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THE GAUSS AND GAUSS-CODAZZI-RICCI EQUATIONS FOR RHEONOMOUS ANHOLONOMIC MANIFOLD

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In this paper we shall follow mostly the terminology and notations of [2] and [3]. So we shall make use of Einstein's summation convention, the indices $\alpha, \beta, \gamma, \ldots, A, B, C, \ldots, i, j, k$ run through the values $1, 2, \ldots, n$, while the indices a, b, c, d, e, f and p, q, r, s run through the values $1, 2, \ldots, m$ and $m + 1, m + 2, \ldots, n$ respectively, where the positive integers m, n satisfy the condition $1 < m < n$.

1. Let L_n be the n-dimensional affinely connected space. For the sake of simplicity we suppose that there exists a one-to-one mapping $f : X \to \{x^\alpha\}$ of the whole space L_n on some domain of parameters O which lies in the linear space P_n of all ordered n-tuples of real numbers. For the point X we shall use also the notation $[x^\alpha]$. Let I be an open interval. Evidently $W_{n+1} = L_n \times I$ is a stratified space. Let us denote the stratum (L_n, t), $t \in I$ by $L_n(t)$. If $[x^\alpha, t]$ is a current point in W_{n+1} then evidently $[x^\alpha, t] \in L_n(t)$. Let $P_{n+1} = P_n \times I$. Obviously, the mapping $[x^\alpha, t] \to \{x^\alpha, t\}$ of the space W_{n+1} into P_{n+1} is one-to-one mapping of W_{n+1} on the domain $\Omega = O \times I \subset P_{n+1}$. If we suppose that the admissible transformations of parameters $\{x^\alpha, t\} \to \{x^{\alpha^*}, t^*\}$ are described by all possible functions of the class C_∞

$$(1,1) \qquad x^{\alpha^*} = x^{\alpha^*}(x^\beta)\,, \quad t^* = t + C$$

($C \in (-\infty, +\infty)$) which realize the regular mappings of the domains O or interval I on another domain $O^* \subset P_n$ or another interval I^* respectively, then we shall denote W_{n+1} by $r - L_n(t)$ and we shall call it the *rheonomous stationary* affinely connected space (shortly the rheonomous space $r - L_n(t)$).

Let us suppose that by means of functions of the class C_2

$$(1,2) \qquad B_a^\alpha = B_a^\alpha(x^\omega, t)\,, \quad {}_pn^\alpha = {}_pn^\alpha(x^\omega, t)\,; \quad \{x^\omega, t\} \in \Omega$$

and there is defined for every $t \in I$ an anholonomic manifold $L_n^m(t)$ in $L_n(t) \subset r - L_n(t)$ (cf. also [3], p. 154 and foll.). Let the following suppositions be satisfied:

1. There are given a domain $\Lambda \subset P_n$ and such n functions of the class C_2

$$(1,3) \qquad x^\alpha = x^\alpha(u^A, t), \quad \{u^A\} \in \Lambda, \quad t \in I$$

that, for every $t \in I$, the functions (1,3) realize a regular one-to-one mapping $\{u^A\} \rightarrow$ $\rightarrow \{x^\alpha\}$ of the domain Λ on the domain O.

2. The admissible transformations $B_a^\alpha \rightarrow B_{a*}^\alpha$, ${}_pn^\alpha \rightarrow {}_{p*}n^\alpha$ are described in $r - L_n(t)$ by the equations

$$(1,4) \qquad B_{a*}^\alpha = A_{a*}^a B_a^\alpha, \quad {}_{p*}n^\alpha = {}_{p*}^{p}\delta_p n^\alpha,$$

while A_{a*}^a are all possible arbitrary functions of the class C_∞ defined on the domain Ω, Det $|A_{a*}^a| \neq 0$ everywhere, and further

$$(1,5) \qquad \{x^\omega\} \in O, \quad t_1, t_2 \in I \Rightarrow A_{a*}^a(x^\omega(u^A, t_1), t_1) = A_{a*}^a(x^\omega(u^A, t_2), t_2)$$

holds true.

Then we say that there is defined a *rheonomous anholonomic* manifold $r - L_n^m(t)$ in $r - L_n(t)$. If the parameter t does not occur explicitly in the functions (1,2) and (1,3) we say that $r - L_n^m(t)$ is *stationary*.

