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# OPTIMAL UNIVERSAL APPROXIMATIONS OF FOURIER COEFFICIENTS IN SPACES OF CONTINUOUS PERIODIC FUNCTIONS

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## 1. INTRODUCTION

The computation of the Fourier coefficients, i.e. the integrals

$$I_p(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-ipx}\, dx \tag{1.1}$$

is a problem occurring frequently in practice. As a rule, the problem is numerically solved by successive calculation of the coefficients needed using some quadrature formula (see e.g. [4]). The effort to lower the amount of calculations in the case of the computation of a large number of the integrals (1.1) has resulted in methods based on the approximation of $I_p(f)$ for a given function $f$ by an expression of the type

$$\sum_{k=1}^{n} a_k(f)\, g_k(p) \tag{1.2}$$

and on the successive substitution of the values of $p$ [7]. The point of the procedure is that the number of the terms of this expression (and also of the functionals $a_k(f)$) is lower than that of the Fourier coefficients computed. In this way, the number of the evaluated functionals is reduced.

The question is which method is the most suitable one for solving the problem described. Recently, great attention has been paid to the optimal approximations of linear functionals (see e.g. [3], [5]). Especially, BABUŠKA [1], [2] has dealt with the optimal quadrature formulae for the computation of the Fourier coefficients. It is necessary to keep in mind that the question of optimality is a rather ambiguous one. Namely, the optimality problem is always studied relatively, i.e. with respect to a given functional space. The optimal approximation and its error depend on this space naturally and, as shown in [2], this dependence can be very strong. The available information on $f(x)$ does not allow us usually to determine a definite space where we could choose from a given class of approximations the optimal one. This implies the importance of finding so called universal approximations. These are the approximations the error of which does not differ "too much" (in a precisely defined manner) from that of the optimal approximation in a wide class of spaces. The universal approximation need not be optimal in any space from the class given, but it provides us with the desirable independence of the choice of the numerical method on the choice of the space.

In this paper, we shall treat the universal approximations of the integrals (1.1) supposing that we want to calculate their values for a given function $f$ and a set $P$ of subscripts $p$. We shall assume the function $f$ to belong to a Babuška-introduced space of continuous $2\pi$-periodic functions. We shall give our attention to the methods of the type (1.2) where we shall study the problem of the choice of the functions $g_k(p)$, which is of decisive importance for the universality of these methods. For our considerations, the choice of the functionals $a_k = a_k(f)$ is not relevant though we shall touch it in some discussions.

The paper is divided into six sections, the introduction being the first one.

The properties of Hilbert spaces of periodic functions the Fourier coefficients of which are to be computed are summarized in the second section. Sec. 3, which has rather auxiliary character, describes some ways of the approximation of the integrals (1.1) by the expressions of the type (1.2). Some of the given approximations are, as shown in the following sections, of practical importance; others are used as examples and as counterexamples in the proofs of the theorems in the following sections.

The theoretical basis of the paper is the fourth section where the criterion of the optimality for our problem is formulated and a lower bound on the error of the optimal approximation is derived. This bound is used frequently later. The concept of the universal approximation is introduced here, too. The study of the universality is the subject of Sec. 5. The classes of spaces in which universal approximations do not exist are given. An approximation universal with respect to a wide class of the spaces of periodic functions is constructed. Further, the necessary conditions which the functions $g_k(p)$ must satisfy for (1.2) to be a universal approximation are derived and the optimal approximations are studied in the class of universal approximations.

The sixth section surveys the practical aspects of the results proved in the paper.

## 2. THE SPACES OF PERIODIC FUNCTIONS

In this section, we describe to the necessary extent the Hilbert spaces of continuous $2\pi$-periodic functions used in our considerations. A function is meant as a complex-valued function of one real argument in this paper unless otherwise stated. The properties of the periodic spaces will not be proved, the theorems of this section being particular cases of the theorems contained in [6].

First, let us define a general periodic space.

**Definition 2.1.** A Hilbert space $H$ the elements of which are continuous $2\pi$-periodic functions is called *periodic* if the following conditions are satisfied:

(a) For all $f \in H$,

$$\|f\|_C \leqq B(H) \cdot \|f\|$$

where $\|\cdot\|_C$ is the usual norm in the space $C_{2\pi}$ of continuous $2\pi$-periodic functions and the constant $B(H)$ does not depend on $f$.

(b) Let $f \in H$; then $g(x) = f(x + c) \in H$ for every real number $c$ and $\|f\| = \|g\|$.

A trivial example of a periodic space is the linear space of all the functions of the form

$$ae^{ikx}$$

where $a$ is an arbitrary complex number and $k$ is a fixed integer. The scalar product is defined as

$$(a_1 e^{ikx}, a_2 e^{ikx}) = a_1 \bar{a}_2 .$$

Thus the functions $e^{ikx}$, $k$ integer, can be elements of the periodic spaces. Therefore, we introduce a set $A_H$ of integers for every periodic space $H$ describing those functions $e^{ikx}$ which are in $H$.

**Definition 2.2.** Let $H$ be a periodic space. An integer $k$ is said to belong to the set $A_H$ if $e^{ikx} \in H$.

The basic structure of periodic spaces is characterized by

**Theorem 2.1.** *Let $H$ be a periodic space which does not consist only of the zero function. Then $A_H \neq \emptyset$ and the system $\{e^{ikx}\}$, $k \in A_H$ forms an orthogonal basis of the space $H$. Furthermore,*

$$\sum_{k \in A_H} \|e^{ikx}\|^{-2} < \infty . \tag{2.1}$$

It will prove useful to introduce a convenient notation by the following

**Definition 2.3.** Let $H$ be a periodic space, $k \in A_H$. We denote

$$\eta_k = \|e^{ikx}\| .$$

We shall make use also of the following theorem justifying the rearrangements of Fourier series and the interchanges of limits.

**Theorem 2.2.** *Let $H$ be a periodic space, $f \in H$. Then the Fourier expansion of the function $f(x)$ converges absolutely and uniformly with respect to $x$.*

Theorem 2.1 asserts that each non-trivial periodic space determines a sequence $\eta = \{\eta_k\}_{k \in A_H}$ of positive numbers satisfying (2.1). It may be shown that there exists even a one-to-one correspondence between the class of periodic spaces and the class of such sequences.

**Theorem 2.3.** *Let $A \neq \emptyset$ be a set of integers. Let $\eta_k > 0$, $k \in A$ be real numbers satisfying the condition*

$$\sum_{k \in A} \eta_k^{-2} < \infty . \tag{2.2}$$

*Then there exists a periodic space $H$ such that $e^{ikx} \in H$, $k \in A$ and $\|e^{ikx}\| = \eta_k$. This space is the completion of the linear hull generated by $e^{ikx}$, $k \in A$ with respect to the scalar product*

$$\begin{aligned} (e^{ikx}, e^{isx}) &= 0 , \quad k, s \in A , \quad k \neq s , \\ (e^{ikx}, e^{ikx}) &= \eta_k^2 , \quad k \in A . \end{aligned} \tag{2.3}$$

The mutual relation of the norm of an element $f \in H$ where $H$ is a periodic space, and its Fourier coefficients is given by Parseval's identity having the form

$$\|f\|^2 = \sum_{k \in A_H} |I_k(f)|^2 \eta_k^2 \tag{2.4}$$

in our notation.

In Sec. 5, we shall prove that for the class of periodic spaces no universal approximation exists in the set of approximating functionals that will be under consideration. Therefore, we introduce the subclass of strongly periodic spaces the properties of which are sufficient for the universal approximation to exist. Our definition of the strongly periodic space differs somewhat from that of Babuška [2], [6]. We omit namely one condition regarding the character of the increase of $\eta_k$ as $k \to \infty$ which is not necessary for the considerations of this paper.

**Definition 2.4.** A periodic space $H$ is said to be *strongly periodic* if the following conditions are satisfied:

(c) $e^{ikx} \in H$ for all integers $k$ and $\eta_k = \eta_{-k}$.
(d) $\eta_k \geqq \eta_j$ for $|k| \geqq |j|$, $k, j$ integers.

When referring to the conditions (a)–(d) from Definitions 2.1, 2.4 in what follows we shall denote them only by the corresponding letters. Note that Definition 2.4 implies that $A_H$ is the set of all integers for strongly periodic spaces.

Examples of strongly periodic spaces are the periodic spaces $H_\gamma$ [2] with the scalar products

$$(g, h)_\gamma = \sum_{j=0}^{\infty} \gamma_j \int_0^{2\pi} g^{(j)}(x) \overline{h^{(j)}(x)} \, dx$$

where $\gamma_0, \gamma_1, \ldots$ are real numbers satisfying

(1) $\gamma_j \geqq 0$ for all integers $j$, $\gamma_0 > 0$;
(2) there exists $j_0 > 0$ such that $\gamma_{j_0} \neq 0$;
(3) $\lim_{j \to \infty} \gamma_j^{1/j} = 0$.

## 3. THE APPROXIMATIONS OF FOURIER COEFFICIENTS IN PERIODIC SPACES

Let $H$ be a periodic space, $f \in H$. Let us assume that we want to calculate the values of $I_p(f)$ for all $p$ from a given set $P$ of integers. Let $P$ consist of $r$ distinct elements. (This assumption refers to the whole paper and will not be repeated afterwards.)

We shall study $r$-tuples of the approximating functionals of the form

$$(3.1) \qquad \{G_p\}_{p \in P}, \quad G_p(f) = \sum_{k=1}^{n} a_k(f)\, g_k(p), \quad n \geqq 1 .$$

We suppose $g_k(p)$, $k = 1, 2, \ldots, n$ to be complex-valued functions of an integer argument $p$ defined on $P$. For given $n$ and $P$ we construct the set of all $\{G_p\}$ such that $a_k(f)$ are bounded complex functionals defined on $H$. This set will be denoted by $M_n(P)$. The elements of the set $M_n(P)$ will be called "approximations" shortly.

Remark 3.1. A *bounded functional* $G$ means here a functional with the property that there exists a constant $K$ such that

$$|G(f)| \leqq K\|f\|$$

is valid for all $f \in H$. The smallest constant $K$ of this kind will be called the *norm* of the functional $G$ and designated by $\|G\|$. For linear functionals, this definition coincides with the usual definition of the norm. If we need to emphasize that the norm of $f$ is considered just in the space $H$ we use the notation $\|f\|_H$. Similarly, in order to emphasize that the norm of a functional $G$ is taken just over the space $H$ we use the symbol $\|G\|_H$.

In the numerical methods, $a_k(f)$ are usually linear combinations of the values of $f$ (and possibly also of its derivatives) in certain points of the interval $[0, 2\pi]$. Therefore, we shall pay attention to the asymptotic properties of the sequences of approximations

$$(3.2) \qquad \{G_p^{(l)}\}, \quad p \in P, \quad l = 1, 2, \ldots, \quad \text{where} \quad G_p^{(l)}(f) = \sum_{k=1}^{n} a_k^{(l)}(f)\, g_k(p)$$

and define the set $\tilde{M}_n(P)$ of approximations analogously to $M_n(P)$, but with

$$(3.3) \qquad a_k^{(l)}(f) = \sum_{s=1}^{j(l)} \alpha_{s,j}^{(k)} f(x_{s,j}^{(k)})$$

where $x_{s,j}^{(k)} \in [0, 2\pi]$ and $\alpha_{s,j}^{(k)}$ are complex numbers. The results obtained for the approximations from $M_n$ are related in a very simple manner to the corresponding asymptotic statements for the sequences of approximations from $\tilde{M}_n$. Moreover, the fact that we have confined ourselves in (3.3) only to the linear combinations of the values of $f$ is, as the reader will find, not substantial for the considerations of this paper. Our considerations will be concerned primarily with the properties of the functions $g_k(p)$. The distinction between $M_n$ and $\tilde{M}_n$ is made rather for a better insight and for the practical applications of the results of the paper.

The functionals $a_k^{(l)}(f)$ defined by (3.3) are linear. Their additivity and homogeneity is obvious. The boundedness follows from the boundedness of the functionals in question on $C_{2\pi}$ and from the continuous imbedding of $H$ into $C_{2\pi}$ stated in (a). The boundedness of the above $a_k^{(l)}(f)$ implies $\tilde{M}_n(P) \subset M_n(P)$.

