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STABILITY IN CONTINUOUS LOCAL SEMI-FLOWS[1])

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In this paper several stability properties of subsets of a solution space of a continuous local semi-flow on a uniform space will be investigated. The corresponding results are generalizations of some known results concerning relations between several notions of stability in the theory of differential equations, as they are treated e.g. in [2].

1. ABSTRACT LOCAL SEMI-FLOWS

1.1. Notation. In this paragraph P will denote an arbitrary abstract set, R the naturally ordered set of reals, I the set of integers, N the set of all positive integers. We shall investigate partial maps $t : R \times P \times R \to P$ and use the following notation: domain t will denote the set $\{(\theta, x, \alpha) \in R \times P \times R : t(\theta, x, \alpha) \text{ is defined}\}$. The value of the map t in the point $(\theta, x, \alpha) \in$ domain t will be denoted by ${}_{\beta}t_{\alpha}x$. To every $\alpha, \beta \in R$, $\alpha \leqq \beta$, there corresponds a partial map ${}_{\beta}t_{\alpha} : P \to P : {}_{\beta}t_{\alpha}(x) = {}_{\beta}t_{\alpha}x$ whenever $(\beta, x, \alpha) \in$ $\in$ domain t. For every triple $\alpha, \beta, \gamma \in R$ such that $\alpha \leqq \beta \leqq \gamma$ the symbol ${}_{\gamma}t_{\beta} \circ {}_{\beta}t_{\alpha}$ will denote the composition of the maps ${}_{\gamma}t_{\beta}$ and ${}_{\beta}t_{\alpha}$. To every partial map t there correspond the following sets:

$$\begin{aligned}
D &\equiv \{(x, \alpha) \in P \times R : (\alpha, x, \alpha) \in \text{domain } t\}\,,\\
C &\equiv \{x \in P : (x, \alpha) \in D \text{ for some } \alpha \in R\}\,,\\
B &\equiv \{\alpha \in R : (x, \alpha) \in D \text{ for some } x \in P\}\,,\\
D_A &\equiv \{(x, \gamma) \in D : x \in P,\ \gamma \in A \subset R\}\,.
\end{aligned}$$

Finally, we shall define a map

$$\varepsilon : D \to R \cup \{+\infty\} : \varepsilon(x, \alpha) = \sup \{\theta \in R : (\theta, x, \alpha) \in \text{domain } t\}\,,$$

which will play very important role in the following investigations. This notation will be used to define the following notion.

[1]) The main part of this paper was lectured at the 2nd EQUADIFF Symposium held in September 1966 in Bratislava.

1.2. Definition. A partial map $t : R \times P \times R \to P$ will be called an *abstract local semi-flow on P* iff the following conditions are satisfied:

(i) ${}_{\alpha}t_{\alpha}x = x$ holds for each $(x, \alpha) \in D$;
(ii) $\varepsilon(x, \alpha) > \alpha$ holds for each $(x, \alpha) \in D$;
(iii) ${}_{\gamma}t_{\beta} \circ {}_{\beta}t_{\alpha}x = {}_{\gamma}t_{\alpha}x$ holds for each $x \in P$ and $\gamma \geqq \beta \geqq \alpha$ such that at least one side of this equality is defined.

An abstract local semi-flow t on P will be called global iff $\varepsilon(x, \alpha) = +\infty$ holds for each $(x, \alpha) \in D$.

1.3. In what follows, al semi-flow (ag semi-flow) will be written instead of abstract local semi-flow (abstract global semi-flow). From 1.2. (iii) there follows directly the following proposition: if $(\gamma, x, \alpha) \in$ domain t, $\gamma > \alpha$, then $(\theta, x, \alpha) \in$ domain t holds for all $\theta \in \langle\alpha, \gamma\rangle$. Hence, using 1.2. (ii) there follows that corresponding to each $(x, \alpha) \in D$ there is a $\beta > \alpha$ such that $(\theta, x, \alpha) \in$ domain t for each $\theta \in \langle\alpha, \beta\rangle$. Clearly, if $y = {}_{\beta}t_{\alpha}x$, then $x = {}_{\alpha}t_{\alpha}x$, $y = {}_{\beta}t_{\beta}y$ and for each $z \in P$ such that $z \leqq {}_{\beta}t_{\alpha}x$ there holds $z \leqq y$. Finally, if $y = {}_{\beta}t_{\alpha}x$, then for each $\theta \geqq \beta$ there holds $(\theta, x, \alpha) \in$ domain t iff $(\theta, y, \beta) \in$ domain t, hence $\varepsilon(x, \alpha) \leqq \varepsilon(y, \beta)$.

1.4. Example. Let $P = R$ and define $t : R \times R \times R \to R : {}_{\theta}t_{\alpha}x = x/[1 + x(\theta - \alpha)]$ for $x, \alpha \in R$, and $\theta \in \langle\alpha, +\infty)$ for $x \geqq 0$, $\theta \in \langle\alpha, \alpha - 1/x)$ for $x < 0$. Clearly, t is al semi-flow on R, $\varepsilon(x, \alpha) = +\infty$ for $x \geqq 0$, $\varepsilon(x, \alpha) = \alpha - 1/x$ for $x < 0$.

1.5. Definition. Let t be an al semi-flow on P, $(x, \alpha) \in D$. A partial map $s : R \to P$ will be called a *solution* of the al semi-flow t through the point (x, α) iff the following conditions are satisfied:

(i) domain s is a nondegenerate interval in R;
(ii) $s\theta = {}_{\theta}t_{\alpha}x$ holds for all θ in domain s.

A *characteristic solution* s of al semi-flow t through (x, α) will be called every solution of t through (x, α) such that domain $s = \langle\alpha, \varepsilon(x, \alpha))$.

1.6. Remark. Clearly, domain $s \subset \langle\alpha, \varepsilon(x, \alpha))$ holds for every solution s through a point $(x, \alpha) \in D$.

1.7. Definition. Let t be an al semi-flow on P, $\tau \in R$. The al semi-flow t is said to *admit the period* τ iff

$${}_{\beta-\tau}t_{\alpha-\tau} = {}_{\beta}t_{\alpha} = {}_{\beta+\tau}t_{\alpha+\tau}$$

holds whenever ${}_{\beta}t_{\alpha}$ is defined.