Remark. In L_n the equations (1,2)–(1,5) describe simultaneously a one-parametric system of anholonomic manifolds $L_n^m(t)$ with a special transformation group. All results which we shall deduce for the rheonomous manifold $r - L_n^m(t)$ may be directly interpreted for this system.

Let us suppose that by means of functions

$$(1,6) \qquad Z^{\alpha_1,\dots,\alpha_u,a_1,\dots,a_y}_{\beta_1,\dots,\beta_v,b_1,\dots,b_z} = Z^{\alpha_1,\dots,\alpha_u,a_1,\dots,a_y}_{\beta_1,\dots,\beta_v,b_1,\dots,b_z}(x^\omega, t), \ \{x^\omega, t\} \in \Omega \ ^{1})$$

there is given for every $t \in I$ a mixed tensor field in $L_n^m(t) \subset L_n(t)$. For the functions (1,6) we shall use the "matrix" form of notation $Z = Z(x^\omega, t)$ and we shall say that by means of these functions there is given a mixed tensor field in $r - L_n^m(t)$. The *covariant* derivate of this field will be denoted $(D_c Z)$. It is defined in the usual way for every $t \in I$ (cf. [2], p. 93 and [3], p. 155). By means of "matrix" notation we shall write shortly

$$(1,7) \qquad D_c Z = \partial_c Z + B_c^\gamma \Gamma_\gamma Z + \Gamma_c Z, \quad (\text{where } \partial_c Z = B_c^\gamma \partial_\gamma Z).$$

By means of functions (1,3) and (1,6) let us construct the composite functions $Z = Z(x^\alpha(u^A, t), t)$ and denote them shortly $Z = \tilde{Z}(u^A, t)$. Furthermore we shall use the notation

$$\partial_t Z = \frac{\partial Z(x^\alpha, t)}{\partial t}, \quad \partial_{\tilde{t}} Z = \frac{\partial \tilde{Z}(u^A, t)}{\partial t}.$$

[1]) Everywhere we suppose the differentiability of a "necessary" order of these functions.

It is easily shown that

$$(1,8) \qquad \partial_{\bar{t}} Z = B^{\gamma}_{t}\, \partial_{\gamma} Z + \partial_{t} Z\,, \quad \text{where} \quad B^{\gamma}_{t} = \frac{\partial x^{\alpha}(u^{A}, t)}{\partial t}\,.$$

Now we shall show that by means of function

$$(1,9) \qquad D_{t} Z = \partial_{\bar{t}} Z + B^{\gamma}_{t} \Gamma_{\gamma} Z$$

there is defined a tensor field $(D_t Z)$ on the rheonomous variety which is of the same kind as the tensor field (Z). Thus we want to show that by (1,1) or (1,4)

$$(1,10) \qquad D_{t} Z^{\alpha_1{}^*,\ldots,\alpha_u{}^*,a_1,\ldots,a_y}_{\beta_1{}^*,\ldots,\beta_v{}^*,b_1\ldots,b_z} = A^{\alpha_1{}^*,\ldots,\beta_v}_{\alpha_1,\ldots,\beta_v{}^*} D_{t} Z^{\alpha_1,\ldots,\alpha_u,a_1,\ldots,a_y}_{\beta_1,\ldots,\beta_v,b_1,\ldots,b_z}$$

or

$$(1,11) \qquad D_{t} Z^{\alpha_1,\ldots,\alpha_u,a_1{}^*,\ldots,a_y{}^*}_{\beta_1,\ldots,\beta_v,b_1{}^*,\ldots,b_z{}^*} = A^{a_1{}^*,\ldots,b_z}_{a_1,\ldots,b_z{}^*} D_{t} Z^{\alpha_1,\ldots,\alpha_u,a_1,\ldots,a_y}_{\beta_1,\ldots,\beta_v,b_1,\ldots,b_z}\,,$$

respectively, hold true.

A) Let us choose $y + z$ arbitrary but fixed Latin indices of the tensor field and denote them by a bar. Along every curve $x^{\alpha} = x^{\alpha}({}_{\circ}u^{A}, t)$, $t \in I$ the functions $Z^{\alpha_1,\ldots,\alpha_u,\bar{a}_1,\ldots,\bar{a}_y}_{\beta_1,\ldots,\beta_v,\bar{b}_1,\ldots,\bar{b}_z}$ define in L_n a tensor $\bar{Z}$ ($\{{}_{\circ}u^{A}\}$ is an arbitrary but fixed point in O). The absolute derivate $(\nabla_t \bar{Z})$ of this field along the considered curve satisfies the relation $\nabla_t \bar{Z} = D_t \bar{Z}$. Hence the correctness of the equation (1,10) follows.

