t) = 1$ for $\alpha \leqq t \leqq 1$ provided $\alpha > 0$; if $\alpha = 0$ then we set $\varphi_i(0) = 1$, $\varphi_i(t) = 0$, $0 < t \leqq 1$. Let us put $\Phi = \varkappa(\varphi)$. Then Proposition 2,1 yields $\langle \mathbf{x}, \Phi \rangle = \int_0^1 x_i(t)\, d\varphi_i(t) = x_i(\alpha) \neq 0$ in the case $\alpha > 0$ and similarly $\langle \mathbf{x}, \Phi \rangle = -x_i(0) \neq 0$ if $\alpha = 0$. Hence $\langle .,. \rangle$ separates points of V_n.

(ii) Let $\boldsymbol{\Phi} \neq 0$ (i.e. $\boldsymbol{\Phi} \neq S_n$). For any $\boldsymbol{\varphi} \in V_n$, $\varkappa(\boldsymbol{\varphi}) = \boldsymbol{\Phi}$ it holds either

1) there exists an $\alpha \in (0, 1)$ such that $\varphi_i(\alpha+) \neq \varphi_i(\alpha-)$ for some $i = 1, 2, \ldots, n$ or

2) for each $\alpha \in (0, 1)$ it is $\boldsymbol{\varphi}(\alpha+) = \boldsymbol{\varphi}(\alpha-)$ and there exist two points $\beta, \gamma \in \langle 0, 1\rangle$, $\beta < \gamma$ such that $\varphi_i(\beta) \neq \varphi_i(\gamma)$ for some $i = 1, \ldots, n$ where β, γ are points of continuity of $\varphi_i(t)$, i.e. $\varphi_i(\beta) = \varphi_i(\beta-)$, $\varphi_i(\gamma) = \varphi_i(\gamma-)$.

In the case 1) we set $x_i(t) = 0$ for $t \in \langle 0, 1\rangle$, $t \neq \alpha$, $x_i(\alpha) = 1$, $x_j(t) = 0$ for $t \in \langle 0, 1\rangle$ if $j \neq i$. Then Corollary 2,1 yields

$$\langle \mathbf{x}, \boldsymbol{\Phi}\rangle = \int_0^1 x_i(t)\,\mathrm{d}\varphi_i(t) = \varphi_i(\alpha+) - \varphi_i(\alpha-) \neq 0\,.$$

In the case 2) it suffices to set $x_i(t) = 1$ for $t \in \langle \beta, \gamma\rangle$, $x_i(t) = 0$ for $t \in \langle 0, 1\rangle - \langle \beta, \gamma\rangle$, $x_j(t) = 0$ for $t \in \langle 0, 1\rangle$, $j \neq i$. Then we obtain from Proposition 2,1

$$\langle \mathbf{x}, \boldsymbol{\Phi}\rangle = \int_0^1 x_i(t)\,\mathrm{d}\varphi_i(t) = \varphi_i(\gamma) - \varphi_i(\beta) \neq 0\,.$$

Hence $\langle .,.\rangle$ separates points of V_n/S_n.

Remark 5,1. Since the bilinear form $\langle .,.\rangle$ separates points of V_n/S_n we can subjoin the following addition to Corollary 2,2: If $g \in V(a, b)$ and $\int_a^b f(t)\,\mathrm{d}g(t) = 0$ for all $f \in V(a, b)$ then necessarily $g \in S(a, b)$, i.e. $\Delta g(t) = 0$ for all $t \in (a, b)$, $\Delta^+ g(a) = \Delta^- g(b) = 0$. This means that if $g \in V(a, b)$ then $\int_a^b f(t)\,\mathrm{d}g(t) = 0$ for every $f \in V(a, b)$ if and only if $g \in S(a, b)$.