1.8. Remark. If an al semi-flow t admits a period τ, then it admits also the period $k\tau$ for each $k \in I$. Hence there follows: if $(x, \alpha) \in D$, then $(x, \alpha + k\tau) \in D$ for each $k \in I$, so that $y = {}_{\beta}t_{\alpha}x$ implies $(y, \beta + l\tau) \in D$ for each $l \in I$. In the case of the al semi-flow admitting a period τ the map ε has the following interesting property.

$$\begin{aligned}\varepsilon(x, \alpha + k\tau) &= \sup\{\theta \in R : (\theta, x, \alpha + k\tau) \in \text{domain } t\} = \\ &= \sup\{\theta \in R : (\theta - k\tau, x, \alpha) \in \text{domain } t\} = \\ &= \sup\{\zeta + k\tau \in R : (\zeta, x, \alpha) \in \text{domain } t\} = \varepsilon(x, \alpha) + k\tau .\end{aligned}$$

We can characterise al semi-flows admitting a period in the following manner.

1.9. Lemma. *An al semi-flow t admits the period $\tau \in R$ iff for every solution s of t the partial maps $s_{\pm\tau} : R \to P : s_{\pm\tau}\theta = s(\theta \pm \tau)$ are also the solutions of t.*

Proof. First we prove the necessity of this condition.

The sets domain $s_{\pm\tau}$ are clearly nondegenerate intervals in R, and $s_{\pm\tau}\theta = s(\theta \pm \tau) = {}_{\theta\pm\tau}t_{\alpha\pm\tau}s(\alpha \pm \tau) = {}_{\theta}t_{\alpha}s_{\pm\tau}\alpha$ for all $\theta \geqq \alpha$ in domain s, hence s_τ and $s_{-\tau}$ are also the solutions of t.

Now we prove the sufficiency.

Let ${}_{\beta}t_{\alpha}$ be defined. We have to prove that the partial maps ${}_{\beta\pm\tau}t_{\alpha\pm\tau}$ are also defined and that there holds ${}_{\beta-\tau}t_{\alpha-\tau} = {}_{\beta}t_{\alpha} = {}_{\beta+\tau}t_{\alpha+\tau}$. Take $(x, \alpha) \in D_\alpha$. Let s be a characteristic solution of t through (x, α). Then $s_{\pm\tau}$ are also the solutions of t, domain $s_{\pm\tau} = \langle\alpha \mp \tau, \varepsilon(x, \alpha) \mp \tau)$, $s\theta = s_{-\tau}(\theta + \tau) = s_\tau(\theta - \tau)$ for all $\theta \in$ domain s, and $x = s\alpha = s_{\pm\tau}(\alpha \mp \tau)$. Further, there holds $\beta \pm \tau \in$ domain $s_{\mp\tau}$ so that ${}_{\beta\pm\tau}t_{\alpha\pm\tau}$ is defined and there is

$$ {}_{\beta\pm\tau}t_{\alpha\pm\tau}x = {}_{\beta\pm\tau}t_{\alpha\pm\tau}s_{\mp\tau}(\alpha \pm \tau) = s_{\mp\tau}(\beta \pm \tau) = s\beta = {}_{\beta}t_{\alpha}x ,$$

so that ${}_{\beta-\tau}t_{\alpha-\tau} = {}_{\beta}t_{\alpha} = {}_{\beta+\tau}t_{\alpha+\tau}$, i.e. t admits the period τ.

1.10. Definition. Let an al semi-flow t admits the period τ. A pair $(x, \alpha) \in D$ is called a *τ-periodic pair* iff there holds ${}_{\alpha+|\tau|}t_{\alpha}x = x$.

1.11. Remark. A τ-periodic pair $(x, \alpha) \in D$ can be characterised also in the following way. (x, α) is a τ-periodic pair iff ${}_{\theta+|\tau|}t_{\alpha}x = {}_{\theta}t_{\alpha}x$ holds for each $\theta \geqq \alpha$. Hence there follows that every solution s of t through (x, α) is a periodic map with the period $|\tau|$, i.e. $s(\theta + |\tau|) = s\theta$ for each $\theta \geqq \alpha$. Every τ-periodic pair is also a $k\tau$-periodic pair for each $k \in I$, hence, $\varepsilon(x, \alpha) = +\infty$ holds for each τ-periodic pair (x, α) with $\tau \neq 0$. If $(x, \alpha) \in D$ is a τ-periodic pair and $y = {}_{\beta}t_{\alpha}x$, then the pairs (y, β) and $(x, \alpha + k\tau)$ for each $k \in I$, are also τ-periodic.

1.12. Definition. An al semi-flow t on P is called *stationary* iff it admits all the periods $\tau \in R$.

1.13. Remark. From 1.9. one can easily obtain the following characterisation of stationary al semi-flows: an al semi-flow t is stationary iff for each solution s of t a partial map s_σ, for each $\sigma \in R$, is a solution of t. Clearly, if $(x, \alpha) \in D$ is a τ-periodic pair of a stationary al semi-flow, for some $\tau \in R$, then for each $\beta \in R$ all the pairs (x, β) are also τ-periodic. For stationary al semi-flows the following criterion of globality takes place.

1.14. Lemma. *If t is a stationary al semi-flow and there exist numbers $\alpha, r \in R$ such that $\alpha < r < \varepsilon(x, \alpha)$ for all $(x, \alpha) \in D_\alpha$, then t is global.*

Proof. Suppose that there is a pair $(y, \beta) \in D$ such that $\varepsilon(y, \beta) < +\infty$. Let $\gamma > 0$ be such that $\beta < \varepsilon(y, \beta) + \alpha - r + \gamma < \varepsilon(y, \beta)$ and set $z = {}_{\alpha+\varepsilon(y,\beta)-r+\gamma}t_\beta y$. Then, corresponding to 1.3. and 1.8., there follows

$$\varepsilon(y, \beta) = \varepsilon(z, \alpha + \varepsilon(y, \beta) - r + \gamma) = \varepsilon(z, \alpha) + \varepsilon(y, \beta) - r + \gamma,$$

hence $\varepsilon(z, \alpha) = r - \gamma < r$, contradicting the assumption $\varepsilon(z, \alpha) > r$ for $(z, \alpha) \in D_\alpha$.

1.15. Corollary. *For every stationary al semi-flow t either t is global or for each $\alpha \in R$ there holds* $\inf \{\varepsilon(x, \alpha) : (x, \alpha) \in D_\alpha\} = 0$.

The following example indicates that lemma 1.14. need not hold for non-stationary semi-flows admitting the period $\tau > 0$.