Each approximation from $M_n(P)$ is assigned an $r$-tuple of the error functionals (also "the error of the approximation" in what follows) defined as

$$J_p(f) = I_p(f) - G_p(f) \tag{3.4}$$

where $f \in H$ and $p \in P$.

It will be necessary for the further study to know the errors of some simple approximations. Thus, in this section we shall prove some statements that will be used mostly as examples and counterexamples. Their proofs are based on the Riesz representation theorem.

**Theorem 3.1.** *Let $H$ be a periodic space. The functionals $I_p$, $p$ integer, are linear on $H$. If $p \notin A_H$ then $I_p(f) = 0$ for all $f \in H$ (the null functional).*

Proof. Obviously, $I_p$ is additive and homogeneous. For $p \in A_H$, (2.4) implies

$$|I_p(f)| \leqq \frac{1}{\eta_p} \|f\|$$

for all $f \in H$, which proves the boundedness in this case. For $p \notin A_H$, the statement of the theorem is obtained through a simple calculation employing Theorem 2.2.

Now we may calculate the norm of the functional $I_p$.

**Theorem 3.2.** *Let $H$ be a periodic space, $p$ integer. Then*

$$\|I_p\| = \frac{1}{\eta_p} \quad \text{for} \quad p \in A_H\,,$$
$$\qquad 0 \quad \text{for} \quad p \notin A_H\,.$$

Proof. Theorem 3.1 justifies using the Riesz representation theorem according to which for $f \in H$

$$I_p(f) = (f, F_p)\,, \quad \|I_p\| = \|F_p\|\,. \tag{3.5}$$

It is easy to verify that

$$F_p(x) = \frac{1}{\eta_p^2}\, e^{ipx} \quad \text{for} \quad p \in A_H\,, \tag{3.6}$$
$$\qquad 0 \qquad \text{for} \quad p \notin A_H\,.$$

The statement of the theorem follows immediately through calculating $\|F_p\|$ from the relation

$$\|F_p\|^2 = (F_p, F_p)\,.$$

In the following sections we shall also use the functionals

$$K_s = g(s)\,(\alpha I_p + \beta I_q) \tag{3.7}$$

where $\alpha$, $\beta$ are real numbers, $s$, $p$, $q$ are integers, $p \neq q$ and $g(s)$ is a real function of an integer argument $s$. We shall take them for approximations of $I_s$. The point of this way of approximation is roughly as follows: We know (for a given $f \in H$) the value of some linear combination of two Fourier coefficients. We approximate the Fourier coefficients of $f$ by this fixed linear combination in such a way that for the coefficient $I_s$ we multiply the value of the above combination by the number $g(s)$, which depends on $s$ but not on $f$. Obviously $\{K_s\} \in M_1$.

When studying the approximations of $I_s$ we shall limit ourselves to the case where $s \in A_H$ (for $s \notin A_H$, $I_s$ is the null functional). The norm of the error of the functional $K_s$ is given by

**Theorem 3.3.** *Let $H$ be a periodic space, $s, p, q \in A_H$, $p \neq q$. Denote*

$$\gamma(s) = g^2(s)\left(\frac{\alpha^2}{\eta_p^2} + \frac{\beta^2}{\eta_q^2}\right) + \frac{1}{\eta_s^2}\,.$$

*Then*

$$\|I_s - K_s\|^2 = \begin{cases} \gamma(s) & \text{for} \quad s \neq p, q\,, \\ \gamma(p) - 2g(p)\dfrac{\alpha}{\eta_p^2} & \text{for} \quad s = p\,, \\ \gamma(q) - 2g(q)\dfrac{\beta}{\eta_q^2} & \text{for} \quad s = q\,. \end{cases}$$

Proof. Clearly, $K_s$ is linear and we can use the Riesz representation theorem. Using (3.5) and (3.6), we get for $s, p, q \in A_H$ and $f \in H$

$$K_s(f) = (f, E_s) \tag{3.8}$$

where

$$E_s(x) = g(s)\left(\frac{\alpha}{\eta_p^2}\,e^{ipx} + \frac{\beta}{\eta_q^2}\,e^{iqx}\right). \tag{3.9}$$

Similarly

$$I_s(f) - K_s(f) = (f, \Phi_s)$$

and using (3.5), (3.6) and (3.8) we get

$$\Phi_s(x) = \frac{e^{isx}}{\eta_s^2} - E_s(x)\,.$$

The norm of $I_s - K_s$ is obtained again from

$$\|I_s - K_s\|^2 = (\Phi_s, \Phi_s)\,.$$

The theorem is proved.

It is possible to prove analogous statements regarding the trapezoidal rule with $j$ equally spaced abscissae, i.e. the functional

$$L_p^{(j)}(f) = \frac{1}{j}\sum_{k=1}^{j} e^{-i(2\pi/j)kp} f\left(\frac{2\pi}{j}k\right). \tag{3.10}$$

Before so doing, however, we introduce a convenient notation.

**Definition 3.1.** Let $H$ be a periodic space, $p \in A_H$. Let $j$ be a positive integer. We denote

$$C(j, p, \eta) = \big(\eta_p^2 \sum_{\substack{t=-\infty \\ p-tj\in A_H}}^{+\infty} \eta_{p-tj}^{-2}\big)\,. \tag{3.11}$$

The series in the above definition converges by virtue of Theorem 2.1. The quantity $C(j, p, \eta)$ appears in the expressions for the errors of the approximations using the trapezoidal rule. First we shall investigate its asymptotic behaviour as $j \to \infty$.

**Lemma 3.1.** *Let $H$ be a periodic space, $p \in A_H$. Then*

$$\lim_{j\to\infty} \sum_{\substack{t;p-tj\in A_H \\ t\neq 0}} \eta_{p-tj}^{-2} = 0 \tag{3.12}$$

*and*

$$\lim_{j\to\infty} C(j, p, \eta) = 1\,. \tag{3.13}$$

Proof. Denote the sum occurring on the left-hand side of (3.12) by $\Sigma'$. Let $p \in A_H$ be given. We wish to prove that for every $\varepsilon > 0$ there exists an integer $j_0$ such that

$$\sum{}' \frac{1}{\eta_{p-tj}^2} < \varepsilon$$

is valid for all $j \geqq j_0$.

For arbitrary $\varepsilon > 0$, Theorem 2.1 implies the existence of $K$ such that

$$\sum_{\substack{|k|>K \\ k\in A_H}} \frac{1}{\eta_k^2} < \varepsilon\,. \tag{3.14}$$

Clearly, for this $K$ we may find $j_0$ such that

$$|p - tj| > K$$

is valid for all $j \geqq j_0$ and all $t \neq 0$. Thus for $j \geqq j_0$

$$\sum' \frac{1}{\eta_{p-tj}^2} = \sum_{\substack{t;p-tj\in A_H \\ |p-tj|>K}} \frac{1}{\eta_{p-tj}^2} \leqq \sum_{\substack{|k|>K \\ k\in A_H}} \frac{1}{\eta_k^2} .$$

Using (3.14) now, we get the first statement of the lemma. The second one follows immediately from (3.12) if we write

$$C(j, p, \boldsymbol{\eta}) = \left(1 + \eta_p^2 \sum' \frac{1}{\eta_{p-tj}^2}\right)^{-1} .$$

The lemma is proved.

Given $p$ and $j$, we find now the norm of the error functional of the trapezoidal rule.

**Theorem 3.4.** *Let $H$ be a periodic space, $p \in A_H$. Let $j$ be a positive integer. Then*

$$\|I_p - L_p^{(j)}\|^2 = \eta_p^{-2} \frac{1 - C(j, p, \boldsymbol{\eta})}{C(j, p, \boldsymbol{\eta})} . \tag{3.15}$$

*Thus, as $j \to \infty$, $L_p^{(j)}$ converges to $I_p$ in the norm.*

Proof is exactly parallel to that of Theorem 3.2 in [2] where the author assumes $A_H$ to be the set of all integers. According to the Riesz representation theorem, for $p \in A_H$ and all $f \in H$

$$I_p(f) - L_p^{(j)}(f) = (f, \Phi_p^{(j)}) ,$$

where

$$\Phi_p^{(j)}(x) = \frac{e^{ipx}}{\eta_p^2} - \sum_{\substack{t=-\infty \\ p-tj\in A_H}}^{+\infty} \frac{e^{i(p-tj)x}}{\eta_{p-tj}^2} .$$

The norm of the error functional is now obtained from

$$\|I_p - L_p^{(j)}\|^2 = (\Phi_p^{(j)}, \Phi_p^{(j)}) .$$

The convergence of $L_p^{(j)}$ follows immediately from (3.15) and Lemma 3.1, and the theorem is proved.

If we take care of the trapezoidal rule, the analogue of the functional (3.7) is the functional

$$N_s^{(j)} = g(s)(\alpha L_p^{(j)} + \beta L_q^{(j)}) \tag{3.16}$$

where $\alpha$, $\beta$ are real numbers, $s$, $p$, $q$ are integers, $j$ is a positive integer and $g(s)$ is a real function of an integer argument. Clearly $\{N_s^{(j)}\} \in \tilde{M}_1$. Since we shall be concerned only with the asymptotic properties of the functionals (3.16) as $j \to \infty$, our interest here is not in deriving the formula for the error in case of a given $j$, but rather in finding the limit of the norm of the error functional.

**Theorem 3.5.** *Let $H$ be a periodic space, $p, q, s \in A_H$, $p \neq q$. Let there be given real numbers $\alpha$, $\beta$, a function $g(s)$ and the functionals $K_s$ and $N_s^{(j)}$ according to* (3.7) *and* (3.16). *Then*

$$\lim_{j \to \infty} \|I_s - N_s^{(j)}\| = \|I_s - K_s\| .$$

Proof follows immediately from (3.7), (3.16) and Theorem 3.4.

In the remainder of this section, we shall consider another approximation connected with the following problem. Suppose we know the values of $I_p(f)$ for $p \in P_1$ where $P_1$ is a subset of $P$. Now, the question is how to approximate $I_p$ for $p \in P - P_1$. The reader will see that the approximation by zero functionals for $p \in P - P_1$ will play an important role in the study of universality.

We give a precise formulation restricting ourselves to the cases where $P \subset A_H$. For each positive integer $n$, $n < r$ we define the approximation $\{B_p\} \in M_n$ as follows: We divide $P$ into two disjoint sets, $P = P_1 \cup P_2$, such that $P_1$ has $n$ elements. We write $P_1 = \{p_1, p_2, \ldots, p_n\}$, $P_2 = \{p_{n+1}, p_{n+2}, \ldots, p_n\}$. Setting

$$(3.17) \qquad \begin{aligned} a_k &= I_{p_k}, \quad k = 1, 2, \ldots, n ; \\ g_k(p_j) &= 1 \quad \text{for} \quad k = j , \\ &\phantom{=}\ 0 \quad \text{for} \quad k \neq j , \quad j = 1, 2, \ldots, r \end{aligned}$$

in (3.1), we have clearly

$$\begin{aligned} B_{p_j} &= I_{p_j}, \quad j = 1, 2, \ldots, n ; \\ &\phantom{=}\ O , \quad j = n + 1, n + 2, \ldots, r \end{aligned}$$

where $O$ denotes the null functional, and

$$(3.18) \qquad \max_{p \in P} \|I_p - B_p\| = \max_{p \in P_2} \|I_p - B_p\| = \max_{p \in P_2} \frac{1}{\eta_p} .$$

Setting

$$(3.19) \qquad a_k = L_{p_k}^{(j)}, \quad k = 1, 2, \ldots, n ,$$

we obtain approximations $\{B_p^{(j)}\} \in \tilde{M}_n$, which form an analogue of $\{B_p\}$ important in practice.

**Theorem 3.6.** *Let $H$ be a periodic space, $P \subset A_H$. Let $n$ be a positive integer, $n > r$. Then there exists an integer $j_0$ such that*

$$\max_{p \in P} \|I_p - B_p^{(j)}\| = \max_{p \in P} \|I_p - B_p\| = \max_{p \in P_2} \frac{1}{\eta_p} \tag{3.20}$$

*for every $j \geqq j_0$.*

Proof follows immediately from (3.18) using Theorem 3.4 to find the number $j_0$ such that

$$\max_{p \in P_1} \|I_p - B_p^{(j)}\| = \max_{p \in P_1} \|I_p - L_p^{(j)}\| \leqq \max_{p \in P_2} \frac{1}{\eta_p}$$

for every $j \geqq j_0$.