Since $\langle \mathbf{x}, \boldsymbol{\Phi}\rangle$ from (5,4) is independent of the choice of $\boldsymbol{\varphi} \in V_n$, $\varkappa(\boldsymbol{\varphi}) = \boldsymbol{\Phi}$ and

$$\left|\int_0^1 \mathbf{x}'(t)\,\mathrm{d}\boldsymbol{\varphi}(t)\right| \leqq \sup_{t\in\langle 0,1\rangle} \|\mathbf{x}'(t)\|\ \mathrm{var}_0^1\,\boldsymbol{\varphi} \leqq n\|\mathbf{x}\|_{V_n}\,\mathrm{var}_0^1\,\boldsymbol{\varphi}$$

holds we have

$$|\langle \mathbf{x}, \boldsymbol{\Phi}\rangle| \leqq n\|\mathbf{x}\|_{V_n} \inf_{\varkappa(\boldsymbol{\varphi})=\boldsymbol{\Phi}} \mathrm{var}_0^1\,\boldsymbol{\varphi} = n\|\mathbf{x}\|_{V_n}\,\|\boldsymbol{\Phi}\|_{V_n/S_n} \tag{5,5}$$

Theorem 5,2. *Let* $\mathbf{K}(s, t) : I = \langle 0, 1\rangle \times \langle 0, 1\rangle \to L(R^n \to R^n)$, $v(\mathbf{K}) < +\infty$, $\mathrm{var}_0^1\,\mathbf{K}(0, .) < +\infty$, $\mathrm{var}_0^1\,\mathbf{K}(., 0) < +\infty$.

Then either the Fredholm-Stieltjes integral equation

$$\mathbf{x}(s) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\,\mathbf{x}(t) = \mathbf{x}^\circ(s)\,, \quad \mathbf{x}^\circ \in V_n \tag{5,6}$$

admits a unique solution for any $\mathbf{x}^\circ \in V_n$ *or the homogeneous equation*

$$\mathbf{x}(s) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\,\mathbf{x}(t) = 0 \tag{5,7}$$

admits r *linearly independent solutions* $\mathbf{x}_1, \ldots, \mathbf{x}_r \in V_n$.

In the first case the equation

$$(5,8) \qquad \varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = \varphi^\circ(t)\,, \quad \varphi^\circ \in V_n$$

has a solution for any $\varphi^\circ \in V_n$ *(this solution is not necessarily unique). In the second case the equation* (5,6) *has a solution in* V_n *if and only if*

$$(5,9) \qquad \langle \mathbf{x}^\circ, \boldsymbol{\varphi} \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, \mathrm{d}\varphi(t) = 0$$

for any solution $\boldsymbol{\varphi} \in V_n$ *of the equation*

$$(5,10) \qquad \varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = 0$$

and symmetrically (5,8) *has a solution if and only if*

$$(5,11) \qquad \langle \mathbf{x}, \varphi^\circ \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, \mathrm{d}\varphi^\circ(t) = 0$$

for any solution $\mathbf{x} \in V_n$ *of the equation* (5,7).

Proof. By Theorems 3,1 and 5,1 the operators

$$\boldsymbol{K}\mathbf{x} = \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) : V_n \to V_n\,, \quad \varkappa(\varphi) = \boldsymbol{\Phi}\,,$$

$$\boldsymbol{L\Phi} = \varkappa\left(\int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s)\right) : V_n/S_n \to V_n/S_n$$

are completely continuous. $\langle .\,, . \rangle$ from (5,4) represents a bilinear form on $V_n \times V_n/S_n$ which separates points of V_n and V_n/S_n (cf. Lemma 5,1) and (5,5) holds.

Further by (2,28) we have

$$\langle \boldsymbol{K}\mathbf{x}, \boldsymbol{\Phi} \rangle = \langle \boldsymbol{K}\mathbf{x}, \varphi \rangle_{(0,1)} = \left\langle \int_0^1 \mathrm{d}_t[\mathbf{K}(.\,, t)]\, \mathrm{d}\mathbf{x}(t), \varphi \right\rangle_{(0,1)} =$$

$$= \left\langle \mathbf{x}, \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s) \right\rangle_{(0,1)} = \left\langle \mathbf{x}, \varkappa\left(\int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s)\right)\right\rangle = \langle \mathbf{x}, \boldsymbol{L\Phi} \rangle$$

for any $\mathbf{x} \in V_n$, $\boldsymbol{\Phi} \in V_n/S_n$. All assumptions of Theorem 4,1 are satisfied and using this Theorem we obtain the first part of Theorem 5,2 viz. (the alternative for Eq. (5,6) resp. (5,7)). Further by Theorem 4,1 the equation

