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COUNTABLE INFINITY OF EIGENVALUES OF SYMMETRIC ALGEBRAIC INTEGRAL EQUATION

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1. INTRODUCTION

The notion of the symmetric homogeneous algebraic integral equation was introduced by W. SCHMEIDLER in [1] as a generalization of the linear homogeneous integral equation with a symmetric kernel. The existence of a real eigenvalue and the corresponding eigenfunction [1,2] and the countable infinity of eigenvalues [1] were proved under certain assumptions. In the following we shall show how it is possible to prove the countable infinity of the set of eigenvalues of the symmetric algebraic integral equation using the theory of branching of solutions [3].

2. THE CASE OF LINEAR SYMMETRIC INTEGRAL EQUATION

The linear symmetric integral equation

$$\mu\, y(s) - \int_{\mathscr{A}} K(s, t)\, y(t)\, \mathrm{d}t = 0\,, \tag{1}$$

where μ is a real parameter, the variables s, t vary over a finite one- or more-dimensional region $\mathscr{A}$ and the kernel $K(s, t) = K(t, s)$ is a real continuous function, has at most countably infinite set of eigenvalues. This is caused by the fact that the eigenfunctions corresponding to any two different eigenvalues are mutually orthogonal. The simple proof of this assertion cannot be directly generalized to the symmetric algebraic integral equation. In [1] the proof of the countable infinity of simple eigenvalues of (1) is taken over to the proof of the non-existence of such sequences of eigenvalues (different from the studied eigenvalue) and corresponding eigenfunctions which converge to a given eigenvalue and eigenfunction respectively. The countable infinity of eigenvalues in the algebraic case is proved under certain assumptions in the same way. Now we shall prove the countable infinity of eigenvalues of (1) on the basis of the following consideration: Let μ_0 be an eigenvalue and let $y_0(s)$ be the corresponding eigenfunction of the kernel $K(s, t)$. It is necessary to find out whether there exist $\delta > 0$ and $\varepsilon(\delta) > 0$ such that for every μ from the δ-neighbourhood of the point $\mu_0(|\mu - \mu_0| < \delta)$ equation (1) has at least one non-zero solution $y(s)$ such that $\|y(s) - y_0(s)\| < \varepsilon$ (i.e. lying in the ε-neighbourhood of the solution $y_0(s)$). In the affirmative case it means that equation (1) is also nontrivially solvable in some neighbourhood of the point μ_0 and so the set of the eigenvalues is innumerable.

Let $\mu_0 \neq 0$ be a p-multiple eigenvalue of the kernel $K(s, t)$ with eigenfunctions $z_i(s)$ $(i = \overline{1, p})$ continuous in $\mathscr{A}$ for which $(z_i, z_j) = \delta_{ij}$ holds. Then

$$y_0(s) = \sum_{i=1}^{p} A_i z_i(s) \tag{2}$$

where A_i are arbitrary constants is the general solution of (1). If we substitute

$$\mu = \mu_0 + \lambda, \quad y(s) = y_0(s) + v(s) \tag{3}$$

in (1), we obtain the equation for $v(s)$

$$\mu_0 v(s) - \int_{\mathscr{A}} K(s, t) v(t) \, dt = -\lambda(y_0(s) + v(s)). \tag{4}$$

Introducing the kernel $L(s, t)$

$$L(s, t) = K(s, t) - \mu_0 \sum_{i=1}^{p} z_i(s) z_i(t)$$

instead of $K(s, t)$, equation (4) can be written in the form

$$v(s) = \sum_{i=1}^{p} C_i z_i(s) - \frac{\lambda}{\mu_0} (I + E)(y_0(s) + v(s)) \tag{5}$$

where

$$C_i = \int_{\mathscr{A}} z_i(t) v(t) \, dt, \quad E = \int_{\mathscr{A}} E(s, t) \ldots dt; \tag{6}$$