B) Obviously

$$(1,12) \qquad Z^{\alpha_1,\ldots,\alpha_u,a_1{}^*,\ldots,a_y{}^*}_{\beta_1,\ldots,\beta_v,b_1{}^*,\ldots,b_z{}^*} = A^{a_1{}^*,\ldots,b_z}_{a_1,\ldots,b_z{}^*}\, Z^{\alpha_1,\ldots,\alpha_u,a_1,\ldots,a_y}_{\beta_1,\ldots,\beta_v,b_1,\ldots,b_z}\,.$$

A analogical transformation law holds also for the functions $\partial_{\bar{t}} Z$ and $B^{\gamma}_{t} \Gamma_{\gamma} Z$. In the first case we shall verify it by differentiating with respect to t both terms of (1,12) and using relations $\partial_{\bar{t}} A^{a}_{a*} = 0$, $\partial_{\bar{t}} A^{b^*}_{b} = 0$ which follow from (1,5). In the second case we shall verify our assertion by a direct simple calculation. Hence (1,11) holds.

Let us add that we call the tensor field $(D_t Z)$ the *absolute t-derivate* of the tensor field (Z) and that for any two tensor fields (U), (V) of the rheonomous manifold $r - L^{m}_{n}(t)$ the following relations hold:

$$(1,13) \qquad D_t(U + V) = D_t U + D_t V\,[2]\,, \quad D_t(UV) = (D_t U)\, V + U D_t V\,,$$
$$D_t U^{\alpha}_{\beta} = (D_t U)^{\alpha}_{\beta}\,, \quad D_t U^{a}_{b} = (D_t U)^{b}_{a}\,.[3]$$

Let J be an open interval, $J \subset I$, and C a regular curve in $r - L_n(t)$ described by the parametric equations

$$(1,14) \qquad x^{\alpha} = x^{\alpha}(T)\,, \quad t = T\,, \quad T \in J\,.$$

[2]) We suppose that U, V are of the same kind.

[3]) By means of indices α, β or a, b the "contraction" tensor operation is outlined.

For every $T \in J$ let us have m numbers $\mathrm{d}u^a/\mathrm{d}T$ such that

$$(1{,}15) \qquad \frac{\mathrm{d}x^\alpha}{\mathrm{d}T} = B^\alpha_a \frac{\mathrm{d}u^a}{\mathrm{d}T} + B^\alpha_t \,.$$

Then we call C a trajectory which lies in $r - L^m_n(t)$. The vector (1,15) is so-called tangential vector of this trajectory (at its point $[x^\alpha(T), T]$). The trajectory $x^\alpha = x^\alpha({}_0u_A, T)$, $t = T$, $T \in I$ (${}_0u^A =$ konst., $\{{}_0u^A\} \in \Lambda$) is called a *parametric t-line*. Obviously the vector (B^α_t) is tangential vector of the parametric t-line. Also we may easily see that the notion of a parametric t-line is invariant with respect to the admissible transformations (1,4) of the rheonomous manifold $r - L^m_n(t)$.

At the point $[x^\alpha, t] \in r - L_n(t)$ let us construct the n-dimensional tangential space of the manifold $L_n(t)$. In this space, by means of equations

$$(1{,}16) \qquad y^\alpha = v^a B^\alpha_a \,, \quad v^a \in (-\infty, +\infty)$$

or

$$(1{,}17) \qquad z^\alpha = w^a B^\alpha_a + B^\alpha_t \,, \quad w^a \in (-\infty, +\infty) \,,$$

there are defined two m-dimensional linear spaces denoted by $W[x^\alpha, t]$ or $T[x^\alpha, t]$ and called the *virtual* or *tangential* space of the rheonomous manifold $r - L^m_n(t)$, respectively. Let us agree that a vector, wich may be placed so that its starting point is $[x^\alpha, t]$ and its end point lies in the virtual space $W[x^\alpha, t]$ or in the tangential space $T[x^\alpha, t]$, will be called a *virtual* vector or a *tangential* vector respectively of the rheonomous manifold $r - L^m_n(t)$.