$$(5,12) \qquad \boldsymbol{\Phi} - \boldsymbol{L\Phi} = \boldsymbol{\Phi}^\circ\,, \quad \boldsymbol{\Phi}^\circ \in V_n/S_n$$

has a unique solution for any $\boldsymbol{\Phi}^\circ \in V_n/S_n$. For an arbitrary $\varphi^\circ \in V_n$ we denote $\boldsymbol{\Phi}^\circ = \varkappa(\varphi^\circ) \in V_n/S_n$. Let $\varphi \in V_n$ be a representant of the (unique) solution of (5,12) with this $\boldsymbol{\Phi}^\circ$. Then we have

$$\varkappa\left(\varphi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s)\right) = \varkappa(\varphi^\circ)\,,$$

i.e.

$$\varkappa\left(\varphi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s) - \varphi^\circ)\right) = 0 \in V_n/S_n\,.$$

Hence

$$\varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) - \varphi^\circ(t) = \mathbf{w}(t) \in S_n$$

for all $t \in \langle 0, 1\rangle$. Since $\int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}(\varphi(s) - \mathbf{w}(s))$ we have

$$\varphi(t) - \mathbf{w}(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}(\varphi(s) - \mathbf{w}(s)) = \varphi^\circ(t)$$

for all $t \in \langle 0, 1\rangle$, i.e. the function $\varphi - \mathbf{w} \in V_n$ is a solution of Eq. (5,8). (The unicity of $\mathbf{w} \in S_n$ is not quaranteed.)

For the second case we know by Theorem 4,1 that (5,6 has a solution if and only if for any solution $\boldsymbol{\Phi} \in V_n/S_n$ of the equation

$$\boldsymbol{\Phi} - \boldsymbol{L}\boldsymbol{\Phi} = 0 \tag{5,13}$$

we have $\langle \mathbf{x}^\circ, \boldsymbol{\Phi}\rangle = 0$.

Obviously the following assertion holds: for any solution $\boldsymbol{\Phi} \in V_n/S_n$ of Eq. (5,13) there is a $\varphi \in V_n$, $\varkappa(\varphi) = \boldsymbol{\Phi}$ such that φ is a solution of (5,10). In fact, for any representant $\psi \in V_n$ of $\boldsymbol{\Phi}$ ($\varkappa(\psi) = \boldsymbol{\Phi}$) we have

$$\varkappa\left(\psi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\psi(s)\right) = 0 \in V_n/S_n\,, \quad \text{i.e.}$$

$$\psi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\psi(s) = \mathbf{w}(t) \in S_n\,.$$

If we set $\varphi = \psi - \mathbf{w}$, then $\varkappa(\varphi) = \varkappa(\psi) = \boldsymbol{\Phi}$ and φ is a solution of (5,10). It is easy to prove also the converse statement: If $\varphi \in V_n$ is a solution of (5,10), then $\boldsymbol{\Phi} = \varkappa(\varphi)$ is a solution of (5,13).

Since $\langle \mathbf{x}, \boldsymbol{\Phi}\rangle = \langle \mathbf{x}, \varphi\rangle_{(0,1)}$ is independent of the choice of the representant $\varphi \in V_n$, $\varkappa(\varphi) = \boldsymbol{\Phi}$, we conclude that in the second case (5,6) has a solution if and only if $\langle \mathbf{x}^\circ, \varphi\rangle_{(0,1)} = \int_0^1 \mathbf{x}^{\circ\prime}(t)\, \mathrm{d}\varphi(t) = 0$ for all solutions of Eq. (5,10). The symmetrical statement about Eq. (5,8) can be proved similarly.

Remark 5,2. Theorem 5,2 is a Fredholm type theorem for Fredholm-Stieltjes integral equations (5,6). Let us mention that Eq. (5,8) as well as the equation $\boldsymbol{\Phi} - \boldsymbol{L\Phi} = \boldsymbol{\Phi}^\circ$ are not the adjoint equations to (5,6) in the usual sense. We have not a satisfactory description of the dual space V_n' to V_n which would make it possible to derive the analytic form of the adjoint operator $\boldsymbol{K}^*$. Nevertheless conditions for solvability of Eq. (5,6) are obtained in a form which is closely related to the well known Fredholm alternative for Fredholm integral equations of the second kind in L_2 – spaces.

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