1.16. Example. Let P be a singleton $\{a\}$ and let $\tau > 0$ be given. Define an al semi-flow t such that ${}_\beta t_\alpha a = a$ iff $\beta \geqq \alpha$ in R and there exists $k \in I$ such that $\alpha, \beta \in$ $\in \langle (k - \frac{1}{2}) \tau, (k + \frac{1}{2}) \tau)$.

1.17. Definition. Let t be an al semi-flow. A pair $(x, \alpha) \in D$ will be called a *stationary pair* of the al semi-flow t iff ${}_\theta t_\alpha x \leqq x$ holds for all $\theta \in \langle \alpha, \varepsilon(x, \alpha))$.

1.18. Remark. Finally, let us investigate in some details a structure of the set D for al semi-flows admitting the period τ. Let $\langle a, b\rangle$ and $\langle c, d\rangle$ are intervals in R such that $b - a = d - c$ and let $c = a + k\tau$ for some $k \in I$. According to 1.8., there holds $(x, \alpha) \in D$ iff $(x, \alpha + k\tau) \in D$ for each $k \in I$. Hence one obtains $(x, \alpha) \in D_{\langle a,b\rangle}$ iff $(x, \alpha + k\tau) \in D_{\langle c,d\rangle}$, i.e. $D_{\langle c,d\rangle} = \{(x, \beta) : \beta = \alpha + k\tau,\ (x, \alpha) \in D_{\langle a,b\rangle}\}$. Loosely spaking, D consists of "strips" obtained by shifting "the strip" $D_{\langle 0,\tau\rangle}$ along the θ-axis by an integer multiple of τ. Hence there follows the corollary: if an al semi-flow admits the period $\tau > 0$ and $D_{\langle 0,\tau\rangle} = A \times \langle 0, \tau\rangle$ for some $A \subset P$, then $D = A \times R$. Especially, if an al semi-flow is stationary, then there holds $D = C \times R$ (see 1.1.).

2. CONTINUOUS LOCAL SEMI-FLOWS AND BOUNDEDNESS OF SOLUTIONS

2.1. Notation. In the remaining part of this paper P will denote a uniform space with a uniformity $\mathfrak{U}$, R will denote a uniform space of reals with the natural uniformity (induced in the natural way by the Euclidean metric on the one-dimensional Euclidean space) and $\mathfrak{C}$ will denote the set of all compact subsets of the space P. For $A, B \subset R$,

$X \subset P$, the set $\{_\theta t_\alpha x : (\theta, x, \alpha) \in \text{domain } t \cap (A \times X \times B)\}$ will be denoted by $_A t_B X$. Finally, for $X \subset P$ and $U \in \mathfrak{U}$ let $U[X]$ denote the set $\{y \in P : (x, y) \in U$ for some $x \in X\}$.

2.2. Definition. A partial map $t : R \times P \times R \to P$ will be called a *continuous local semi-flow* (a *continuous global semi-flow*) on a uniform space P iff the following three conditions are satisfied:

(i) t is an abstract local (global) semi-flow on P;
(ii) t is continuous (in the corresponding uniform topologies);
(iii) the map ε is lower semi-continuous, i.e.

$$\varepsilon(x, \alpha) \leqq \liminf_{\substack{(y,\beta)\to(x,\alpha)\\ (y,\beta)\in D}} \varepsilon(y, \beta)\,.$$

2.3. Remark. We shall write cl (cg) semi-flow instead of continuous local (global) semi-flow.

From 2.2. (ii) there follows directly that, for each $(x, \alpha) \in D$, the partial map $t' : R \to$ $\to P : t'(\theta) = {_\theta t_\alpha} x$ is continuous, hence every solution of a cl semi-flow on a uniform space P is continuous.

In what follows, the term "a solution of a cl semi-flow through (x, α)" will denote "a characteristic solution of a cl semi-flow through (x, α)".

2.4. Lemma. *Let P be a uniform space, t a cl semi-flow on P, $(x, \alpha) \in D$. If $\varepsilon(x, \alpha) <$ $< +\infty$ and numbers $\theta_j \in \langle\alpha, \varepsilon(x, \alpha))$ are such that the sequence $\{\theta_j\}$ tends to $\varepsilon(x, \alpha)$ for $j \to +\infty$, then the sequence $\{({_{\theta_j} t_\alpha} x, \theta_j)\}$ has no accumulation point in D.*

Proof. Suppose that $(y, \varepsilon(x, \alpha)) \in D$ is an accumulation point of this sequence. According to 1.3. $\varepsilon({_{\theta_j} t_\alpha} x, \theta_j) = \varepsilon(x, \alpha)$, hence by 2.2. (iii) one obtains

$$\varepsilon(y, \varepsilon(x, \alpha)) \leqq \liminf_{({_{\theta_j} t_\alpha} x, \theta_j)\to(y,\varepsilon(x,\alpha))} \varepsilon({_{\theta_j} t_\alpha} x, \theta_j) = \liminf \varepsilon(x, \alpha) = \varepsilon(x, \alpha)\,,$$

but $\varepsilon(y, \varepsilon(x, \alpha)) \leqq \varepsilon(x, \alpha)$ contradicts 1.2. (ii) as $(y, \varepsilon(x, \alpha)) \in D$, which proves the lemma.

2.5. Corollary. *Let $(x, \alpha) \in D$ be such that $\varepsilon(x, \alpha) < +\infty$, and let $L_n = \{{_\theta t_\alpha} x :$ $: \max(\alpha, \varepsilon(x, \alpha) - 1/n) \leqq \theta < \varepsilon(x, \alpha)\}$. Then for each $y \in \bigcap_{n\in N} \bar{L}_n$ there holds $(y, \varepsilon(x, \alpha)) \notin D$.*

2.6. Definition. Let P be a uniform space. A set $X \subset P$ is said to be *bounded* iff its closure $\bar{X}$ in P (with the corresponding uniform topology) is compact.

2.7. Lemma. *Let P be a locally compact uniform space. Then to every bounded set $X \subset P$ there corresponds a $U \in \mathfrak{U}$ such that $U[X]$ is bounded.*

Proof. Since P is locally compact, there exists a family $\{V(x)\}$ of open relatively compact neighbourhoods of points $x \in \bar{X}$ such that $\bar{X} \subset \bigcup_{x \in X} V(x)$. As $\bar{X}$ is compact, there exists a finite subset $\{V(x_1), \ldots, V(x_n)\} \subset \{V(x)\}$ such that the set $V = \bigcup_{j=1}^{n} V(x_j)$ is an open relatively compact neighbourhood of $\bar{X}$. Hence (see e.g. [1], chap. II, § 4;3,1), there is a $U \in \mathfrak{U}$ such that $U[X] \subset U[\bar{X}] \subset V \subset \bar{V} \in \mathfrak{C}$, so that the set $U[X]$ is relatively compact.