The tensor field $(D_T Z)$ that is defined along the trajectory (1,14) by means of the equation

$$(1{,}18) \qquad D_T Z = \frac{\mathrm{d}u^a}{\mathrm{d}T} D_a Z + D_t Z \,, \quad T \in J$$

is called the *absolute derivative* of the tensor field (Z) along the trajectory (1,14). If the trajectory (1,14) is a parametric t-line or its part then $\mathrm{d}u^a/\mathrm{d}T = 0$ for all $T \in J$ and the equation (1,18) is of the form $D_T Z = D_t Z$. It is easy to show that the absolute differentiation along the given trajectory obeys the rules which we shall obtain from (1,13) by replacing the operator D_t by the operator D_T.

We shall write the equations (1,18) in another form. By means of (1,8) and (1,9) and with the notation (1,7) the equations (1,18) may be easily written in the form

$$(1{,}19) \qquad D_T Z = \frac{\mathrm{d}Z}{\mathrm{d}T} + \frac{\mathrm{d}x^\gamma}{\mathrm{d}T} \Gamma_\gamma Z + \frac{\mathrm{d}u^a}{\mathrm{d}T} \Gamma_a Z \,.$$

Thus to calculate the absolute derivative along a trajectory it is sufficient that the tensor field (Z) is defined only at the points of this trajectory.

2. At every point of a rheonomous manifold so-called *Gauss equations* hold:

$$D_a B_b^\alpha = {}^p h_{ab}\, {}_p n^\alpha\,, \quad D_a\, {}_p n^\alpha = -{}_p l_a^b\, B_b^\alpha + {}_p^q v_a\, {}_q n^\alpha \tag{2,1}$$

or

$$\begin{aligned} D_t B_a^\alpha &= w_a^b B_b^\alpha + {}^p m_a\, {}_p n^\alpha\,, \quad D_t\, {}_p n^\alpha = -{}_p m^a B_a^\alpha + {}_p^q Q\, {}_q n^\alpha\,, \\ \dot{D}_a B_t^\alpha &= {}'w_a B_b^\alpha + {}^{p'} m_a\, {}_p n^\alpha\,, \quad D_t B_t^\alpha = W^a B_a^\alpha + {}^p W\, {}_p n^\alpha \end{aligned} \tag{2,2}$$

where

$${}^p h_{ab} = {}^p n_\alpha D_a B_b^\alpha\,, \quad -{}_p l_a^b = B_\alpha^b D_a\, {}_p n^\alpha\,, \quad {}_p^q v_a = {}^q n_\alpha\, D_a\, {}_p n^\alpha \tag{2,3}$$

or

$$\begin{aligned} w_a^b &= B_\alpha^b D_t B_a^\alpha\,, \quad {}^p m_a = {}^p n_\alpha D_t B_a^\alpha\,, \quad -{}_p m^a = B_\alpha^a D_t\, {}_p n^\alpha\,, \quad {}_p^q Q = {}^q n_\alpha D_t\, {}_p n^\alpha\,, \\ {}'w_a^b &= B_\alpha^b D_a B_t^\alpha\,, \quad {}^{p'} m_a = {}^p n_\alpha D_a B_t^\alpha\,, \quad W^a = B_\alpha^a D_t B_t^\alpha\,, \quad {}^p W = {}^p n_\alpha D_t B_t^\alpha\,, \end{aligned} \tag{2,4}$$

respectively.

The proof of the Gauss equations (2,1) or (2,2) is not difficult. If we arrange these equations by means of (2,3) or (2,4) respectively, we shall easily see that they are satisfied identically for all $\{x^\omega, t\} \in \Omega$. Let us add that on a "fixed" manifold there exists an analogy to the Gauss equations (2,1) only.

We introduce now the following notation:

$$Q_{cb}{}^a = \partial_{\bar{t}} \Gamma_{cb}{}^a\,, \tag{2,5}$$

$$\Omega_a^\alpha = \tfrac{1}{2}(\partial_a B_t^\alpha - \partial_{\bar{t}} B_a^\alpha)\,. \tag{2,6}$$

The geometrical object $(Q_{cb}{}^a)$ will be called the *anholonomic object* of the rheonomous manifold $r - L_n^m(t)$. It may be shown that in the holonomic case $(Q_{cb}{}^a)$ is a tensor field. It is easy to calculate that (Ω_a^α) is a tensor field. This field will be called the *field of rheonomous curvature* of the manifold $r - L_n^m(t)$. It is not difficult to verify that

$$\partial_{[a} \partial_{\bar{t}]} Z = \Omega_a^\gamma\, \partial_\gamma Z \tag{2,7}$$

holds for every tensor field (Z).