Theorems 3.1 to 3.6 provide us with the necessary auxiliary material. Now we can pass to discussing the questions regarding the choice of the method.

## 4. THE OPTIMAL APPROXIMATIONS IN PERIODIC SPACES

Let there be given a periodic space $H$ and the set $P$. Let $\{G_p\} \in M_n$ for a positive integer $n$. As a criterion for judging the quality of this approximation we shall use the quantity

$$\omega_H(P, G_p) = \max_{p \in P} \|J_p\|_H \tag{4.1}$$

where $J_p$, $p \in P$ are the error functionals (3.2). (The boundedness of $J_p$ follows from that of $I_p$ and $a_k$, $k = 1, 2, \ldots, n$.) Now let $M \subset M_n$ for some $n$. Then the best possible approximation from $M$ is, for the given $H$ and $P$, characterized by the quantity

$$\Omega_H(M, P) = \inf_M \omega_H(P, G_p) . \tag{4.2}$$

Further, we denote

$$\Omega_H(n, P) \equiv \Omega_H(M_n, P) , \tag{4.3}$$

$$\tilde{\Omega}_H(n, P) \equiv \Omega_H(\tilde{M}_n, P) .$$

Obviously, $\Omega_H(M, P) \geqq \Omega_H(n, P)$ for any $M \subset M_n$.

**Definition 4.1.** Let $n$ be a positive integer, $M \subset M_n$. The approximation $\{G_p\}_{p \in P} \in M$ is said to be an *optimal approximation* from the set $M$ and for given $P$ and $H$ if

$$\omega_H(P, G_p) = \Omega_H(M, P) .$$

For the sequences of approximations from $\tilde{M}_n$ the notion of asymptotical optimality is defined. We point out that we treat only such sequences that $g_k(p)$, $k = 1, 2, \ldots, n$ are the same for all the elements of a given sequence (cf. (3.2)). Clearly, this corresponds to the practical meaning of our considerations.

**Definition 4.2.** Let $n$ be a positive integer, $M \subset \tilde{M}_n$. The sequence of approximations $\{G_p^{(j)}\}_{p \in P} \in M, j = 1, 2, \ldots$ is said to be *asymptotically optimal* if

$$\lim_{j \to \infty} \omega_H(P, G_p^{(j)}) = \Omega_H(M, P) .$$

Since for every other sequence $\{F_p^{(j)}\}_{p \in P} \in M, j = 1, 2, \ldots$

$$\liminf_{j \to \infty} \omega_H(P, F_p^{(j)}) \geqq \Omega_H(M, P) ,$$

$\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ will be also called an asymptotically optimal sequence in the class of the sequences of approximations from $M$.

Denote the number of the elements of the set $P \cap A_H$ by $m$ and suppose $m > 1$ in what follows. As indicated in the introduction we confine the considerations of the paper to the approximations from $M_n$ with $n < m$.

Remark 4.1. For $n \geqq m$, the whole problem has diverse character, since in this case also $\{I_p\}$ belongs to $M_n$. As far as the set $\tilde{M}_n$ is concerned the problem of finding the optimal approximation for a fixed $j$ may be transferred to that solved in [2]. Moreover, the results of [2] may be applied for the study of further aspects (asymptotical optimality, universality).

Now, for the reader's convenience, we sum up the assumptions made on the space $H$ and the set $P$ and regarding the remainder of the paper:

(4.4) (1) $H$ is a periodic space (which does not consist only of the zero function).
(2) The elements of $P$ are distinct; their number is denoted by $r$.
(3) $P \cap A_H$ has $m$ elements, $m > 1$.
(4) $0 < n < m$, $n$ integer.

Under these assumptions we show in Theorem 4.1 that $\Omega_H(n, P) > 0$. Therefore, for each approximation from $M \subset M_n$ we can form the quotient

$$Q_H(M, P, G_p) = \frac{\omega_H(P, G_p)}{\Omega_H(M, P)} \tag{4.5}$$

and say that the approximation $\{G_p\}$ is the "better" in a given space and with respect to the set $M$, the smaller $Q_H(M, P, G_p)$ is. Clearly $Q_H \geqq 1$. Analogously to (4.3) we shall use the notation $Q_H(n, P, G_p)$ and $\tilde{Q}_H(n, P, G_p)$. The subscript $H$ will be used only where it is essential for understanding the text.

In deriving the upper bounds for the quantity $Q$ we shall need a lower bound on $\Omega$. To find the latter we shall use

**Lemma 4.1.** *Let $H$ be a periodic space with the zero element $\Theta$. Let the set $P$ and a positive integer $n$ be given. If for all $f \in H$ and all approximations $\{G_p\} \in M_n$*

$$\inf_{a_k} \max_{P} |I_p(f) - G_p(f)| \geqq C_H(f, g_1, g_2, \ldots, g_n) \tag{4.6}$$

*is valid, then*

$$\Omega_H(n, P) \geqq \inf_{g_k} \sup_{\substack{f \in H \\ f \neq \Theta}} \left( \frac{C_H(f, g_1, g_2, \ldots, g_n)}{\|f\|} \right). \tag{4.7}$$

Proof. The inequality

$$\|J_p\| \geqq \frac{|J_p(f)|}{\|f\|},$$

where $J_p$, $p \in P$ are the error functionals, holds clearly for all approximations from the set $M_n$, all $p \in P$ and all functions $f \in H, f \neq \Theta$.

Therefore,

$$\max_{P} \|J_p\| \geqq \|f\|^{-1} \max_{P} |J_p(f)|$$

for all $f \in H, f \neq \Theta$ and all $\{G_p\} \in M_n$. Moreover, obviously

$$\max_{P} \|J_p\| \geqq \|f\|^{-1} \inf_{a_k} \max_{P} |J_p(f)|.$$

Using (4.6) we get now

$$\max_{P} \|J_p\| \geqq \|f\|^{-1} C_H(f, g_1, g_2, \ldots, g_n)$$

for all $f \in H, f \neq \Theta$ and all $\{G_p\} \in M_n$, where the right-hand side is independent of the functionals $a_k$.

Further, from this inequality we obtain

$$\max_{P} \|J_p\| \geqq \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, g_2, \ldots, g_n)) \tag{4.8}$$

for all $\{G_p\} \in M_n$ and finally

$$\inf_{M_n} \max_{P} \|J_p\| \geqq \inf_{M_n} \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, \ldots, g_n)) =$$
$$= \inf_{g_k} \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, \ldots, g_n)),$$

which completes the proof.

Now we are in position to derive the lower bound for $\Omega(n, P)$.

**Theorem 4.1.** *Let $H$ be a periodic space. Given the set $P$, denote $R = P \cap A_H$ and let $\{p_1, p_2, \ldots, p_{n+1}\}$ be an $(n + 1)$-tuple of integers $p_j \in R$. Then*

$$\Omega(n, P) \geqq \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} . \tag{4.9}$$

Proof. Our basic tool here will be Lemma 4.1. Therefore, we need a lower bound on

$$\inf_{a_k} \max_{P} |J_p(f)| = \inf_{a_k} \max_{P} |I_p(f) - \sum_{k=1}^{n} a_k(f)\, g_k(p)|$$

satisfying the hypothesis of the above lemma.

We choose some $(n + 1)$-tuple $\{p_1, p_2, \ldots, p_{n+1}\}$ of numbers $p_j \in R$ and denote

$$g_k(p_j) = g_{kj}\,, \quad I_{p_j}(f) = \hat{I}_j(f)\,,$$

and

$$J_{p_j}(f) = \hat{J}_j(f)\,,$$

$k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n + 1$. We note that

$$\max_{p \in P} |J_p(f)| \geqq \max_{p \in R} |J_p(f)| \geqq \max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)| . \tag{4.10}$$

Now, let us pay attention to $n$-dimensional vectors

$$[g_{1j}, g_{2j}, \ldots, g_{nj}]\,, \quad j = 1, 2, \ldots, n + 1 .$$

Since there is $(n + 1)$ of them, they are linearly dependent and thus we can find numbers $\lambda_j$, $j = 1, 2, \ldots, n + 1$ such that

$$\sum_{j=1}^{n+1} \lambda_j g_{kj} = 0\,, \quad k = 1, 2, \ldots, n \tag{4.11}$$

and

$$\sum_{j=1}^{n+1} |\lambda_j| = 1 . \tag{4.12}$$

For these $\lambda_j$'s we calculate

$$\sum_{j=1}^{n+1} \lambda_j \hat{J}_j(f) = \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f) - \sum_{j=1}^{n+1} \lambda_j \left(\sum_{k=1}^{n} a_k(f)\, g_{kj}\right) =$$
$$= \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f) - \sum_{k=1}^{n} a_k(f) \left(\sum_{j=1}^{n+1} \lambda_j g_{kj}\right) .$$

By (4.11), the second term on the right-hand side is zero and we have

$$\sum_{j=1}^{n+1} \lambda_j \hat{J}_j(f) = \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)$$

for every $f \in H$.

For the absolute values we obtain

$$|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)| \leqq \sum_{j=1}^{n+1} |\lambda_j| \, |\hat{J}_j(f)| \leqq \sum_{j=1}^{n+1} |\lambda_j| \max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)| .$$

Substituting from (4.12), we have

$$\max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)| \geqq |\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)| .$$

The $\lambda_j$'s depend only on $g_k(p_j)$. Therefore, the right-hand side of the above inequality is independent of $a_k(f)$ and we get, using (4.10),

$$\inf_{a_k} \max_{P} |J_p(f)| \geqq |\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f))| .$$

This inequality holds for all $f \in H$ and all approximations $\{G_p\} \in M_n$ and satisfies the hypothesis of Lemma 4.1.

Using (4.7) we shall need to know

$$S \equiv \sup_{f \in H, f \neq \theta} \frac{|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)|}{\|f\|} .$$

According to (3.5) and (3.6)

$$S = \sup_{f \in H, f \neq \theta} \frac{|\sum_{j=1}^{n+1} \lambda_j (f, \hat{F}_j)|}{\|f\|}$$

and

$$\hat{F}_j(x) = \frac{1}{\hat{\eta}_j^2} e^{ip_j x}$$

where we have denoted $\hat{\eta}_j \equiv \eta_{p_j}$ for $p_j \in R$. We may easily find

$$S = \sup_{f \in H, f \neq \theta} \frac{|(f, \sum_{j=1}^{n+1} \bar{\lambda}_j \hat{F}_j)|}{\|f\|} = \|\sum_{j=1}^{n+1} \bar{\lambda}_j \hat{F}_j\|$$

where the bar denotes complex conjugation. From (2.3),

$$(\hat{F}_j, \hat{F}_k) = 0 \quad \text{for} \quad k \neq j \, ; \quad k, j = 1, 2, \ldots, n+1 \, ,$$

$$(\hat{F}_j, \hat{F}_j) = \frac{1}{\hat{\eta}_j^2} , \quad j = 1, 2, \ldots, n+1$$

and we may easily compute

(4.13) $$S = \sqrt{\left( \sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2} \right)} .$$

Now, it remains only to find

$$\inf_{g_k} \sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2}\right)}.$$

The expression the exact lower bound of which we are looking for depends only on the values of $g_k(p)$, $k = 1, 2, \ldots, n$ in the points $p = p_j$, $j = 1, 2, \ldots, n + 1$. As stated at the beginning of the proof, each matrix $[g_{kj}]$, $k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n + 1$ determines some numbers $\lambda_j$, $j = 1, 2, \ldots, n + 1$ satisfying (4.12). And, conversely, every $(n + 1)$-tuple of numbers $\lambda_j$ satisfying (4.12) corresponds to some matrices $[g_{kj}]$. Thus, the problem becomes that of finding

$$\inf_{\substack{[\lambda_j], j=1,2,\ldots,n+1; \\ \sum_{j=1}^{n+1} |\lambda_j| = 1}} \left(\sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2}\right)}\right).$$

As we shall now show using the Cauchy inequality, the expression under consideration attains on the set of the vectors $[\lambda_j]$, $j = 1, 2, \ldots, n + 1$ satisfying (4.12) its exact lower bound, and that for

$$\lambda_j = \frac{\hat{\eta}_j^2}{\sum_{k=1}^{n+1} \hat{\eta}_k^2}, \quad j = 1, 2, \ldots, n + 1 .$$

We have namely

$$\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2} \cdot \sum_{j=1}^{n+1} \hat{\eta}_j^2 \geqq \left(\sum_{j=1}^{n+1} |\lambda_j|\right)^2 = 1$$

and therefore

$$\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2} \geqq \left(\sum_{j=1}^{n+1} \hat{\eta}_j^2\right)^{-1}$$

for all $[\lambda_j]$'s from the set mentioned above. That the lower bound is attained may be verified by substituting for $\lambda_j$.