2.8. Definition. A solution of a cl semi-flow t through $(x, \alpha) \in D$ is said to be *bounded* iff the set ${}_R t_\alpha x$ is bounded.

2.9. Lemma. *Let t be a cl semi-flow on a uniform space P with D closed in $P \times R$. If a solution of t through $(x, \alpha) \in D$ is bounded, then $\varepsilon(x, \alpha) = +\infty$.*

Proof. Let numbers $\beta_j \in \langle \alpha, \varepsilon(x, \alpha))$ are such that $\{\beta_j\} \to \varepsilon(x, \alpha)$ for $j \to +\infty$. According to the assumption ${}_R t_\alpha x$ is relatively compact, and we can assume that the sequence $\{{}_{\beta_j} t_\alpha x\}$ converges to some $y \in P$. Since $({}_{\beta_j} t_\alpha x, \beta_j) \in D$, and D is closed, there holds $(y, \varepsilon(x, \alpha)) \in D$, what contradicts 2.4. and proves the lemma.

2.10. Corollary. *Let t be a cl semi-flow on a uniform space P with D closed in $P \times R$.*

(i) *If the solution through (x, α) is bounded for each $(x, \alpha) \in D$, then cl semi-flow t is global.*

(ii) *If C (see 1.1.) is compact, then cl semi-flow t is global.*

2.11. Remark. From 2.2. (ii) there follows immediately the following simple proposition, which we shall use often in what follows. Let t be a cl semi-flow on P, $X \in \mathfrak{C}$, $\alpha \leqq \beta \leqq \delta$ reals such that $\varepsilon(x, \alpha) > \delta$ for all $(x, \gamma) \in X \times \langle \alpha, \beta \rangle$. Then ${}_{\langle \alpha, \delta \rangle} t_{\langle \alpha, \beta \rangle} X \in \mathfrak{C}$. Hence there follows easily that every solution through some τ-periodic pair, for $\tau \neq 0$, is bounded. Especially, a solution of t through some stationary pair is bounded.

Now we shall formulate some simple propositions concerning boundedness of solutions of cl semi-flows admitting a period.

2.12. Lemma. *Let cg semi-flow t admits the period $\tau \neq 0$, and let the solution through $(x, 0)$, for each $(x, 0) \in D$, is bounded. Then each solution of t is bounded.*

Proof. Let $(x, \alpha) \in D$ and let $\beta = \alpha - k\tau \in \langle 0, \tau)$ for some $k \in I$. Clearly,

$${}_R t_\alpha x = {}_{\langle \alpha, k\tau \rangle} t_\alpha x \cup {}_{\langle k\tau, +\infty)} t_\alpha x \,,$$

where ${}_{\langle \alpha, k\tau \rangle} t_\alpha x \in \mathfrak{C}$ according to 2.11., and from the relation

$${}_{\langle k\tau, +\infty)} t_\alpha x = \{{}_\theta t_{k\tau} \circ {}_{k\tau} t_\alpha x : \theta \geqq k\tau\} = \{{}_\sigma t_0 y : y = {}_{k\tau} t_\alpha x, \sigma \geqq 0\}$$

there follows that ${}_{\langle k\tau, +\infty)} t_\alpha x$ is bounded, hence the lemma easily follows.

If the cl semi-flow t in 2.12. is stationary, one can omit there the assumption of globality of t, which will be proved now.

2.13. Lemma. *Let a cl semi-flow t be stationary and let there be $(x, \alpha) \in D$ such that the solution of t through (x, α) is bounded. Then for each $\beta \in R$ such that $(x, \beta) \in D$, the solution of t through (x, β) is bounded.*

Proof. Since t is stationary, there holds ${}_{\theta}t_{\beta}x = {}_{\theta-\beta+\alpha}t_{\alpha}x$ and $\varepsilon(x, \beta) = \varepsilon(x, \alpha) + \beta - \alpha$, so that

$$ {}_{R}t_{\beta}x = \{{}_{\theta}t_{\beta}x : \beta \leqq \theta < \varepsilon(x, \alpha)\} = \{{}_{\theta-\beta+\alpha}t_{\alpha}x : \alpha \leqq \theta - \beta + \alpha < \varepsilon(x, \beta) - \beta - \alpha\} = $$
$$ = \{{}_{\sigma}t_{\alpha}x : \alpha \leqq \sigma < \varepsilon(x, \alpha)\} = {}_{R}t_{\alpha}x\,, $$

and according to the assumption ${}_{R}t_{\alpha}x$ is bounded, what proves the lemma.

2.14. Corollary. *Let a cl semi-flow t be stationary and let there exists $\alpha \in R$ such that for each $(x, \alpha) \in D_{\alpha}$ the solution of t through (x, α) is bounded. Then all solutions of t are bounded.*

Now we shall remember several further notions of boundedness.

2.15. Definition. Let t be a cl semi-flow on a uniform space P.

(i) Solutions of t are said to be *equi-bounded for a given $\alpha \in R$* iff for each $X \in \mathfrak{C}$ the set ${}_{R}t_{\alpha}X$ is bounded.

(ii) Solutions of t are said to be *equi-bounded* iff they are equi-bounded for each $\alpha \in R$.

(iii) Solutions of t are said to be *uniformly bounded* iff for each $X \in \mathfrak{C}$ the set ${}_{R}t_{R}X$ is bounded.

2.16. Theorem. *Let a cl semi-flow t admit the period $\tau > 0$ and let $\varepsilon(x, \alpha) > \tau$ hold for each $(x, \alpha) \in D$ with $0 \leqq \alpha \leqq \tau$. Then the following three propositions are equivalent.*

(i) *Solutions of t are equi-bounded for $\alpha = 0$.*
(ii) *Solutions of t are equi-bounded.*
(iii) *Solutions of t are uniformly bounded.*

Proof. It suffices to prove that for each $X \in \mathfrak{C}$ the set ${}_{R}t_{R}X$ is bounded whenever ${}_{R}t_{0}X$ is bounded.