For the sake of completeness let us state that we shall use the well-known notations (cf. [3], p. 155, Formula (15,6), p. 285, Exc. (15,5)):

$$\begin{aligned} \Omega_{cb}{}^a &= -B_\alpha^a\, \partial_{[c} B_{b]}^\alpha \equiv (\partial_{[c} B_{|\alpha|}^a)\, B_{b]}^\alpha\,, \\ \Gamma_{cb}{}^a &= \Gamma_{\gamma\beta}{}^\alpha B_c^\gamma B_b^\beta B_\alpha^a - B_c^\gamma B_b^\beta\, \partial_\gamma B_\beta^a\,, \\ S_{cb}{}^a &= S_{\gamma\beta}{}^\alpha B_c^\gamma B_b^\beta B_\alpha^a\,. \end{aligned} \tag{2,8}$$

Let us remark that is easy to deduce from (2,8) the important equation

$$\Gamma_{[cb]}{}^a = S_{cb}{}^a - \Omega_{cb}{}^a\,. \tag{2,9}$$

We shall show that the following equations are satisfied

(2,10) $$D_{[b}D_{t]}v^\alpha = \tfrac{1}{2}K_{\delta\gamma\beta}{}^\alpha B_b^\delta B_t^\gamma v^\beta + \Omega_b^\gamma \nabla_\gamma v^\alpha ,$$

(2,11) $$D_{[b}D_{t]}v_a = \Omega_b^\gamma \partial_\gamma v_a + \tfrac{1}{2}Q_{ba}{}^c v_c ,$$

(2,12) $$D_{[b}D_{t]}v_a^\alpha = \tfrac{1}{2}K_{\delta\gamma\beta}{}^\alpha B_b^\delta B_t^\gamma v_a^\beta + \Omega_b^\gamma \nabla_\gamma v_a^\alpha + \tfrac{1}{2}Q_{ba}{}^c v_c^\alpha ,$$

(2,13) $$D_{[d}D_{c]}v^\alpha = \tfrac{1}{2}K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_c^\gamma v^\beta - S_{dc}{}^b D_b v^\alpha$$

for the tensor fields (v^α), (v_a), (v_a^α) of the rheonomous manifold $r - L_n^m(t)$.

We calculate easily $D_b D_t v^\alpha$ and $D_t D_b v^\alpha$. From it and in virtue of the formula $K_{\delta\gamma\beta}{}^\alpha = 2\partial_{[\delta}\Gamma_{\gamma]\beta}{}^\alpha + 2\Gamma_{[\delta|\varrho|}{}^\alpha\Gamma_{\gamma]\beta}{}^\varrho$ and by (2,6) and (2,7) we obtain (2,10). In a similar way we may deduce the equations (2,11), (2,12) and (2,13). Let us add that in (2,12) we made use of the notation $\nabla_\gamma v_a^\alpha = \partial_\gamma v_a^\alpha + \Gamma_{\gamma\beta}{}^\alpha v_a^\beta$. This $(\nabla_\gamma v_a^\alpha)$ is not a mixed tensor. Nevertheless, the equation (2,12) is of tensor-like character because, as may be shown, $(\Omega_\beta^\gamma \nabla_\gamma v_a^\alpha + \tfrac{1}{2}Q_{ba}{}^c v_c^\alpha)$ is a tensor.

3. We may interpret the Gauss equations (2,1) and (2,2) as a system of differential equations of the first order for the unknown functions $B_a^\alpha(x^\omega, t)$, ${}_pn^\alpha(x^\omega, t)$, $B_t^\alpha(x^\omega, t)$. We shall call the equations that express the conditions of integrability of this system, the *Gauss-Codazzi-Ricci equations.* Making the usual modifications (cf. e.g. [2], p. 120) we begin with the equations, the left-hand members of wich are of the form $D_{[c}D_{b]}B_a^\alpha$ or $D_{[c}D_{b]}\,{}_pn^\alpha$, respectively. For the conditions of integrability we shall obtain the equations wich are formally identical with the equations in [3], p. 160, Formulae 15.28a), b), c), d). We shall call them the Gauss-Codazzi-Ricci equations I. If, by mentioned modifications the left-hand member of the equations is of the form $D_{[c}D_{t]}B_a^\alpha$ or $D_{[c}D_{t]}\,{}_pn^\alpha$ or $D_{[c}D_{t]}B_t^\alpha$ or $D_{[c}D_{t]}B_t^\alpha$, respectively, we obtain the equations which we shall call the Gauss-Codazzi-Ricci equations II. We shall show that these equations may be written in the following form:

(3,1) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_t^\gamma B_b^\beta B_\alpha^a + 2\Omega_d^\delta \nabla_\gamma B_b^\alpha B_\alpha^a + Q_{db}{}^a = \\ = D_d w_b^a - {}_ql_d^a\,{}^qm_b + {}^qh_{db}\,{}_qm^a ,$$

(3,2) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_t^\gamma B_b^\beta\,{}^pn_\alpha + 2\Omega_d^\gamma \nabla_\gamma B_b^\alpha\,{}^pn_\alpha = {}^ph_{de}w_b^e + D_d\,{}^pm_b + \\ + {}_q^pv_d\,{}^qm_b - D_t\,{}^ph_{db} - {}^qh_{db}\,{}_q^pQ ,$$

(3,3) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_t^\gamma\,{}_pn^\beta B_\alpha^a + 2\Omega_d^\gamma \nabla_\gamma\,{}_pn^\alpha B_\alpha^a = -D_d\,{}_pm^a - {}_ql_d^a\,{}_p^qQ + \\ + D_t\,{}_pl_d^a + {}_pl_d^e w_e^a + {}_p^qv_d\,{}_qm^a ,$$

(3,4) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_t^\gamma\,{}_pn^\beta\,{}^qn_\alpha + 2\Omega_d^\gamma \nabla_\gamma\,{}_pn^\alpha\,{}^qn_\alpha = -{}^qh_{de}\,{}_pm^e + D_d\,{}_p^qQ + \\ + {}_r^qv_d\,{}_p^rQ + {}_pl_d^e\,{}^qm_e - D_t\,{}_p^qv_d - {}_p^rv_d\,{}_r^qQ ,$$

(3,5) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_c^\gamma B_t^\beta B_\alpha^a - 2S_{dc}{}^e\,{}'w_e^a = D_{[d}{}'w_{c]}^a - {}_ql_{[d}^a\,{}^qm_{c]} ,$$

(3,6) $$K_{\delta\gamma\beta}{}^\alpha B_d^\delta B_c^\gamma B_t^\beta\,{}^pn_\alpha - 2S_{dc}{}^e\,{}^{p\prime}m_e = {}^ph_{[d|e|}{}'w_{c]}^e + \\ + D_{[d}{}^pm_{c]} + {}_q^pv_{[d}{}^qm_{c]} ,$$

$$(3,7)\qquad K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{t}B^{\beta}_{t}B^{a}_{\alpha} + 2\Omega^{\gamma}_{d}\,\nabla_{\gamma}B^{\alpha}_{t}B^{a}_{\alpha} = D_{d}W^{a} - {}_{q}l^{a}_{d}\,{}^{q}W + \\ - D_{t}{}'w^{a}_{d} - {}'w^{e}_{d}w^{a}_{e} + {}^{q\prime}m_{d}\,{}_{q}m^{a}\,,$$

$$(3,8)\qquad K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{t}B^{\beta}_{t}\,{}^{p}n_{\alpha} + 2\Omega^{\gamma}_{d}\,\nabla_{\gamma}B^{\alpha}_{t}\,{}^{p}n_{\alpha} = {}^{p}h_{de}W^{e} + D_{d}{}^{p}W + {}^{p}_{q}v_{d}{}^{q}W + \\ - {}'w^{e}_{d}\,{}^{p}m_{e} + D_{t}{}^{p\prime}m_{d} - {}^{q\prime}m_{d}\,{}^{p}_{q}Q\,.$$