This completes the proof of the theorem.

We now give three remarks regarding Theorem 4.1 and its proof.

Remark 4.2. One may see from Theorem 4.1 that for $m > n + 1$ we can obtain different bounds on $\Omega$ choosing different $(n + 1)$-tuples of numbers from $R$. The best choice yielding the most realistic bound depends on the space $H$.

Remark 4.3. To find a lower bound on $\Omega$ it was not inevitably necessary to use the second inequality in (4.10). If we had bounded, however, $\max_{p \in P} |J_p(f)|$ in the same way, we should have obtained a worse bound on $\Omega$ as a consequence of our further procedure.

Remark 4.4. Let there be given functions $g_1(p), g_2(p), \ldots, g_n(p)$. Denote the set of all approximations from $M_n$ employing these $g_k(p)$'s by $M_n^g$. Then, on the assumptions of Theorem 4.1, (4.8) and (4.13) yield

$$\Omega(M_n^g, P) \geqq \sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_{p_j}^2}\right)} \tag{4.14}$$

where the $\lambda_j$'s are given by (4.11), (4.12).

A reasonable question to ask at this point is whether the result of Theorem 4.1 with its rather general formulation can be improved on substantially. The answer is negative. First we shall show that the equality sign stands in (4.9) in a special case.

**Theorem 4.2.** *Let $H$ be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Then*

$$\Omega(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2}\,.$$

Proof. We have $n = 1$, $r = m = 2$ and the assumptions of Theorem 4.1 are satisfied. We shall find an optimal approximation in $M_1(P^*)$, i.e. a couple of functionals $\{K_p, K_q\}$ such that

$$\max_{s=p,q} \|I_s - K_s\| = (\eta_p^2 + \eta_q^2)^{-1/2}\,.$$

If $K_s$ is given by (3.7) the couple $\{K_p, K_q\}$ belongs to $M_1(P^*)$ evidently. We denote by $\{K_p^*, K_q^*\}$ the approximation obtained by setting

$$\text{a) } \alpha = \beta = A\,, \quad A \text{ real}\,, \quad A \neq 0\,; \tag{4.15}$$

$$\text{b) } g(p) = \frac{\eta_q^2}{A(\eta_p^2 + \eta_q^2)}\,, \quad g(q) = \frac{\eta_p^2}{A(\eta_p^2 + \eta_q^2)}$$

in (3.7). The error of this approximation is given by Theorem 3.3. After substitution for $\alpha$, $\beta$ and $g(s)$ we get

$$\|I_p - K_p^*\|^2 = \|I_q - K_q^*\|^2 = \frac{1}{\eta_p^2 + \eta_q^2}\,. \tag{4.16}$$

Thus, $\{K_p^*, K_q^*\}$ is indeed an optimal approximation from $M_1(P^*)$ for given $P^*$ and $H$ and we have

$$\Omega^2(1, P^*) = \frac{1}{\eta_p^2 + \eta_q^2}\,,$$

as required.

Clearly, Theorem 4.1 holds also in the case of the set $\tilde{M}_n$ and the quantity $\tilde{\Omega}$. Even in this case it yields an acceptable bound, for we can show the lower bound in (4.9) to be attainable asymptotically as $j \to \infty$.

**Theorem 4.3.** *Let $H$ be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Then*

$$\tilde{\Omega}(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2}.$$

Proof is by contradiction. We have $n = 1$, $r = m = 2$ and the assumptions of Theorem 4.1 are satisfied. Since $\tilde{\Omega}(1, P^*) \geqq \Omega(1, P^*)$, (4.9) is true also for $\tilde{\Omega}(1, P^*)$.

Let

$$\tilde{\Omega}(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2} + \varepsilon$$

where $\varepsilon > 0$. Denote by $N_s^{*(j)}$, $s = p, q$ the functionals given by (3.16) where $\alpha$, $\beta$ and $g(s)$ are the same as those in the proof of Theorem 4.2. The approximations $\{N_p^{*(j)}, N_q^{*(j)}\}$ belong to $\tilde{M}_1(P^*)$ obviously for all positive integers $j$.

By Theorem 3.5 and with respect to (4.16), there exists a positive integer $j_0$ such that

$$\max_{s=p,q} \|I_s - N_s^{*(j_0)}\| < (\eta_p^2 + \eta_q^2)^{-1/2} + \varepsilon,$$

which contradicts the definition of $\tilde{\Omega}(1, P^*)$. The theorem is proved.

We may easily prove an assertion supplementing Theorem 4.2 in a sense.

**Theorem 4.4.** *Let the set $P$ have $r$ elements, $r > 1$. Given an integer $n$, $0 < n < r$, for every $\varepsilon > 0$ there exists a periodic space $H_\varepsilon$ such that $A_{H_\varepsilon} \supset P$ and that for each $(n + 1)$-tuple $\{p_1, p_2, \ldots, p_{n+1}\}$ of numbers $p_j \in P$*

$$\Omega_{H_\varepsilon}(n, P) < \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} + \varepsilon$$

*is valid.*

Proof. Given an approximation $\{G_p\} \in M_n$,

$$\Omega(n, P) \leqq \omega(P, G_p)$$

holds in every periodic space. Let us consider the class of all periodic spaces with the property that $A_H \supset P$ and take for $\{G_p\}$ especially $\{B_p\}$ from Sec. 3 dividing $P = P_1 \cup P_2$ arbitrarily.

We get for each space from the class considered

$$\Omega(n, P) \leqq \max_{p \in P_2} \frac{1}{\eta_p}. \tag{4.17}$$

We look for the space where

$$\max_{p \in P_2} \frac{1}{\eta_p} < \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} + \varepsilon. \tag{4.18}$$

Such a space does exist, for it is sufficient to require

$$\max_{p \in P_2} \frac{1}{\eta_p} < \varepsilon$$

and use Theorem 2.3. The theorem is proved.

An analogous statement regarding $\tilde{\Omega}(n, P)$ might be proved using $\{B_p^{(j)}\}$ from Sec. 3.

Proving Theorems 4.2 and 4.3 we constructed an optimal approximation and an asymptotically optimal sequence of approximations, respectively. Constructing the optimal functionals we utilized the information on the space we worked in. Therefore, we may expect the properties of the optimal approximation constructed to depend strongly on the space in which we work. That is, the $Q$ of our approximation, which is 1 in a given space, may be expected to be large in other spaces. In fact, this *is* the case. More precisely, the whole situation is illustrated by the following

**Theorem 4.5.** *Let $H$ be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Construct the optimal approximation $\{K_s^*\}_{s=p,q}$ from $M_1(P^*)$ in the given space (cf. (4.15)). Then for every $D > 0$ there exists a periodic space $H'$ such that*

$$Q_{H'}(1, P^*, K_s^*) > D\,.$$

Proof. It is sufficient to find, for a given $D > 0$, a periodic space $H'$ such that

$$\frac{\|J_p^*\|_{H'}^2}{\Omega_{H'}^2(1, P^*)} > D^2$$

where $J_p^* = I_p - K_p^*$ is the error functional. Denote

$$\|e^{ikx}\|_H = \eta_k\,,$$
$$\|e^{ikx}\|_{H'} = \varepsilon_k$$

for $k \in A_H$, $k \in A_{H'}$ respectively. We shall take into consideration only the spaces $H'$ with the property $P^* \subset A_{H'}$.

By Theorem 3.3 we get, substituting for $\alpha$, $\beta$, $g(s)$ from (4.15),

$$\|J_p^*\|_{H'}^2 = \frac{\eta_q^4}{(\eta_p^2 + \eta_q^2)^2} \cdot \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p^2 \varepsilon_q^2} + \frac{1}{\varepsilon_p^2}\left[1 - \frac{2\eta_q^2}{\eta_p^2 + \eta_q^2}\right].$$

Introducing a convenient notation, we may write

$$\|J_p^*\|_{H'}^2 = \frac{1}{\varepsilon_p^2}\left[C_1\left(1 + \frac{\varepsilon_p^2}{\varepsilon_q^2}\right) + C_2\right],$$

where $C_1$, $C_2$ are constants given by the space $H$, $C_1 > 0$, $C_1 + C_2 > 0$. In virtue of Theorem 4.2 we have

$$\Omega^2_{H'}(1, P^*) = \frac{1}{\varepsilon_p^2 + \varepsilon_q^2} .$$

Thus

$$\frac{\|J_p^*\|^2_{H'}}{\Omega^2_{H'}(1, P^*)} \geqq \left(1 + \frac{\varepsilon_q^2}{\varepsilon_p^2}\right)(C_1 + C_2) .$$

From this we can see that $H'$ may be an arbitrary periodic space such that $A_{H'} \supset P^*$ and

$$\frac{\varepsilon_q^2}{\varepsilon_p^2} > \frac{D^2}{C_1 + C_2} - 1 .$$

According to Theorem 2.3, such a space does exist, and the theorem is proved.

Evidently, if $|p| \neq |q|$ the theorem holds even though we confine ourselves to strongly periodic spaces $H'$. (If $|p| = |q|$ then $g(p) = g(q) = 1/(2A)$ in the class of strogly periodic spaces independently of the space, and $\{K_s^*\}$, $s \in P^*$ is optimal in every strongly periodic space.)

Analogous statements regarding the functionals $N_s^{*(j)}$ from Theorem 4.3, which are asymptotically optimal for given $P^*$ and $H$, hold asymptotically.

**Theorem 4.6.** *Let $H$ be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Construct the asymptotically optimal sequence of approximations $\{N_s^{*(j)}\}_{s=p,q} \in \tilde{M}_1(P^*)$,*

$$N_s^{*(j)} = g(s)\left(\alpha L_p^{(j)} + \beta L_q^{(j)}\right),$$

*in the space given (cf. (4.15)). Then for every $\tilde{D} > 0$ there exists a periodic space $H'$ such that*

$$\lim_{j\to\infty} \tilde{\Omega}_{H'}(1, P^*, N_s^{*(j)}) > \tilde{D} .$$

Proof. It is true in every periodic space $H'$ that

$$\lim_{j\to\infty} \tilde{Q}_{H'}(1, P^*, N_s^{*(j)}) = \frac{\lim_{j\to\infty} \omega_{H'}(P^*, N_s^{*(j)})}{\tilde{Q}_{H'}(1, P^*)} = \frac{\omega_{H'}(P^*, K_s^*)}{\Omega_{H'}(1, P^*)} = Q_{H'}(1, P^*, K_s^*) ,$$

as follows from Theorems 4.2, 4.3 and 3.5. The statement of Theorem 4.6 follows now from Theorem 4.5.

As a rule in practice, the information available on $f(x)$ is not so detailed as to enable us to determine the space in which the computation is to be carried out. Thus, we cannot use the optimal approximation without the risk illustrated by Theorems 4.5

and 4.6, the integrand being inserted only in an indefinite space from a class of spaces. It is of considerable importance, therefore, to search for such approximations that their $Q$ is bounded on some class of periodic spaces.

This implies the purpose of the following

**Definition 4.3.** An approximation $\{G_p\}_{p \in P} \in M_n$ is termed a *universal approximation* for a given $P$ and a class $\mathfrak{H}$ of periodic spaces if

$$Q_H(n, P, G_p) \leqq D(n, P)$$

holds in every space $H \in \mathfrak{H}$ and $D(n, P)$ is a constant independent of $H$.

This constant has also its quantitative meaning understandably. Similarly we have

**Definition 4.4.** A sequence of approximations $\{G_p^{(j)}\}_{p \in P}$, $j = 1, 2, \ldots$ from $\tilde{M}_n$ is termed a *universal sequence of approximations* for a given $P$ and a class $\mathfrak{H}$ of periodic spaces if

$$\limsup_{j \to \infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq \tilde{D}(n, P)$$

holds in every space $H \in \mathfrak{H}$ and $\tilde{D}(n, P)$ is a constant independent of $H$.