First we shall prove that ${}_{R}t_{R}X \subset {}_{R}t_{\langle 0,\tau\rangle}X$. Let $\beta = \beta' + k\tau \in R$, $k \in I$, $\beta' \in \langle 0, \tau\rangle$. Then there holds

$$ {}_{R}t_{\beta}X = \{{}_{\theta}t_{\beta}x : x \in X,\ \theta \geqq \beta\} = $$
$$ = \{{}_{\theta-k\tau}t_{\beta-k\tau}x : x \in X,\ \beta - k\tau \leqq \theta - k\tau < \varepsilon(x, \beta - k\tau)\} = $$
$$ = \{{}_{\sigma}t_{\beta'}x : x \in X,\ \beta \leqq \sigma < \varepsilon(x, \beta')\} = {}_{R}t_{\beta'}X \subset {}_{R}t_{\langle 0,\tau\rangle}X\,. $$

Now we shall prove that ${}_R t_{\langle 0,\tau\rangle} X$ is bounded. Clearly,

$$ {}_R t_{\langle 0,\tau\rangle} X = {}_{\langle 0,\tau\rangle} t_{\langle 0,\tau\rangle} X \cup {}_{\langle \tau,+\infty)} t_{\langle 0,\tau\rangle} X \,, $$

where the first member of the union is compact. Let $H = {}_\tau t_{\langle 0,\tau\rangle} X$. As H is compact, the set ${}_R t_0 H$ is bounded according to the assumption and there holds

$$ {}_R t_0 H = \{{}_\theta t_0 y : y \in H,\ 0 \leqq \theta < \varepsilon(y, 0)\} = $$
$$ = \{{}_{\theta+\tau} t_\tau y : y \in H,\ \tau \leqq \theta + \tau < \varepsilon(y, \tau)\} = $$
$$ = \{{}_\sigma t_\tau \circ {}_\tau t_\beta x : x \in X,\ 0 \leqq \beta \leqq \tau \leqq \sigma < \varepsilon(x, \beta)\} = {}_{\langle \tau,+\infty)} t_{\langle 0,\tau\rangle} X \,, $$

hence the second member of the union is also bounded, thus the set ${}_R t_R X$ is bounded, i.e. (iii) holds, what finishes the proof.

2.17. Theorem. *Let t be a stationary cl semi-flow. Then the following three propositions are equivalent*:

(i) *solutions of t are equi-bounded for some $\alpha \in R$*;
(ii) *solutions of t are equi-bounded*;
(iii) *solutions of t are uniformly bounded.*

Proof. Again we shall prove only that (i) implies (iii). Let $X \in \mathfrak{C}$. For any $x \in X$ and $\alpha, \beta \in R$ there holds ${}_R t_\alpha x = {}_R t_\beta x$ (see proof of 2.13.) so that ${}_R t_\beta X = {}_R t_\alpha X$ and hence ${}_R t_R X = {}_R t_\alpha X$. Since ${}_R t_\alpha X$ is bounded according to the assumption, then ${}_R t_R X$ is also bounded, i.e. (iii) holds and the proof is finished.

3. STABILITY

3.1. Notation. In this paragraph P will denote a complete locally compact uniform space with a uniformity $\mathfrak{U}$, $\mathfrak{C}$ the set of all its compact subsets, t a cl semi-flow on P. Given any $X \subset P$ and $\alpha \in R$, denote X^α the set $\{x \in X : (x, \alpha) \in D\}$. In what follows, very important role will be played by a partial map

$$ m : R \to \exp P \tag{1} $$

and by a corresponding one

$$ \bar{m} : R \to \exp P : \bar{m}\theta = \overline{m\theta}\,. \tag{2} $$

The further assumptions on the maps (1) and (2) will be given later. We shall investigate several properties of maps (1) and (2) concerning stability and we shall use often the following simple lemma.

3.2. Lemma. *Let P be a uniform space with a uniformity $\mathfrak{U}$. Corresponding to every $U \in \mathfrak{U}$ there is a $V \in \mathfrak{U}$ such that for each $X \subset P$ there holds $V[\overline{X}] \subset U[X]$.*

3.3. Definition. Let there be given a cl semi-flow t on P and a partial map $m : R \to \to \exp P$. m is said to be *invariant* with respect to t iff for each $(x, \alpha) \in D$ such that $x \in m\alpha$ there holds ${}_{\theta}t_{\alpha}x \in m\theta$ for each $\theta \in \langle \alpha, \varepsilon(x, \alpha))$.

3.4. Lemma. *Let m be invariant with respect to t and let $(x, \alpha) \in D$ hold for each $\alpha \in$ domain m and $x \in m\alpha$. Then $\bar{m}$ is also invariant with respect to t.*

Proof. Let us suppose that there is an $(x, \alpha) \in D$ such that $x \in \bar{m}\alpha$, and for some $\beta \in \langle \alpha, \varepsilon(x, \alpha))$ there holds ${}_{\beta}t_{\alpha}x \notin \bar{m}\beta$. Let $x_{\gamma} \in m\alpha_{\gamma}$ be such that the sequence $\{(x_{\gamma}, \alpha_{\gamma})\}$ converges to (x, α), and $\alpha_{\gamma} \leqq \beta < \varepsilon(x_{\gamma}, \alpha_{\gamma})$ (the existence of such elements follows easily from the lower semi-continuity of ε). Since the uniform topology of P is completely regular, there exists $U \in \mathfrak{U}$ such that $U[{}_{\beta}t_{\alpha}x] \cap U[\bar{m}\beta] = \emptyset$, and corresponding to the U there exist $V \in \mathfrak{U}$ and γ_0 such that $(x_{\gamma_0}, x) \in V$ implies $({}_{\beta}t_{\alpha}x_{\gamma_0}, {}_{\beta}t_{\alpha}x) \in U$, i.e. ${}_{\beta}t_{\alpha}x \in U[{}_{\beta}t_{\alpha}x_{\gamma_0}]$, hence ${}_{\beta}t_{\alpha}x_{\gamma_0} \notin U[m\beta]$, although $x_{\gamma_0} \in m\alpha_{\gamma_0}$ and m is invariant, which is a contradiction proving the lemma.

If the condition, $(x, \alpha) \in D$ for each $\alpha \in$ domain m and $x \in m\alpha$, is not satisfied, lemma 3.4. need not take place, as will be shown in the following example.