I is the identity operator and $E(s, t)$ is the resolving kernel of the kernel $\mu_0^{-1} L(s, t)$. Let us look for the solution of (5) in the form of the series

$$v(s) = \sum_{l_1 + \ldots + l_p + l = 1}^{\infty} C_1^{l_1} \ldots C_p^{l_p} \lambda^l v_{l_1 \ldots l_p l}(s). \tag{7}$$

Substituting (7) into (5) and equating coefficients of $C_1^{l_1} \ldots C_p^{l_p} \lambda^l$ we obtain the following expressions for $v_{l_1 \ldots l_p l}(s)$

$$v_{\underset{\text{i-th index}}{0 \ldots 010 \ldots 0} l}(s) = -\left(\frac{1}{\mu_0}\right)^l z_i(s), \quad i = \overline{1, p}, \quad l = \overline{0, \infty}, \tag{8}$$

$$v_{0\ldots 0l}(s) = \left(-\frac{1}{\mu_0}\right)^l \sum_{i=1}^{p} A_i\, z_i(s)\,, \quad l = \overline{1, \infty}\,,$$

$$v_{l_1 \ldots l_p l}(s) = 0\,, \quad \sum_{i=1}^{p} l_i \geqq 2\,, \quad l = \overline{0, \infty}\,.$$

Choose such numbers D, Y_0, V and Z_i ($i = \overline{1, p}$) that

(9)
$$\|I + E\| = D\,, \quad \|y_0(s)\| = Y_0\,, \quad \|v(s)\| = V\,,$$
$$\|z_i(s)\| = Z_i\,, \quad i = \overline{1, p}$$

where $\|v(s)\| = \max\limits_s |v(s)|$. From (9) and (5) we have the equation for V

(10)
$$V = \sum_{i=1}^{p} |C_i|\, Z_i + \left|\frac{\lambda}{\mu_0}\right| D(Y_0 + V)$$

from which we determine V uniquely for $|\lambda| < |\mu_0|/D$. Looking for the solution of equation (10) in the form

$$V = \sum_{l_1 + \ldots + l_p + l = 1}^{\infty} |C_1|^{l_1} \ldots |C_p|^{l_p} |\lambda|^l\, V_{l_1 \ldots l_p l}\,,$$

we easily find that

$$\|v_{l_1 \ldots l_p l}(s)\| \leqq V_{l_1 \ldots l_p l}\,;$$

it means that the series (7), which can be written with respect to the special form of the functions $v_{l_1 \ldots l_p l}(s)$ in the form

(11)
$$v(s) = \sum_{i=1}^{p} z_i(s) \left[C_i + (A_i + C_i) \sum_{l=1}^{\infty} \left(\frac{-\lambda}{\mu_0}\right)^l \right],$$

converges absolutely and uniformly according to s to the unique solution of equation (5) for $|\lambda| = |\mu - \mu_0| < |\mu_0| \min(1, 1/D)$.

Substituting (11) into (6) instead of $v(s)$ we get the system of equations determining the constants C_i

(12)
$$\frac{\lambda}{\mu_0 + \lambda} (A_i + C_i) = 0\,, \quad i = \overline{1, p}\,,$$

from which there follows ($\lambda \neq 0$)

(13)
$$C_i = -A_i\,;$$

this is not a small solution tending to zero with $\lambda \to 0$. The unique solution of equation (1) in the neighbourhood of the point μ_0 mentioned above is $y(s) \equiv 0$ as follows from (13) and (11). Hence, the assertion is proved.