The following simple lemma will be of use in constructing the universal sequences of approximations.

**Lemma 4.2.** *Let $\{G_p\}$ be a universal approximation from $M_n$ for a given set $P$ and a given class $\mathfrak{H}$ of periodic spaces. Let $\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ be a sequence of approximations from $\tilde{M}_n$. If*

$$\lim_{j \to \infty} \max_P \|I_p - G_p^{(j)}\| = \max_P \|I_p - G_p\| \tag{4.19}$$

*is valid in all spaces from $\mathfrak{H}$, then $\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ is a universal sequence of approximations and*

$$\lim_{j \to \infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq Q_H(n, P, G_p)$$

*holds for every $H \in \mathfrak{H}$.*

Proof. Given $H \in \mathfrak{H}$,

$$\tilde{Q}_H(n, P, G_p^{(j)}) \leqq Q_H(n, P, G_p^{(j)})$$

holds for every positive integer $j$. Using (4.19) we get

$$\limsup_{j \to \infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq \lim_{j \to \infty} Q_H(n, P, G_p^{(j)}) = Q_H(n, P, G_p) \leqq D(n, P)$$

for every $H \in \mathfrak{H}$, as required.

In the following section we shall treat universal approximations in classes of periodic spaces, our interest being in the choice of the functions $g_k(p)$ primarily. The class of all periodic spaces will be denoted by $\mathfrak{H}_1$, the class of periodic spaces such that $A_H$ is the set of all integers will be denoted by $\mathfrak{H}_2$ and the class of all strongly periodic spaces by $\mathfrak{H}_3$. Clearly, $\mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \mathfrak{H}_3$.

## 5. THE UNIVERSAL APPROXIMATIONS IN PERIODIC SPACES

In the foregoing section we gave an example of an approximation not universal with respect to any of the classes $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$. Naturally, the question arises whether for these classes a universal approximation exists at all.

**Theorem 5.1.** *For any n and P there exists no approximation in $M_n(P)$ universal with respect to $\mathfrak{H}_1$ or $\mathfrak{H}_2$.*

Proof. Since $\mathfrak{H}_2 \subset \mathfrak{H}_1$, it is sufficient to prove the theorem for the class $\mathfrak{H}_2$. We shall show that for an arbitrary approximation $\{G_p\} \in M_n$ and an arbitrary number $D$ there exists a space $H \in \mathfrak{H}_2$ such that

$$\varrho_H(n, P, G_p) > D .$$

Choose $(n + 1)$ elements $p_1, p_2, \ldots, p_{n+1}$ from $P$ arbitrarily. Now, the approximation $\{G_p\}$ determines some numbers $\lambda_j$, $j = 1, 2, \ldots, n + 1$ satisfying (4.11), (4.12) and independent of $H$. The $\lambda_j$'s cannot vanish simultaneously; denote by $\lambda_s$ a non-zero one. By (4.14), we get for the approximation given

$$\omega(P, G_p) \geqq \left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_{p_j}^2}\right)^{1/2} \geqq \frac{|\lambda_s|}{\eta_{p_s}} .$$

By (4.17),

$$\Omega(n, P) \leqq \max_{p \in P_2} \frac{1}{\eta_p} ,$$

having split $P$ arbitrarily into two disjoint subsets $P_1$ and $P_2$ such that $P_1$ has $n$ elements.

Make this division in such a way that $p_s \in P_1$. Then

$$\varrho_H(n, P, G_p) \geqq |\lambda_s| \frac{\eta_q}{\eta_{p_s}}$$

where $1/\eta_q = \max_{p \in P_2} 1/\eta_p$ and the subscript $q$ depends on $H$ of course. In virtue of Theorem 2.3, for every $D$ it is possible to find a space $H \in \mathfrak{H}_2$ such that

$$\frac{\eta_q}{\eta_{p_s}} > \frac{D}{|\lambda_s|} ,$$

which completes the proof.

Remark 5.1. The proofs of further theorems regarding the non-existence of universal approximation (or the conditions necessary for a given approximation to be universal) are analogous to the proof of Theorem 5.1 in outline. The basis of the latter proof is formed by the bounds (4.14) and (4.17). But, (4.14) holds also for the approximations from $\tilde{M}_n$, and using (3.20) we have

$$\Omega(n, P) \leqq \tilde{\Omega}(n, P) \leqq \max_{P_2} \frac{1}{\eta_p} \tag{5.1}$$

where $P_2$ was described in the proof of the foregoing theorem. Therefore, if we prove some theorem of the kind mentioned above for the universal approximations from $M_n$, we can simply obtain an analogous asymptotic statement regarding the sequences of approximations from $\tilde{M}_n$.

The proof of the following theorem concerning $\tilde{M}_n$ will be yet indicated, the passage from $M_n$ to $\tilde{M}_n$ being left to the reader throughout the remainder of the section.

**Theorem 5.2.** *For any n and P there exists no sequence of approximations from $\tilde{M}_n(P)$ universal with respect to $\mathfrak{H}_1$ or $\mathfrak{H}_2$.*

Proof. Again, it is sufficient to prove the theorem for $\mathfrak{H}_2$. As we have stated, by (3.20) the bound (4.17) used in the proof of Theorem 5.1 is true also for $\tilde{\Omega}(n, P)$. Besides this bound we used only the properties of the functions $g_k(p)$ (i.e. those of the $\lambda_j$'s and the bound (4.14)) in the above proof. Thus, for an arbitrary approximation $\{G_p^{(j)}\} \in \tilde{M}_n$ and number $D$ we can find a space $H \in \mathfrak{H}_2$ such that

$$\tilde{Q}_H(n, P, G_p^{(j)}) > D$$

independently of $j$. Therefore, in this space

$$\limsup_{j \to \infty} \tilde{Q}_H(n, P, G_p^{(j)}) \geqq D\,,$$

as required.

Before we shall consider the existence of a universal approximation for the class $\mathfrak{H}_3$, we rewrite Theorem 4.1 for the case of strongly periodic spaces. Throughout the remainder of the section we shall suppose the set $P$ to be arranged in such a manner that $|p_i| \geqq |p_j|$ holds whenever $i \geqq j$. It follows then from Definition 2.4

$$\eta_{p_i} \geqq \eta_{p_j} \quad \text{for} \quad i \geqq j$$

in strongly periodic spaces.

**Theorem 5.3.** *Let there be given a strongly periodic space H and the set P. Then*

$$\Omega(n, P) \geqq \Big(\sum_{j=1}^{n+1} \eta_{p_j}^2\Big)^{-1/2}\,. \tag{5.2}$$

Owing to the arrangement of $P$ introduced above, this bound is the best one from those obtainable by Theorem 4.1.

Proving Theorem 5.1 on the non-existence of a universal approximation with respect to $\mathfrak{H}_1$ and $\mathfrak{H}_2$ we made use of that in various spaces from these classes the norms $\eta_k$ may behave arbitrarily to some extent. With the class $\mathfrak{H}_3$, however, the $\eta_k$'s have the properties (c), (d). Thus we may expect some change in the situation now.

**Theorem 5.4.** *Given the set $P$ and a positive integer $n$, there exists an approximation from $M_n$ universal with respect to $\mathfrak{H}_3$. It is the approximation $\{B_p\}$ from Sec. 3 given by the division $P = P_1 \cup P_2$ such that $P_1 = \{p_1, p_2, \ldots, p_n\}$, $P_2 = \{p_{n+1}, p_{n+2}, \ldots, p_r\}$, and (3.17). Moreover,*

$$Q(n, P, B_p) \leqq \left(1 + \sum_{k=1}^{n} \left(\frac{\eta_{p_k}}{\eta_{p_{n+1}}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2} \tag{5.3}$$

*holds for every $H \in \mathfrak{H}_3$.*

Proof. By (3.18) and using the properties (c), (d) we get

$$\omega(P, B_p) = \max_{p \in P_2} \frac{1}{\eta_p} = \frac{1}{\eta_{p_{n+1}}}. \tag{5.4}$$

This and (5.2) yield

$$Q(n, P, B_p) \leqq \frac{1}{\eta_{p_{n+1}}} \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{1/2}.$$

The statement of the theorem follows now readily by a trivial modification using (c) and (d).

A practical analogue is

**Theorem 5.5.** *Given the set $P$ and a positive integer $n$, there exists a sequence of approximations from $\tilde{M}_n$ universal with respect to $\mathfrak{H}_3$. They are the approximations $\{B_p^{(j)}\}$ from Sec. 3 with the division $P = P_1 \cup P_2$ such as in Theorem 5.4. Moreover, for every $H \in \mathfrak{H}_3$ there exists an integer $j_0$ such that*

$$\tilde{Q}(n, P, B_p^{(j)}) \leqq \left(1 + \sum_{k=1}^{n} \left(\frac{\eta_{p_k}}{\eta_{p_{n+1}}^2}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2} \tag{5.5}$$

*holds for all $j \geqq j_0$.*

Proof follows immediately from Theorems 5.4, 3.6 and Lemma 4.2.

Remark 5.2. We can see from (5.3) and (5.5) that if $\eta_{p_{n+1}}$ is sufficiently large in comparison with $\eta_{p_n}$, the quotients $Q$ and $\tilde{Q}$ from Theorems 5.4, 5.5 may approach 1 arbitrarily closely.

Now, our interest is whether it is essential for the universality that there was

$$(5.6) \qquad g_k(p_j) = 1 \quad \text{for} \quad k = j\,, \qquad 0 \quad \text{for} \quad k \neq j\,, \quad k = 1, 2, \ldots, n\,; \quad j = 1, 2, \ldots, r$$

in Theorems 5.4 and 5.5 or whether there exist universal approximations (or sequences of approximations) employing other systems $\{g_k(p)\}$. If such approximations exist we shall be interested in bounding their $Q$ (or $\tilde{Q}$). Before proceeding to considerations just suggested observe that, with the above arrangement of $P$, the upper bound (5.1) of the quantities $\Omega$ and $\tilde{\Omega}$ has the form

$$(5.7) \qquad \Omega(n, P) \leqq \tilde{\Omega}(n, P) \leqq \frac{1}{\eta_{p_{n+1}}}$$

in strongly periodic spaces.

Because of (4.14), it will prove reasonable to carry out further investigation not with the systems $\{g_k(p)\}_{k=1}^n$, $p \in P$, but right with the numbers $\lambda_j$ occurring in (4.14).

For a given $n$, consider all the subsets of $P$ of the form

$$P_s = \{p_1, p_2, \ldots, p_n, p_s\}\,, \quad n + 1 \leqq s \leqq r\,.$$

Each of these subsets together with the system $\{g_k(p)\}$ determines the numbers $\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)}$ (cf. (4.11), (4.12)). There may be more than one such an $(n + 1)$-tuple for a given $\{g_k(p)\}$. Altogether, they represent all the solutions of the system

$$(5.8) \qquad \lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_n^{(s)}\, g_k(p_k) + \lambda_{n+1}^{(s)}\, g_k(p_s) = 0\,, \qquad k = 1, 2, \ldots, n$$

satisfying the condition

$$(5.9) \qquad \sum_{j=1}^{n+1} |\lambda_j^{(s)}| = 1\,.$$

We denote by $W_s$ the linear space of all the solutions of (5.8), its dimension being designated by dim $W_s$. Further, we shall write

$$\mathbf{e}_j = [\delta_{1j}, \delta_{2j}, \ldots, \delta_{n+1,j}]\,, \quad j = 1, 2, \ldots, n + 1$$

where $\delta_{ij}$ is the Kronecker delta*). The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}$ form a basis of the $(n + 1)$-dimensional linear vector space $W$. Given a linearly independent system $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_l\}$ of vectors from $W$, we denote by $V(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_l)$ the linear subspace of $W$ with the above system as a basis. Finally, let $(\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)})$ be a solution of (5.8) satisfying (5.9). We denote

$$\boldsymbol{\lambda}^{(s)} = [|\lambda_1^{(s)}|, |\lambda_2^{(s)}|, \ldots, |\lambda_{n+1}^{(s)}|]$$

*) $\delta_{ij} = 0$ unless $i = j$, in which case $\delta_{ij} = 1$.

and the system of vectors $\{\lambda^{(s)}\}_{s=n+1}^{r}$ will be called a *system of vectors* $\lambda^{(s)}$ *determined by the system of functions* $g_k(p)$.