3.5. Example. Let there be given $\xi > 0$. Define a cl semi-flow t on R as follows:

$$ {}_{\theta}t_{\alpha}x = x + \theta - \alpha\,, \quad (x, \alpha) \in R \times R\,, \quad x - \alpha \geqq \xi \text{ or } x - \alpha \leqq -\xi\,, \quad \theta \geqq \alpha $$

and the map m by the relation

$$ m\theta = \begin{array}{ll} (\theta - \xi, +\infty) & \text{for} \quad \theta \in (-\infty, 0)\,, \\ (\theta - \xi(1 - 2\theta), +\infty) & \text{for} \quad \theta \in \langle 0, 1 \rangle\,, \\ (\theta + \xi, +\infty) & \text{for} \quad \theta \in (1, +\infty)\,. \end{array} $$

Clearly, m is invariant with respect to t while $\bar{m}$ is not.

It is trivial that the invariantness of $\bar{m}$ need not necessarily imply the invariantness of m.

3.6. Definition. Let there be given a cl semi-flow t on P and a partial map m. Then m is said to be *stable* iff there exists a partial map

(3) $$ v : R \times \mathfrak{U} \to \mathfrak{U} $$

such that

(4) $$ V = v(\alpha, U)\,, \quad x \in V[m\alpha]\,, \quad \theta \in \langle \alpha, \varepsilon(x, \alpha)) \quad \text{implies} \quad {}_{\theta}t_{\alpha}x \in U[m\theta]\,. $$

Now we shall investigate relations between stability and invariantness properties of m and $\bar{m}$.

3.7. Lemma. *m is stable iff $\overline{m}$ is stable.*

Proof. If m is stable, then there is a map (3) such that (4) holds. According to 3.2., there is a $W \in \mathfrak{U}$ such that $W[\overline{m}\alpha] \subset V[m\alpha]$. Since $U[m\theta] \subset U[\overline{m}\theta]$ for all $\theta \in$ $\in$ domain m, setting $W = v(\alpha, U)$, one obtains that $x \in W[\overline{m}\alpha]$, $\theta \in \langle \alpha, \varepsilon(x, \alpha))$ implies ${}_\theta t_\alpha x \in U[\overline{m}\theta]$, which proves the stability of $\overline{m}$.

Now, let $\overline{m}$ be stable. Let there be given $(\alpha, U) \in R \times \mathfrak{U}$ and $W \in \mathfrak{U}$ such that $W \subset U$. Then $W[\overline{m}\theta] \subset U[m\theta]$ holds for all $\theta \in$ domain m. According to the assumption there is a map (3) such that

$$V = v(\alpha, W)\,, \quad x \in V[\overline{m}\alpha]\,, \quad \theta \in \langle \alpha, \varepsilon(x, \alpha)) \quad \text{implies} \quad {}_\theta t_\alpha x \in W[\overline{m}\theta]\,.$$

Hence, setting $v(\alpha, U) = V$ and noticing that $V[m\alpha] \subset V[\overline{m}\alpha]$ one has $x \in V[m\alpha]$, $\theta \in \langle \alpha, \varepsilon(x, \alpha))$ implies ${}_\theta t_\alpha x \in W[\overline{m}\theta] \subset U[m\alpha]$, which denotes the stability of m and finishes the proof.

3.8. Lemma. *If m is stable, than $\overline{m}$ is invariant.*

Proof. Suppose that $\overline{m}$ is not invariant. Then there are $(z, \alpha) \in D$, $z \in \overline{m}\alpha$ and $\beta \in \langle \alpha, \varepsilon(x, \alpha))$ such that ${}_\beta t_\alpha z \notin \overline{m}\beta$. Hence there is $W \in \mathfrak{U}$ such that $W[{}_\beta t_\alpha z] \cap$ $\cap\, W[\overline{m}\beta] = \emptyset$, i.e.

$$(5) \qquad {}_\beta t_\alpha z \notin W[m\beta]\,.$$

Corresponding to the assumption there is a partial map (3) such that

$$(6) \qquad V = v(\alpha, W)\,, \quad y \in V[m\alpha]\,, \quad \theta \in \langle \alpha, \varepsilon(x, \alpha)) \quad \text{implies} \quad {}_\beta t_\alpha y \in W[m\beta]\,.$$

Since $z \in \overline{m}\alpha$, there holds $z \in V[m\alpha]$ for each $V \in \mathfrak{U}$, so that according to (6) there holds ${}_\beta t_\alpha z \in W[m\beta]$, which contradicts (5) and proves the lemma.

A stability of m does not imply invariantness of m, what can be seen easily. Similarly invariantness of m or $\overline{m}$ does not imply stability of m. Notwithstanding, there holds at least the following proposition.

3.9. Lemma. *Let there be given a cl semi-flow t, a partial map m and reals $\sigma_1 < \sigma_2$ such that the following conditions are satisfied:*

(i) *m is invariant on the interval $\langle \sigma_1, \sigma_2 \rangle$;*

(ii) *corresponding to each $U \in \mathfrak{U}$ there is a $\delta > 0$ such that $|\theta - \theta'| < \delta$ implies $m\theta \subset U[m\theta']$, where $\theta, \theta' \in \langle \sigma_1, \sigma_2 \rangle$;*

(iii) $\bigcup_{\sigma_1 \leqq \theta \leqq \sigma_2} m\theta$ *is closed;*

(iv) $\{(\theta, x, \alpha) \in R \times P \times R : \alpha \in \langle \sigma_1, \sigma_2 \rangle,\ \ x \in \text{Frontier}\ m\alpha,\ \ \theta \in \langle \alpha, \sigma_2 \rangle\} \cap$ $\cap$ domain t *is closed.*

Then there is a partial map

$$(7) \qquad u : \mathfrak{U} \to \mathfrak{U}$$

such that $V = u(U)$, $\sigma_1 \leqq \alpha \leqq \sigma_2$, $x \in V[m\alpha]$, $\alpha \leqq \theta \leqq \sigma_2$, $(\theta, x, \alpha) \in$ domain t *implies* ${}_\theta t_\alpha x \in U[m\theta]$.

Proof. Suppose that the lemma does not hold. Then there is $U \in \mathfrak{U}$ such that corresponding to each $V \in \mathfrak{U}$ there exists $(\theta_V, x_V, \alpha_V) \in$ domain t such that there holds

$$(8) \qquad \sigma_1 \leqq \alpha_V \leqq \theta_V \leqq \sigma_2 \,, \quad x_V \in V[m\alpha_V] \quad \text{and} \quad {}_{\theta_V}t_{\alpha_V}x_V \notin U[m\theta_V] \,.$$

Denote $X_V = \{x_V : V \in U\}$ and prove that there is an $\alpha \in \langle\sigma_1, \sigma_2\rangle$ such that $\bar{X}_V \cap \cap m\alpha \neq \emptyset$.

