Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 103 (1978), No. 2, 157--158

Persistent URL: `http://dml.cz/dmlcz/108620`

Časopis pro pěstování matematiky, roč. 103 (1978), Praha

ON NOWHERE DENSITY OF THE CLASS OF SOMEWHAT CONTINUOUS FUNCTIONS IN $M(X)$

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(Received June 18, 1976)

This paper is closely related to the paper [3] and contains the solution of a problem formulated in [3].

Let X, Y be two topological spaces. The function $f : X \to Y$ is said to be somewhat continuous on X if for each set $G \subset Y$ open in Y the following implication holds:

$$f^{-1}(G) \neq \emptyset \Rightarrow \operatorname{Int} f^{-1}(G) \neq \emptyset$$

(cf. [1]). This implies that every function $f : X \to Y$ continuous on X is also somewhat continuous on X.

Let X be a topological space, let $M(X)$ be the linear normed space (with the norm $\|f\| = \sup_{t \in X} |f(t)|$) of all real-valued functions which are defined and bounded on X. Denote by $S(X)$ and $C(X)$ the set of all $f \in M(X)$ which are somewhat continuous and continuous on X, respectively. A problem was posed in [3] wheter $S(X)$ is a nowhere dense subset of $M(X)$ provided that $S(X) \neq M(X)$.

We shall give an affirmative answer to the foregoing question.

Let us remark that if X is a discrete space then each $f \in M(X)$ is continuous in X and hence $M(X) = S(X) = C(X)$.

Theorem. *Let X be a non discrete topological space. Then $S(X)$ is a nowhere dense subset of $M(X)$.*

Proof.*) If $f \in M(X)$, $\delta > 0$, put $K(f, \delta) = \{h \in M(X);\ \|h - f\| < \delta\}$. According to the assumption there exists an $x_0 \in X$ such that $\{x_0\}$ is not open in X. Given $f \in M(X)$, define a real-valued function g on X in the following way:

1) put $g(x_0) = f(x_0)$;

2) if $x \in X$, $x \neq x_0$, $|f(x) - f(x_0)| < \frac{1}{2}\delta$, put $g(x) = f(x_0) + \frac{1}{2}\delta$;

3) If $x \in X$, $|f(x) - f(x_0)| \geqq \frac{1}{2}\delta$, put $g(x) = f(x)$.

*) The author is thankful to the referee for improving the original version of the proof.

It follows from the previous definition that for each $x \in X$, $x \neq x_0$ we have

(1) $$|g(x) - f(x_0)| \geqq \tfrac{1}{3}\delta .$$

Evidently $g \in M(X)$. Further, $|g(x) - f(x)| < \tfrac{2}{3}\delta$ for all $x \in X$, hence

(2) $$\|g - f\| \leqq \tfrac{2}{3}\delta .$$

We shall show that

(a) $K(g, \tfrac{1}{9}\delta) \subset K(f, \delta)$;

(b) $K(g, \tfrac{1}{9}\delta) \cap S(X) = \emptyset$.

It follows from (a), (b) by virtue of the well-known criterion of nowhere density (cf. [2], p. 37) that $S(X)$ is nowhere dense in $M(X)$.

Proof of (a). Let $h \in K(g, \tfrac{1}{9}\delta)$. Then using (2) we have

$$\|h - f\| \leqq \|h - g\| + \|g - f\| < \tfrac{1}{9}\delta + \tfrac{2}{3}\delta < \delta .$$

Proof of (b). Let $h \in K(g, \tfrac{1}{9}\delta)$. Put $V = (h(x_0) - \tfrac{1}{9}\delta,\ h(x_0) + \tfrac{1}{9}\delta)$. Evidently $x_0 \in h^{-1}(V)$. We shall prove that $h^{-1}(V) = \{x_0\}$, hence $\operatorname{Int} h^{-1}(V) = \emptyset$, therefore $h \notin S(X)$.

Suppose $x \in X$, $x \neq x_0$, $h(x) \in V$. Then $|g(x) - h(x)| < \tfrac{1}{9}\delta$, $|h(x) - h(x_0)| < \tfrac{1}{9}\delta$, $|h(x_0) - g(x_0)| < \tfrac{1}{9}\delta$, $g(x_0) = f(x_0)$ imply $|g(x) - f(x_0)| < \tfrac{1}{3}\delta$, which contradicts (1). This completes the proof.

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