Now, return to Theorems 5.4 and 5.5. From (5.6) we readily find that for the system $\{g_k(p)\}$ used there the following assertion is valid:

(5.10) The system $\{g_k(p)\}$ determines a unique system of vectors $\lambda^{(s)}$, namely

$$\lambda^{(n+1)} = \lambda^{(n+2)} = \ldots = \lambda^{(r)} = \mathbf{e}_{n+1} .$$

We shall investigate the systems $\{g_k(p)\}$ for which (5.10) does not hold. First, we give (5.10) a more lucid form and, moreover, we reformulate it in terms of the spaces $W_s$ for later use.

**Lemma 5.1.** *The condition* (5.10) *is equivalent with*

(5.11) $a)\ g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n + 1,\ n + 2,\ \ldots,\ r\,;$

$b)\ \operatorname{rank}\left([g_k(p_j)]_{k,j=1}^{n}\right) = n$

*and with*

(5.12) $$W_s = V(\mathbf{e}_{n+1}) \quad \textit{for each} \quad s = n + 1,\ n + 2,\ \ldots,\ r\,.$$

Proof. We shall show that (5.10) implies (5.11), (5.11) implies (5.12), and (5.12) implies (5.10).

I. Let (5.10) be true for a system $\{g_k(p)\}$. Substituting $\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_n^{(s)} = 0$, $\lambda_{n+1}^{(s)} = 1$ into (5.8), $s = n + 1, n + 2, \ldots, r$, we get (5.11a) immediately. Therefore, (5.8) becomes

(5.13) $$\lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_n^{(s)}\, g_k(p_n) = 0\,, \qquad k = 1, 2, \ldots, n\,,$$

for any $s = n + 1,\ n + 2,\ \ldots,\ r$. According to (5.10), this set of homogeneous linear equations has only the trivial solution. Thus, (5.11b) holds.

II. If (5.11) is valid for a system $\{g_k(p)\}$, then by (5.11b) $\dim W_s = 1$ for each $s = n + 1, n + 2, \ldots, r$. We have $\mathbf{e}_{n+1} \in W_s$ with respect to (5.11a) and thus $W_s = V(\mathbf{e}_{n+1})$.

III. Let (5.12) hold. Given an integer $s$ such that $n + 1 \leqq s \leqq r$, the space $W_s$ contains exactly all the vectors of the form $\varrho \mathbf{e}_{n+1}$ where $\varrho$ runs through the set of complex numbers. The vectors from $W_s$ satisfying (5.9) are exactly those with $|\varrho| = 1$; thus (5.10) holds.

**Theorem 5.6.** *Let the set* $P$ *be given and let* $n$ *be a positive integer such that* $|p_n| \neq |p_{n+1}|$. *Then a necessary condition for an approximation* $\{G_p\} \in M_n$,

$$G_p(f) = \sum_{k=1}^{n} a_k(f)\, g_k(p)\,,$$

*to be universal for the given P and with respect to $\mathfrak{H}_3$ is that $\{g_k(p)\}$ should satisfy* (5.11).

Proof. Let us suppose that the system $\{g_k(p)\}$ does not satisfy (5.11). Then, by Lemma 5.1, it does not satisfy (5.12) either and, therefore, there exists a vector $\lambda^{(s)}$, $n + 1 \leqq s \leqq r$, determined by $\{g_k(p)\}$ and such that

(5.14) $$\lambda_q^{(s)} \neq 0$$

for some $q$, $1 \leqq q \leqq n$.

From (4.14) we get

$$\omega(P, G_p) \geqq \left(\sum_{k=1}^{n} \frac{|\lambda_k^{(s)}|^2}{\eta_{p_k}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2}\right)^{1/2} \geqq \frac{|\lambda_q^{(s)}|}{\eta_{p_q}}$$

for any approximation $\{G_p\} \in M_n$ with the given $\{g_k(p)\}$. Further, using (5.7) we have

$$Q(n, p, G_p) \geqq |\lambda_q^{(s)}| \frac{\eta_{p_{n+1}}}{\eta_{p_q}}.$$

Since $1 \leqq q \leqq n$ and $|p_n| \neq |p_{n+1}|$, and because of (5.14) and (d), for every $D > 0$ a space $H \in \mathfrak{H}_3$ such that

$$\frac{\eta_{p_{n+1}}}{\eta_{p_q}} > \frac{D}{|\lambda_q^{(s)}|}$$

holds in $H$ may be found by means of Theorem 2.3. Therefore,

$$Q_H(n, P, G_p) > D$$

and $\{G_p\}$ is not universal with respect to $\mathfrak{H}_3$, as required.

An analogous theorem holds for the universal sequences of approximations.

**Theorem 5.7.** *Let the set P be given and let n be a positive integer such that $|p_n| \neq |p_{n+1}|$. Then a necessary condition for a sequence of approximations $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$,*

$$G_p^{(j)}(f) = \sum_{k=1}^{n} a_k^{(j)}(f)\, g_k(p),$$

*to be universal for the given P and with respect to $\mathfrak{H}_3$ is that $\{g_k(p)\}$ should satisfy* (5.11).

For the proof see Remark 5.1.

Theorem 5.6 enables us to judge the error of the universal approximation $\{B_p\}$ from Theorem 5.4 with regard to the class of universal approximations. Theorem 5.7 is of similar importance for the universal sequence of approximations from Theorem 5.5.

Denote by $U_n(P)$ the set of universal approximations from $M_n(P)$ for a given $P$ and the class $\mathfrak{H}_3$. Given $n$ such that $|p_n| \neq |p_{n+1}|$, then by Theorem 5.6, Lemma 5.1

and (4.14)

$$\Omega_H(U_n, P) \geqq \frac{1}{\eta_{p_{n+1}}} \tag{5.15}$$

holds in every $H \in \mathfrak{H}_3$. Because of (5.4) we have proved

**Theorem 5.8.** *Given the set $P$ and a positive integer $n$ such that $|p_n| \neq |p_{n+1}|$, the approximation $\{B_p\} \in U_n(P)$ is an optimal universal approximation from $M_n(P)$ for the class of strongly periodic spaces.*

Similarly, denote by $\tilde{U}_n(P)$ the set of all the approximations from $\tilde{M}_n(P)$ that are elements of a universal sequence of approximations for a given set $P$ and the class $\mathfrak{H}_3$. Again, if $n$ is such that $|p_n| \neq |p_{n+1}|$ then by Theorem 5.7, Lemma 5.1 and (4.14)

$$\Omega_H(\tilde{U}_n, P) \geqq \frac{1}{\eta_{p_{n+1}}} \tag{5.16}$$

holds in every $H \in \mathfrak{H}_3$. Using Theorem 5.5, (5.4) and Theorem 3.6 we obtain

**Theorem 5.9.** *Given the set $P$ and a positive integer $n$ such that $|p_n| \neq |p_{n+1}|$, the sequence of approximations $\{B_p^{(j)}\} \in \tilde{U}_n(P)$, $j = 1, 2, \ldots$, is an asymptotically optimal universal sequence of approximations from $\tilde{M}_n(P)$ for $\mathfrak{H}_3$.*

Now, we shall consider the approximations from $M_n$ where $n$ is such that $|p_n| = |p_{n+1}|$ in the set $P$ given. We have then $\eta_{p_n} = \eta_{p_{n+1}}$ in all $H \in \mathfrak{H}_3$ by (c).

First we shall assume the functions $g_k(p)$, $k = 1, 2, \ldots, n$ to be linearly dependent on the set $P$, i.e.

$$\operatorname{rank}\left([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, r\right) < n\,.$$

Recalling Theorems 5.6 and 5.7 we can see that no approximation employing such a system $\{g_k(p)\}$ is universal with respect to $\mathfrak{H}_3$ in case that $|p_n| \neq |p_{n+1}|$ (and similarly for the sequences of approximations). If $|p_n| = |p_{n+1}|$, however, we may readily prove

**Theorem 5.10.** *Let the set $P$ be given and let $n$ be such that $|p_n| = |p_{n+1}|$. Then the approximation $\{C_p\} \in M_n$ given by*

$$\begin{aligned} a_k &= I_{p_k}\,, \quad k = 1, 2, \ldots, n-1\,, \\ a_n &= O \quad (\textit{the null functional}) \end{aligned} \tag{5.17}$$

*and*

$$\begin{aligned} g_k(p_j) &= \delta_{kj}\,, \quad k = 1, 2, \ldots, n-1\,; \quad j = 1, 2, \ldots, r\,, \\ g_n(p_j) &= 0\,, \quad j = 1, 2, \ldots, r \end{aligned} \tag{5.18}$$

*is universal with respect to $\mathfrak{H}_3$. Moreover,*

$$Q(n, P, C_p) \leqq \left(2 + \sum_{k=1}^{n-1} \left(\frac{\eta_{p_k}}{\eta_{p_n}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2}$$

*holds for every $H \in \mathfrak{H}_3$.*

Proof might be carried out by an argument precisely analogous to that of Theorem 5.4, because

(5.19) $$\omega(P, C_p) = \frac{1}{\eta_{p_n}} = \frac{1}{\eta_{p_{n+1}}}$$

in this case.

The formulation and proof of an analogous theorem regarding the universal sequence of approximations $\{C_p^{(j)}\}$ where we have replaced $I_{p_k}$ by $L_{p_k}^{(j)}$ in (5.17) are left to the reader.

The system $\{g_k(p)\}$ from Theorem 5.10 is linearly dependent on $P$. From (5.18) we easily find that the following assertion concerning this $\{g_k(p)\}$ is valid:

(5.20) The system $\{g_k(p)\}$ determines exactly all the systems of vectors $\boldsymbol{\lambda}^{(s)}$ such that

$$\boldsymbol{\lambda}^{(s)} = \mu_s \mathbf{e}_n + (1 - \mu_s)\, \mathbf{e}_{n+1}\,, \quad n + 1 \leqq s \leqq r\,,$$

where $0 \leqq \mu_s \leqq 1$.

The systems $\{g_k(p)\}$ satisfying (5.20) are characterized by

**Lemma 5.2.** *The condition* (5.20) *is equivalent with*

(5.21) *a)* $g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n, n + 1, \ldots, r\,;$

*b) If* $n > 1$ *then*

$$\operatorname{rank}\left([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n - 1\right) = n - 1$$

*and with the condition*

(5.22) $$W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1}) \quad \textit{for each} \quad s = n + 1, n + 2, \ldots, r\,.$$

Proof. We shall show that (5.20) implies (5.21), (5.21) implies (5.22), and (5.22) implies (5.20).

I. Let (5.20) hold for a system $\{g_k(p)\}$. Setting

$$\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_n^{(s)} = 0,\ \lambda_{n+1}^{(s)} = 1$$

and

$$\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_{n-1}^{(s)} = \lambda_{n+1}^{(s)} = 0,\ \lambda_n^{(s)} = 1$$

in (5.8), we get (5.21a) immediately. The sets of equations (5.8) thus become

(5.23) $$\lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_{n-1}^{(s)}\, g_k(p_{n-1}) = 0\,,$$
$$k = 1, 2, \ldots, n - 1$$

for all $s = n + 1, n + 2, \ldots, r$ (if $n > 1$). According to (5.20), this set of equations has only the trivial solution and, therefore, (5.21b) holds.

II. If (5.21) is true for a system $\{g_k(p)\}$ then the rank of the matrices of (5.8) is $n - 1$ and thus $\dim W_s = 2$ for any $s = n + 1, n + 2, \ldots, r$. By (5.21a), $\mathbf{e}_n \in W_s$, $\mathbf{e}_{n+1} \in W_s$ for all $s$ and we have $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$, as required.

III. Let (5.22) be true. Let there be given an integer $s$, $n + 1 \leqq s \leqq r$. The space $W_s$ contains exactly all the vectors of the form $\varrho \mathbf{e}_n + \sigma \mathbf{e}_{n+1}$ where $\varrho$ and $\sigma$ run through the set of complex numbers. The vectors from $W_s$ which satisfy (5.9) are exactly those with $|\varrho| + |\sigma| = 1$. Thus, (5.20) is valid.

The lemma is proved.

Now we are able to formulate

**Theorem 5.11.** *Let the set $P$ be given and let $n$ be a positive integer such that $|p_n| = |p_{n+1}|$. Let there be given a linearly dependent system $\{g_k(p)\}_{k=1}^n$. Then a necessary condition for an approximation $\{G_p\} \in M_n$ or a sequence of approximations $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ employing the system $\{g_k(p)\}$ to be universal for the given $P$ and with respect to $\mathfrak{H}_3$ is that $\{g_k(p)\}$ should satisfy* (5.21).