Suppose $\bar{X}_V \cap (\bigcup_{\sigma_1 \leqq \alpha \leqq \sigma_2} m\alpha) = \emptyset$. Then, since P is completely regular, there exists $W \in \mathfrak{U}$ such that $W[\bar{X}_V] \cap W[\bigcup m\alpha] = \emptyset$. On the other hand, corresponding to the definition of X_V there is $x_W \in X_V$ and $\alpha_W \in \langle\sigma_1, \sigma_2\rangle$ such that $x_W \in W[m\alpha_W]$. Hence, $x_W \in X_V \cap W[m\alpha_W] \subset W[\bar{X}_V] \cap W[\bigcup m\alpha] = \emptyset$, which gives a contradiction.

Now, let $(y, \alpha) \in (\bar{X}_V \cap m\alpha) \times \langle\sigma_1, \sigma_2\rangle$. From the definition of X_V and the assumption (i) there follows $X_V \cap m\alpha = \emptyset$ for all $\alpha \in \langle\sigma_1, \sigma_2\rangle$ so that y must be an accumulation point of X_V; hence there exists a sequence $\{(\theta_{V_j}, x_{V_j}, \alpha_{V_j})\}$ such that $x_{V_j} \in X_V$ and $(\theta_{V_j}, x_{V_j}, \alpha_{V_j}) \to (\theta, y, \alpha)$, where $\sigma_1 \leqq \alpha_{V_j} \leqq \theta_{V_j} \leqq \sigma_2$ are as in (8). Corresponding to (iv), $(\theta, y, \alpha) \in$ domain t, and from the continuity of t there follows (we write j instead of V_j)

$${}_{\theta_j}t_{\alpha_j}x_j \to {}_{\theta}t_{\alpha}y \quad \text{for} \quad j \to +\infty \,.$$

Let $U_0 \in \mathfrak{U}$ be such that $U_0 \subset U$. According to the assumption (ii) there is a number $\delta > 0$ such that $m\theta \subset U_0[m\theta_j]$ for each j such that $|\theta - \theta_j| < \delta$. Hence, there is j_0 such that for all $j > j_0$ there holds

$${}_{\theta_j}t_{\alpha_j}x_j \in U_0[{}_{\theta}t_{\alpha}y] \subset U_0[m\theta] \subset U_0[m\theta_j] \subset U[m\theta_j] \,,$$

which contradicts (8) and proves the lemma.

The preceding lemma need not hold if the assumption (iii) does not hold, which will be seen from the following example.

3.10. Example. Define a cl semi-flow t on R as follows:

$${}_{\theta}t_{\alpha}x = \frac{x\alpha\theta}{x(\alpha - \theta) + \alpha\theta} \quad \text{for} \quad (x, \alpha) \in ((-\infty, 0\rangle \cup \langle 1, +\infty)) \times (0, +\infty) \,,$$

$$\theta \in \left\langle \alpha, \frac{x}{x - \alpha} \right) \quad \text{if} \quad x > \alpha \,,$$

$$\theta \in \langle \alpha, +\infty) \quad \text{if} \quad x \leqq \alpha \,,$$

and the partial map m by the relation

$$m\theta = \langle -1, 1) \quad \text{for all} \quad \theta \in (0, +\infty) \,.$$

Setting e.g. $\sigma_1 = \varepsilon$, $\sigma_2 = 1$, with $0 < \varepsilon < 1$, one obtains the counterexample.

3.11. Definition. Let there be given a cl semi-flow t on P and a partial map m. m is said to be *uniformly stable* iff there exists a map

$$v_1 : \mathfrak{U} \to \mathfrak{U}$$

such that

$$(9) \qquad (x, \alpha) \in D \quad \text{with} \quad x \in v_1(U)[m\alpha] \quad \text{and} \quad \theta \in \langle \alpha, \varepsilon(x, \alpha)) \quad \text{implies} \quad {}_\theta t_\alpha x \in U[m\theta]\,.$$

3.12. Theorem. *Let there be given a cl semi-flow t admitting a period $\tau > 0$ and a partial map m periodic with the period τ, satisfying the assumptions of lemma 3.9. on the interval $\langle 0, \tau \rangle$. Then, if m is stable, it is uniformly stable.*

Proof. Since m is stable, there is a map (3) such that

$$(10) \qquad V = v(\tau, U)\,, \quad (x, \alpha) \in D \quad \text{with} \quad x \in V[m\tau]\,, \quad \theta \in \langle \tau, \varepsilon(x, \tau)) \quad \text{implies} \quad {}_\theta t_\tau x \in U[m\theta]\,.$$

According to 3.9., there is a map (7) such that

$$(11) \qquad W = u(V)\,, \quad \alpha \in \langle 0, \tau \rangle\,, \quad x \in W[m\alpha]\,, \quad \theta \in \langle \alpha, \tau \rangle\,, \quad (\theta, x, \alpha) \in \text{domain}\ t \quad \text{implies} \quad {}_\theta t_\alpha x \in V[m\theta] \subset U[m\theta]\,.$$

Set $v_1(U) = u(v(\tau, U)) = W$. If $\varepsilon(x, \alpha) \leqq \tau$ for all $(x, \alpha) \in D$ with $\alpha \in \langle 0, \tau \rangle$, $x \in W[m\alpha]$, then, for the case $\alpha \in \langle 0, \tau \rangle$, the relation (9) follows directly from (11).

If there is an $(x, \alpha) \in D$ with $\alpha \in \langle 0, \tau \rangle$, $x \in W[m\alpha]$ and $\varepsilon(x, \alpha) \leqq \tau$, then from (11) there follows ${}_\tau t_\alpha x \in V[m\tau]$, hence, according to (10),

$$(12) \qquad {}_\theta t_\alpha x = {}_\theta t_\tau \circ {}_\tau t_\alpha x \in U[m\theta] \quad \text{for all} \quad \theta \in \langle \alpha, \varepsilon(x, \alpha))\,.$$

Now, let $(x, \alpha) \in D$ with $\alpha \in R$, $x \in W[m\alpha]$, and let $\beta \in \langle 0, \tau)$ and integer k be such that $\alpha = k\tau + \beta$. Then, corresponding to the assumptions on the periodicity and relations (11), (12) there holds

$${}_\theta t_\alpha x = {}_{\theta - k\tau} t_\beta x \in U[m(\theta - k\tau)] = U[m\theta] \quad \text{for each} \quad \theta \in \langle \alpha, \varepsilon(x, \alpha))\,,$$

which finishes the proof.