To prove thé preceding assertion we first deduce by means of (2,2) these equations:

$$(3,9)\qquad 2D_{[d}D_{t]}B^{\alpha}_{b} = B^{\alpha}_{c}(D_{d}w^{c}_{b} - {}_{q}l^{c}_{d}\,{}^{q}m_{b} + {}^{q}h_{db}\,{}_{q}m^{c}) + \\ + {}_{s}n^{\alpha}({}^{s}h_{de}w^{e}_{b} + D_{d}{}^{s}m_{b} + {}^{s}_{q}v_{d}{}^{q}m_{b} - D_{t}{}^{s}h_{db} - {}^{q}h_{db}\,{}^{s}_{q}Q)\,,$$

$$(3,10)\qquad 2D_{[d}D_{t]}\,{}_{p}n^{\alpha} = B^{\alpha}_{c}(-D_{d}\,{}_{p}m^{c} - {}_{q}l^{c}_{d}\,{}^{q}_{p}Q + D_{t}\,{}_{p}l^{c}_{d} + {}_{p}l^{e}_{d}w^{c}_{e} + {}^{q}_{p}v_{d}\,{}_{q}m^{c}) + \\ + {}_{s}n^{\alpha}(-{}^{s}h_{de}\,{}_{p}m^{e} + D_{d}\,{}^{s}_{p}Q + {}^{s}_{q}v_{d}\,{}^{q}_{p}Q + {}_{p}l^{e}_{d}\,{}^{s}m_{e} - D_{t}\,{}^{s}_{p}v_{d} - {}^{q}_{p}v_{d}\,{}^{s}_{q}Q)\,,$$

$$(3,11)\qquad 2D_{[d}D_{c]}B^{\alpha}_{t} = B^{\alpha}_{b}(D_{[d}{}'w^{b}_{c]} - {}_{q}l^{b}_{[d}\,{}^{q}m_{c]}) + \\ + {}_{s}n^{\alpha}({}^{s}h_{[d|b|}{}'w^{b}_{c]} + D_{[d}{}^{s}m_{c]} + {}^{s}_{q}v_{[d}{}^{q}m_{c]})\,,$$

$$(3,12)\qquad 2D_{[d}D_{t]}B^{\alpha}_{t} = B^{\alpha}_{c}(D_{d}W^{c} - {}_{q}l^{c}_{d}\,{}^{q}W - D_{t}{}'w^{c}_{d} - {}'w^{e}_{d}w^{c}_{e} + {}^{q\prime}m_{d}\,{}_{q}m^{c}) + \\ + {}_{s}n^{\alpha}({}^{s}h_{de}W^{e} + D_{d}{}^{s}W + {}^{s}_{q}v_{d}{}^{q}W - {}'w^{e}_{d}\,{}^{s}m_{e} - D_{t}{}^{s\prime}m_{d} - {}^{q\prime}m_{d}\,{}^{s}_{q}Q)\,.$$

From the equations (2,11), (2,10), (2,13) and once more (2,11) we may successively calculate that

$$(3,13)\qquad 2D_{[d}D_{t]}B^{\alpha}_{b} = K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{t}B^{\beta}_{b} + 2\Omega^{\gamma}_{d}\,\nabla_{\gamma}B^{\alpha}_{b} + Q_{db}{}^{c}B^{\alpha}_{c}\,,$$

$$(3,14)\qquad 2D_{[d}D_{t]}\,{}_{p}n^{\alpha} = K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{t}\,{}_{p}n^{\beta} + 2\Omega^{\gamma}_{d}\,\nabla_{\gamma}\,{}_{p}n^{\alpha}\,,$$

$$(3,15)\qquad 2D_{[d}D_{c]}B^{\alpha}_{t} = K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{c}B^{\beta}_{t} - 2S_{dc}{}^{b}D_{b}B^{\alpha}_{t}\,,$$

$$(3,16)\qquad 2D_{[d}D_{t]}B^{\alpha}_{t} = K_{\delta\gamma\beta}{}^{\alpha}B^{\delta}_{d}B^{\gamma}_{t}B^{\beta}_{t} + 2\Omega^{\gamma}_{d}\,\nabla_{\gamma}B^{\alpha}_{t}\,.$$

If we make the tensor product of the diference of equations (3,13) and (3,19) and the functions B^{a}_{α} or ${}^{p}n_{\alpha}$ we obtain after a simple modification the equations (3,1) or (3,2), respectively. In the same way we transform the differences of equations (3,14) and (3,10) or (3,15) and (3,11) or (3,16) and (3,12) and obtain the equations (3,3) and (3,4) or (3,5) and (3,6) or (3,7) and (3,8), respectively.

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