Proof. If the system $\{g_k(p)\}$ does not satisfy (5.21) then, by Lemma 5.2, it does not satisfy (5.22) either. Since $\{g_k(p)\}$ is linearly dependent, we have $\dim W_s \geqq 2$ for each $s = n + 1, n + 2, \ldots, r$. Thus there exists a vector $\lambda^{(s)}$, $n + 1 \leqq s \leqq r$, determined by $\{g_k(p)\}$ and such that

$$(5.24) \qquad \lambda_q^{(s)} \neq 0 \quad \text{for some } q\,, \quad 1 \leqq q \leqq n - 1\,.$$

By (4.14) and (5.7) we get similarly as in the proof of Theorem 5.6

$$Q(n, P, G_p) \geqq |\lambda_q^{(s)}| \frac{\eta_{p_{n+1}}}{\eta_{p_q}}$$

for any approximation $\{G_p\} \in M_n$ employing the system $\{g_k(p)\}$. We have $1 \leqq q \leqq \leqq n - 1$ and $|p_{n-1}| \neq |p_{n+1}|$. Owing to (5.24) and (d), for any $D > 0$ it is possible to find such a space $H \in \mathfrak{H}_3$ by means of Theorem 2.3 that

$$Q_H(n, P, G_p) > D$$

holds in $H$. Hence, $\{G_p\}$ is not universal.

The statement of the theorem regarding the sequences of approximations may be readily obtained by the argument indicated in Remark 5.1. The theorem is proved.

Let $V_n(P) \subset U_n(P)$ be the set of approximations employing linearly dependent systems $\{g_k(p)\}$. Let $n$ be an integer such that $|p_n| = |p_{n+1}|$. Then from Theorem 5.11, Lemma 5.2 and (4.14) we get immediately

$$(5.25) \qquad \Omega_H(V_n, P) \geqq \frac{1}{\eta_{p_n}}$$

in every space $H \in \mathfrak{H}_3$. Owing to (5.19), we have proved

**Theorem 5.12.** *Let the set P be given and let n be such that* $|p_n| = |p_{n+1}|$. *Then the approximation* $\{C_p\} \in V_n(P)$ *(cf.* (5.17), (5.18)) *is optimal in* $V_n(P)$ *for the whole class of strongly periodic spaces.*

Similarly, with the set of approximations $\tilde{V}_n(P) \subset \tilde{U}_n(P)$ employing linearly dependent systems $\{g_k(p)\}$ and with $n$ such that $|p_n| = |p_{n+1}|$, it holds

$$\Omega_H(\tilde{V}_n, P) \geqq \frac{1}{\eta_{p_n}} . \tag{5.26}$$

Considering the sequence of approximations $\{C_p^{(j)}\} \in \tilde{V}_n(P), j = 1, 2, \ldots,$

$$\lim_{j \to \infty} \omega(P, C_p^{(j)}) = \frac{1}{\eta_{p_n}} \tag{5.27}$$

follows immediately from Theorem 3.4 by the same procedure as that of the proof of Theorem 3.6. Owing to (5.26), we have proved

**Theorem 5.13.** *Let the set P be given and let n be such that* $|p_n| = |p_{n+1}|$. *Then the sequence of approximations* $\{C_p^{(j)}\} \in \tilde{V}_n(P)$, $j = 1, 2, \ldots$ *is asymptotically optimal in* $\tilde{V}_n(P)$ *for the whole class* $\mathfrak{H}_3$.

Provided $n > 1$ one may easily find that the approximation $\{C_p\}$ and the sequence of approximations $\{C_p^{(j)}\}$ can be treated also as an approximation from $M_{n-1}(P)$ and a sequence of approximations from $\tilde{M}_{n-1}(P)$, respectively. Since the elements of $P$ are distinct, $|p_n| = |p_{n+1}|$ implies $|p_{n-1}| \neq |p_n|$ and $\{C_p\}$ is an optimal universal approximation from $M_{n-1}(P)$ with respect to $\mathfrak{H}_3$ (similarly for $\{C_p^{(j)}\}, j = 1, 2, \ldots$). To put other way round, if $|p_{n+1}| = |p_{n+2}|$ then the optimal universal approximation $\{B_p\} \in M_n$ is a universal approximation from the set $M_{n+1}$ as well. An analogous statement is true regarding the sequence of approximations $\{B_p^{(j)}\}$.

In the remainder of the section we shall treat the approximations with linearly independent systems $\{g_k(p)\}$. The situation here is more complicated than that in the case of $|p_n| \neq |p_{n+1}|$. (5.11) is no longer a necessary condition for the universality and there exist universal approximations with various systems $\{g_k(p)\}$ not satisfying the above condition. An example of such an approximation is described in the following theorem.

**Theorem 5.14.** *Let the set P be given and let n be a positive integer such that* $|p_n| = |p_{n+1}|$. *Then the approximation* $\{\hat{B}_p\} \in M_n$ *described below is a universal approximation from* $M_n$ *for the set P and with respect to* $\mathfrak{H}_3$:

$$\begin{aligned} a_k &= I_{p_k}, \quad k = 1, 2, \ldots, n - 1, \\ a_n &= I_{p_n} + I_{p_{n+1}}; \end{aligned} \tag{5.28}$$

$$\begin{aligned} g_k(p_j) &= \delta_{kj}, \quad k = 1, 2, \ldots, n - 1; \quad j = 1, 2, \ldots, r, \\ g_n(p_j) &= 0, \quad j = 1, 2, \ldots, r, \quad j \neq n, \quad j \neq n + 1, \\ g_n(p_n) &= g_n(p_{n+1}) = \tfrac{1}{2}. \end{aligned} \tag{5.29}$$

*Moreover, it holds for every $H \in \mathfrak{H}_3$:*
*If $r = n + 1$, or $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} \leqq \eta_{p_{n+2}}$ then*

$$(5.30) \qquad Q(n, P, \hat{B}_p) \leqq \frac{1}{\sqrt{2}} \left( 2 + \sum_{j=1}^{n-1} \left( \frac{\eta_{p_j}}{\eta_{p_{n+1}}} \right)^2 \right)^{1/2} \leqq \frac{1}{\sqrt{2}} (1 + n)^{1/2} .$$

*If $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} > \eta_{p_{n+2}}$ then*

$$(5.31) \qquad Q(n, P, \hat{B}_p) \leqq \left( \sum_{j=1}^{n+1} \left( \frac{\eta_{p_j}}{\eta_{p_{n+2}}} \right)^2 \right)^{1/2} \leqq (1 + n)^{1/2} .$$

Proof. Denote

$$P_0 = \{p_1, p_2, \ldots, p_{n-1}\} , \quad P_1 = \{p_n, p_{n+1}\} , \quad P_2 = (P - P_0) - P_1 .$$

Obviously

$$(5.32) \qquad \hat{B}_p = \begin{cases} I_p & \text{for } p \in P_0 , \\ \frac{1}{2}(I_{p_n} + I_{p_{n+1}}) & \text{for } p \in P_1 , \\ O & \text{for } p \in P_2 . \end{cases}$$

We shall examine $r = n + 1$ and $r > n + 1$ separately.

I. Let $r = n + 1$. Then $P_2 = \emptyset$. In virtue of (5.32) we have

$$\omega(P, \hat{B}_p) = \omega(P_1, \hat{B}_p) .$$

By Theorem 3.3, we get

$$(5.33) \qquad \omega(P, \hat{B}_p) = \frac{1}{\eta_{p_n} \sqrt{2}}$$

after setting $p = p_n$, $q = p_{n+1}$, $\alpha = \beta = 1$ and $g(p_n) = g(p_{n+1}) = \frac{1}{2}$. Thus, with respect to (5.2),

$$Q(n, P, \hat{B}_p) \leqq \frac{1}{\eta_{p_n} \sqrt{2}} \left( \sum_{j=1}^{n+1} \eta_{p_j}^2 \right)^{1/2} ,$$

which yields (5.30) easily.

II. Let $r > n + 1$. Then $P_2 \neq \emptyset$. By (5.32),

$$\omega(P, \hat{B}_p) = \max \left( \omega(P_1, \hat{B}_p), \ \omega(P_2, \hat{B}_p) \right) .$$

In the same fashion as in case I we have

$$\omega(P_1, \hat{B}_p) = \frac{1}{\eta_{p_n} \sqrt{2}} .$$

From (5.32) and Theorem 3.2 it follows

$$\omega(P_2, \hat{B}_p) = \frac{1}{\eta_{p_{n+2}}}$$

and finally

$$(5.34) \qquad \omega(P, \hat{B}_p) = \max\left(\frac{1}{\eta_{p_n}\sqrt{2}}, \frac{1}{\eta_{p_{n+2}}}\right).$$

In case that

$$\eta_{p_n}\sqrt{2} \leqq \eta_{p_{n+2}}$$

is valid in the space $H$, the remainder of the proof is the same as in case I. If this is not the case, using (5.2) we obtain

$$Q(n, P, \hat{B}_p) \leqq \frac{1}{\eta_{p_{n+2}}}\left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{1/2},$$

which proves (5.31) and completes the proof of the theorem.

Actually, the used system $\{g_k(p)\}$ does not satisfy (5.11), which may be easily verified. With the sequences of approximations, the situation is similar.

**Theorem 5.15.** *Let the set P be given and let n be a positive integer such that* $|p_n| = |p_{n+1}|$. *Then the sequence of approximations* $\{\hat{B}_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ *obtained through replacing* $I_p$ *by* $L_p^{(j)}$ *in (5.28) is universal for the set P and with respect to* $\mathfrak{H}_3$.

*In addition, the inequalities (5.30) and (5.31) hold for every* $H \in \mathfrak{H}_3$ *with*

$$\lim_{j\to\infty} \tilde{Q}(n, P, \hat{B}_p^{(j)})$$

*instead of*

$$Q(n, P, \hat{B}_p)$$

*on the left-hand sides.*

Proof. By Theorem 3.4,

$$\lim_{j\to\infty} \omega(P_0, \hat{B}_p^{(j)}) = 0$$

(unless $P_0 = \emptyset$). In virtue of Theorem 3.5

$$\lim_{j\to\infty} \omega(P_1, \hat{B}_p^{(j)}) = \omega(P_1, B_p).$$

Since for $P_2 \neq \emptyset$

$$\omega(P_2, \hat{B}_p^{(j)}) = \omega(P_2, \hat{B}_p)$$

for all $j = 1, 2, \ldots$, it holds also

$$\lim_{j \to \infty} \omega(P, \hat{B}_p^{(j)}) = \omega(P, \hat{B}_p)\,. \tag{5.35}$$

From (5.35) and Theorem 5.14, and using Lemma 4.2, we establish the universality of the sequence in question and the bounds on $\lim_{j \to \infty} \tilde{Q}(n, P, \hat{B}_p^{(j)})$, proving the theorem.

The necessary condition for the universality in the case of $|p_n| = |p_{n+1}|$ and linearly independent systems $\{g_k(p)\}$ is formulated not with the values of $g_k(p)$ but by means of the spaces $W_s$. The reason is the attempt to avoid at this stage a detailed examination of the types of universal approximations possible and to give a statement though weaker, but lucid. The detailed examination of the conditions for $g_k(p)$ will be carried out in the remark following Theorem 5.16.

**Theorem 5.16.** *Let the set $P$ be given and let $n$ be a positive integer such that $|p_n| = |p_{n+1}|$. Let there be given a linearly independent system $\{g_k(p)\}_{k=1}^n$. Then a necessary condition for an approximation $\{G_p\} \in M_n$ or a sequence of approximations $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ employing this $\{g_k(p)\}$ to be universal for the given $P$ and with respect to $\mathfrak{H}_3$ is that*

$$W_s \subset V(\mathbf{e}_n, \mathbf{e}_{n+1}) \tag{5.36}$$

*should hold for each $s = n + 1, n + 2, \ldots, r$.*

Proof. The invalidity of (5.36) implies (5.24). The remainder of the proof would be the same as in the proof of Theorem 5.11. The statement regarding the sequences of approximations may be obtained on the basis of Remark 5.1 immediately.