3.13. Definition. Let there be given a cl semi-flow t and a partial map m.

(i) m is said to be *quasi-asymptotically stable* iff there are maps

$$w : R \to \mathfrak{U}\,, \quad s : R \times \mathfrak{U} \to R^+$$

such that

$$\alpha + s(\alpha, U) < \varepsilon(x, \alpha) \quad \text{holds for each} \quad U \in \mathfrak{U} \quad \text{and} \quad (x, \alpha) \in D \quad \text{with} \quad x \in w(\alpha)[m\alpha]\,,$$

and

$$(x, \alpha) \in D, \quad x \in w(\alpha)[m\alpha], \quad \alpha + s(\alpha, U) \leqq \theta < \varepsilon(x, \alpha) \quad \text{implies} \quad {}_{\theta}t_{\alpha}x \in U[m\theta].$$

(ii) m is said to be *asymptotically stable* iff it is stable and quasi-asymptotically stable.

3.14. Lemma. *m is quasi-asymptotically stable (asymptotically stable) iff $\overline{m}$ is quasi-asymptotically stable (asymptotically stable respectively).*

Proof is nearly the same as in 3.7.

3.15. Theorem. *Let there be given a cl semi-flow t and a partial map m satisfying conditions of lemma 3.9. on each compact interval in R. If m is quasi-asymptotically stable, it is asymptotically stable.*

Proof. Let there be given $(\alpha, U) \in R \times \mathfrak{U}$. Since m is quasi-asymptotically stable, there are maps w and s such that $V_0 = w(\alpha)$, $S = s(\alpha, U)$, $x \in V_0[m\alpha]$, $\theta \in \langle \alpha + S, \varepsilon(x, \alpha))$ implies ${}_{\theta}t_{\alpha}x \in U[m\theta]$. According to 3.9., corresponding to the interval $\langle \alpha, \alpha + S \rangle$, there is a map u such that $V_1 = u(U)$, $x \in V_1[m\alpha]$, $\theta \in \langle \alpha, \alpha + S \rangle$ implies ${}_{\theta}t_{\alpha}x \in U[m\theta]$. Now, setting $v(\alpha, U) = V = V_0 \cap V_1$, there holds

$${}_{\theta}t_{\alpha}x \in U[m\theta] \quad \text{whenever} \quad x \in V[m\alpha], \quad \alpha \leqq \theta < \varepsilon(x, \alpha),$$

which proves the stability of m and finishes the proof.

3.16. Definition. Let there be given a cl semi-flow t and a partial map m.

(i) m is said to be *quasi-uniform-asymptotically stable* iff there exist

$$V \in \mathfrak{U} \quad \text{and} \quad s_1 : \mathfrak{U} \to R^+$$

such that

$\alpha + s_1(U) < \varepsilon(x, \alpha)$ holds for each $U \in \mathfrak{U}$ and $(x, \alpha) \in D$ with $x \in V[m\alpha]$,

and

$$(x, \alpha) \in D, \quad x \in V[m\alpha], \quad \theta \in \langle \alpha + s_1(U), \varepsilon(x, \alpha)) \quad \text{implies} \quad {}_{\theta}t_{\alpha}x \in U[m\theta].$$

(ii) m is said to be *uniform-asymptotically stable* iff it is uniform stable and quasi-uniform-asymptotically stable.

3.17. Lemma. *m is quasi-uniform-asymptotically stable (uniform-asymptotically stable) iff $\overline{m}$ is quasi-uniform-asymptotically stable (uniform-asymptotically stable respectively).*

3.18. Theorem. *Let there be given a cg semi-flow t admitting a period $\tau > 0$ and a partial map m periodic with the period τ, satisfying the assumptions of 3.9. on $\langle 0, \tau \rangle$. If m is quasi-asymptotically stable, then it is quasi-uniform-asymptotically stable.*

Proof. According to the assumption there are maps w and s such that

$$(x, \alpha) \in D \quad \text{with} \quad x \in w(\alpha)[m\alpha], \quad \theta \geqq \alpha + s(\alpha, U) \quad \text{implies} \quad {}_\theta t_\alpha x \in U[m\theta].$$

Set $V_1 = w(\tau)$. Then

$$\theta \geqq \tau + s(\tau, U), \quad y \in V_1[m\tau] \quad \text{implies} \quad {}_\theta t_\tau y \in U[m\theta].$$

Now, let $0 \leqq \alpha \leqq \tau$. According to 3.9., there is a map u such that $V = u(V_1)$, $x \in V[m\alpha]$, $\alpha \leqq \theta \leqq \tau$ implies ${}_\theta t_\alpha x \in V[m\theta]$. Especially,

$$(x, \alpha) \in D \quad \text{with} \quad x \in V[m\alpha], \quad 0 \leqq \alpha \leqq \tau \quad \text{implies} \quad {}_\tau t_\alpha x \in V[m\tau].$$

Defining $s_1(U) = \tau + s(\tau, U)$, we have

$$(13) \qquad {}_\theta t_\alpha x \in U[m\theta] \quad \text{whenever} \quad 0 \leqq \alpha \leqq \tau, \quad x \in V[m\alpha], \quad (x, \alpha) \in D, \quad \theta \geqq \alpha + s_1(U).$$

Now, let $\alpha \in R$ be arbitrary, $\alpha = \beta \leqq k\tau$, k integer, $\beta \in \langle 0, \tau)$. Then, according to (13) and the assumptions on the periodicity, we have ${}_\theta t_\alpha x = {}_{\theta - k\tau} t_\beta x \in U[m(\theta - k\tau)] = U[m\theta]$ for each $(x, \alpha) \in D$ with $x \in V[m\alpha]$ and $\theta \geqq \alpha + s_1(U)$, which finishes the proof.

3.19. Theorem. *Let there be given a cg semi-flow t admitting a period $\tau > 0$, and a partial map m periodic with the period τ, satisfying the assumptions of lemma 3.9. on the interval $\langle 0, \tau\rangle$. Then, if m is asymptotically stable, it is uniform-asymptotically stable.*

Proof follows directly from 3.12. and 3.18.

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