Remark 5.3. The foregoing theorem requires that $\{g_k(p)\}$ be linearly independent on $P$. This assumption results in further restriction of the spaces $W_s$, which enables us to discuss the condition (5.36) in a rather detailed way.

First of all, $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$ cannot hold for all $s = n + 1, n + 2, \ldots, r$, because this would imply $g_k(p_n) = g_k(p_{n+1}) = \ldots = g_k(p_r) = 0$ for all $k = 1, 2, \ldots, n$, which contradicts the linear independence of $\{g_k(p)\}$. Thus there exists an integer $t$, $n + 1 \leqq t \leqq r$ such that $W_t = V(\mathbf{e}_n)$ or $W_t = V(\mathbf{e}_{n+1})$ or there exist non-zero complex numbers $\varrho_t$, $\sigma_t$ such that $W_t = V(\varrho_t \mathbf{e}_n + \sigma_s \mathbf{e}_{n+1})$.

I. Let $W_t = V(\mathbf{e}_n)$. Then $g_k(p_n) = 0$ for any $k = 1, 2, \ldots, n$ as we may see from the set of equations (5.8) for $s = t$. Since $W_s \subset V(\mathbf{e}_n, \mathbf{e}_{n+1})$ for all $s$ in question, we get

$$\lambda_{n+1}^{(s)}\, g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n + 1, n + 2, \ldots, r$$

where $(0, 0, \ldots, 0, \lambda_n^{(s)}, \lambda_{n+1}^{(s)})$ is an arbitiary solution of (5.8). Therefore, for each $s$ we have either $g_k(p_s) = 0$, $k = 1, 2, \ldots, n$ or $\lambda_{n+1}^{(s)} = 0$ for any solution of (5.8). If for a given $s$ the former takes place we have $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$; if the latter, then $W_s = V(\mathbf{e}_n)$.

II. Let $W_t = V(\mathbf{e}_{n+1})$. Then (5.8) yields $g_k(p_t) = 0$, $k = 1, 2, \ldots, n$ and $\operatorname{rank}([g_k(p_j)]_{k,j=1}^n) = n$. Thus $\dim W_s = 1$, $W_s \neq V(\mathbf{e}_n)$ holds for any $s$ in question. Hence, two cases are possible, namely $W_s = V(\mathbf{e}_{n+1})$ and $W_s = V(\varrho_s \mathbf{e}_n + \sigma_s \mathbf{e}_{n+1})$ where $\varrho_s \sigma_s \neq 0$.

III. Let $W_t = V(\varrho_t \mathbf{e}_n + \sigma_t \mathbf{e}_{n+1})$, $\varrho_t \sigma_t \neq 0$. Then we get

$$g_k(p_t) = -\frac{\varrho_t}{\sigma_t} g_k(p_n), \quad k = 1, 2, \ldots, n$$

from (5.8) and further

$$\operatorname{rank}([g_k(p_j)]_{k,j=1}^n) = n .$$

Thus $\dim W_s = 1$, $W_s \neq V(\mathbf{e}_n)$ for all $s$ and the conclusion is the same as that in II.

Now, we have described the combinations of the spaces $W_s$ possible and formulated (5.36) more precisely. From the above results we can also derive the equivalent necessary conditions that the system $\{g_k(p)\}$ of a universal approximation must satisfy.

There are two possibilities:

A. 1) $g_k(p_n) = 0$, $k = 1, 2, \ldots, n$;

2) there exists an integer $t$, $n + 1 \leqq t \leqq r$,

such that

$$\operatorname{rank}([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n-1, t) = n ;$$

3) for $s \neq t$, $n + 1 \leqq s \leqq r$, either

$$g_k(p_s) = 0 , \quad k = 1, 2, \ldots, n$$

or

$$\operatorname{rank}([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n-1, s) = n$$

holds.

B. 1) $\operatorname{rank}([g_k(p_j)]_{k,j=1}^n) = n$;

2) for each $s$, $n + 1 \leqq s \leqq r$ either

$$g_k(p_s) = 0$$

or there exists a non-zero number $L_s$ such that

$$g_k(p_s) = L_s\, g_k(p_n)$$

for all $k = 1, 2, \ldots, n$.

The condition A is equivalent to the case I, the condition B to the cases II and III. Moreover, it may be shown that there exist universal approximations the systems $\{g_k(p)\}$ of which cover all the possibilities given by the above two conditions. We shall not, however, give examples of such approximations. Our aim here is to prove some optimal properties of $\{\hat{B}_p\}$ and $\{\hat{B}_p^{(j)}\}$, $j = 1, 2, \ldots$

Recall that $U_n(P)$ is the set of universal approximations from $M_n(P)$ for a given $P$ and the class $\mathfrak{H}_3$ in our notation. Further, $V_n(P) \subset U_n(P)$ is the set of approximations employing linearly dependent systems $\{g_k(p)\}$. Obviously

$$\Omega_H(U_n, P) = \min\left(\Omega_H(V_n, P),\ \Omega_H(U_n - V_n, P)\right). \tag{5.37}$$

If $|p_n| = |p_{n+1}|$ and $\{G_p\}$ is an arbitrary approximation from $U_n - V_n$ then using Theorem 5.16 and (4.14) we get

$$\omega_H(P, G_p) \geqq \max_{s=n+1,\dots,r} \left(\frac{|\lambda_n^{(s)}|^2}{\eta_{p_n}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2}\right)^{1/2}$$

where $\lambda_n^{(s)}, \lambda_{n+1}^{(s)}$ are the corresponding components of the solution of (5.8) and $|\lambda_n^{(s)}| + |\lambda_{n+1}^{(s)}| = 1$ for each $s = n + 1, n + 2, \dots, r$. By an argument precisely similar to that in the last part of the proof of Theorem 4.1 it may be shown that

$$\left(\frac{|\lambda_n^{(s)}|^2}{\eta_{p_n}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2}\right)^{1/2} \geqq (\eta_{p_n}^2 + \eta_{p_s}^2)^{-1/2}$$

and therefore

$$\omega_H(P, G_p) \geqq \max_s (\eta_{p_n}^2 + \eta_{p_s}^2)^{-1/2},$$

which implies

$$\Omega_H(U_n - V_n, P) \geqq \frac{1}{\eta_{p_n}\sqrt{2}}$$

in every space $H \in \mathfrak{H}_3$. On account of (5.25) the relation (5.37) yields

$$\Omega_H(U_n, P) \geqq \frac{1}{\eta_{p_n}\sqrt{2}} \tag{5.38}$$

in any $H \in \mathfrak{H}_3$. The same bound is true also for $\Omega_H(\tilde{U}_n, P)$ evidently.

Hence, for the approximation $\{\hat{B}_p\}$ and the sequence of approximations $\{\hat{B}_p^{(j)}\}$, $j = 1, 2, \dots$ we have by (5.33), (5.34) and (5.35)

**Theorem 5.17.** *Let the set $P$ be given and let $n$ be a positive integer such that $|p_n| = |p_{n+1}|$. Let $r = n + 1$. Then the approximation $\{\hat{B}_p\} \in U_n(P)$ is an optimal universal approximation from $M_n(P)$ for the class $\mathfrak{H}_3$. Further, the sequence of approximations $\{\hat{B}_p^{(j)}\} \in \tilde{U}_n(P)$, $j = 1, 2, \dots$ is an asymptotically optimal universal sequence of approximations from $\tilde{M}_n(P)$ with respect to $\mathfrak{H}_3$.*

*If $r > n + 1$ the optimal properties of the above approximations take place only in those spaces $H \in \mathfrak{H}_3$ where $\eta_{p_{n+1}}\sqrt{2} \leqq \eta_{p_{n+2}}$.*

The assertions (5.33) to (5.35) enable us to compare the error of $\{\hat{B}_p\}$ with the error of the optimal universal approximation even in the cases not covered by Theorem 5.17.

**Theorem 5.18.** *Let the set $P$ be given and let $n$ be such that $|p_n| = |p_{n+1}|$. Let $r > n + 1$ and let there be given a space $H \in \mathfrak{H}_3$ such that $\eta_{p_{n+1}} \sqrt{2} > \eta_{p_{n+2}}$. Then*

$$Q_H(U_n, P, \hat{B}_p) \leqq \sqrt{2}\, \frac{\eta_{p_{n+1}}}{\eta_{p_{n+2}}} \leqq \sqrt{2}$$

*and the same inequalities hold also for* $\lim_{j\to\infty} Q_H(\tilde{U}_n, P, \hat{B}_p^{(j)})$.

If we return to the approximation $\{C_p\}$ from Theorem 5.10 we get with the help of (5.38)

**Theorem 5.19.** *Let the set $P$ be given and let $n$ be such that $|p_n| = |p_{n+1}|$. Then for every $H \in \mathfrak{H}_3$ it holds:*

I. *If $r = n + 1$, or $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} \leqq \eta_{p_{n+2}}$ then*

$$\lim_{j\to\infty} Q_H(\tilde{U}_n, P, C_p^{(j)}) = Q_H(U_n, P, C_p) = \sqrt{2}\,.$$

II. *If $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} > \eta_{p_{n+2}}$ then*

$$Q_H(U_n, P, C_p) \leqq \sqrt{2}$$

*and also*

$$\lim_{j\to\infty} Q_H(\tilde{U}_n, P, C_p^{(j)}) \leqq \sqrt{2}\,.$$

Proof. If I is the case then from (5.38) and Theorem 5.17 it follows

$$\Omega_H(U_n, P) = \Omega_H(\tilde{U}_n, P) = \frac{1}{\eta_{p_n} \sqrt{2}}\,.$$

The assertion of the theorem is obtained immediately from (5.19) and (5.27).

The statement of the theorem in the case II follows again from (5.19), (5.27) and from (5.38). The theorem is proved.

## 6. THE PRACTICAL ASPECTS

In this section we summarize briefly the practical conclusions resulting from the theorems of the paper.

Recall the formulation of our problem. Suppose that the function $f$ is known to belong to some indefinite strongly periodic space. Our task is to compute the values of its Fourier coefficients $I_p(f)$ for $p \in P$. If there is no additional information about $f$ available, then the theorems of Sec. 5 imply that the best strategy from our point of view (that is from the standpoint of universality and optimality as defined in the paper) is very simple. We shall concentrate our effort on approximating $I_p(f)$ for the $p$'s with a small absolute value ($p \in P_1$) and replace $I_p(f)$ simply by zero for the other $p \in P$ (the approximation $\{B_p\}$ from Theorem 5.4).

In addition, if we start replacing by zero from $I_{p_{n+1}}$, there is no reason to choose $p_{n+1}$ in such a way that (with the arrangement of the set $P$ introduced in Sec. 5) $p_{n+1} = -p_n$. In such a case it is better either to set $I_{p_n} \approx 0$ as well (the approximation $\{C_p\}$ from Theorem 5.10) or to use the approximation $\{\hat{B}_p\}$ from Theorem 5.14. The latter of the two possibilities means to evaluate one functional $a_k(f)$ more in comparison with the former, but it may decrease the error of the approximation by a factor of $1/\sqrt{2}$.

The question is to what extent this gain is substantial. Another question, which is of more importance and has not yet been settled, is what information about $f$ should be at our disposal in order to approve the use of more complex systems $\{g_k(p)\}$.

To approximate $I_p$ on $P_1$ we have used the trapezoidal rule functionals $L_p^{(j)}$ throughout the paper. Obviously, if we replace $L_p^{(j)}$ in $\{B_p^{(j)}\}$, $\{C_p^{(j)}\}$ or $\{\hat{B}_p^{(j)}\}$ by other functionals converging to $I_p$ in the norm, we obtain an asymptotically optimal universal sequence of approximations again. The choice of $L_p^{(j)}$ was implied by the availability of the proof of convergence in this case [2].

Moreover, if we use not right $I_p$ in $\{B_p\}$ etc., but some other approximating functionals such that the decisive part of the error resulted will be that of replacing by zero, we shall find that most results of Sec. 5 concerning $\{B_p\}$ etc. hold for the new approximations without any change.

Finally, we point out that it may be readily seen that, with a fixed set $P$, the conclusions of Sec. 5 are valid not only for the class $\mathfrak{H}_3$, but also for the class of periodic spaces $\mathfrak{H}_P$ such that every space $H \in \mathfrak{H}_P$ has the properties (c) and (d) for all $k \in P$, $j \in P$.

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