3. THE CASE OF GENERAL SYMMETRIC ALGEBRAIC INTEGRAL EQUATION

Let us apply the previous consideration to the homogeneous algebraic integral equation generalizing equation (1)

$$\mu^n y^n(s) - \sum_{\beta=0}^{n-1} \mu^\beta y^\beta(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu) \prod_{i=1}^{\nu} y^{\beta+1}(t_i)\, dt_i = 0\,, \tag{14}$$

$$(\beta + 1)\, \nu = n - \beta$$

where $n \geqq 2$ is an integer, μ is a real parameter, the variables $s, t_1, \dots, t_\nu$ vary over the same finite one- or more-dimensional region $\mathscr{A}$, $L_\beta(st_1 \dots t_\nu)$ are real functions symmetric with respect to all the variables and continuous in the whole definition region. Let $\mu_0 \neq 0$ be an eigenvalue and let $y_0(s)$ be the corresponding eigenfunction continuous in $\mathscr{A}$ of equation (14). Substituting (3) into (14) and assuming that

$$p(s) \equiv n\mu_0^n\, y_0^{n-1}(s) - \sum_{\beta=0}^{n-1} \beta\mu_0^\beta\, y_0^{\beta-1}(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu) \prod_{i=1}^{\nu} y_0^{\beta+1}(t_i)\, dt_i \tag{15}$$

is different from zero and (without loss of generality) positive, we write equation (14) in the following form:

$$v(s) \sqrt{(p(s))} - \int_{\mathscr{A}} K(s, t)\, v(t) \sqrt{(p(t))}\, dt = f(s) \tag{16}$$

where

$$K(s, t) = \frac{1}{\sqrt{(p(s)\, p(t))}} \sum_{\beta=0}^{n-1} (n - \beta)\, \mu_0^\beta\, y_0^\beta(s)\, y_0^\beta(t) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_2 \dots t_\nu) \prod_{i=2}^{\nu} y_0^{\beta+1}(t_i)\, dt_i\,,$$

$$f(s) = \frac{1}{\sqrt{p(s)}} \Big[-(\lambda + \mu_0)^n (v(s) + y_0(s))^n +$$

$$+ \sum_{\beta=0}^{n-1} (\lambda + \mu_0)^\beta (v(s) + y_0(s))^\beta \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu) \prod_{i=1}^{\nu} (v(t_i) + y_0(t_i))^{\beta+1}\, dt_i \Big]'\,;$$

the sign $'$ means that $f(s)$ does not include expressions with $\lambda^0 v^0(s)$ and $\lambda^0 v(s)$. The kernel $K(s, t)$ is symmetric and it has at least one real eigenfunction corresponding to the eigenvalue 1. Let number 1 be a p-multiple eigenvalue of the kernel $K(s, t)$ with the eigenfunctions $z_i(s)$ $(i = \overline{1, p})$ continuous in $\mathscr{A}$. At the same time there is $z_i(s) = \varphi_i(s) \sqrt{p(s)}$ $(i = \overline{1, p})$ and – as it can be easily proved – $z_1(s) = y_0(s) \sqrt{p(s)}$. Substituting

$$K(s, t) = L(s, t) + \sum_{i=1}^{p} z_i(s)\, z_i(t)$$

into equation (16), we can write

$$v(s) = \sum_{i=1}^{p} C_i\, \varphi_i(s) + \frac{1}{\sqrt{p(s)}} (I + E) f(s) \tag{17}$$

where

(18) $$C_i = \int_{\mathscr{A}} z_i(t)\, v(t) \sqrt{(p(t))}\, dt$$

and the notation $(I + E)$ has the same meaning as in (5), $E(s, t)$ being the resolving kernel of $L(s, t)$. If we look for the solution of equation (17) in the form (7) we obtain for the functions $v_{l_1 \dots l_p l}(s)$ the system of relations

(19) $$v_{0\dots 0\underset{i}{1}0\dots 0}(s) = \varphi_i(s)\,, \quad i = \overline{1, p}\,,$$

$$v_{0\dots 01}(s) = \frac{-y_0(s)}{\mu_0}\,,$$

$$v_{0\dots 0\underset{i}{2}0\dots 0}(s) = \frac{1}{\sqrt{p(s)}}(I + E)\frac{1}{\sqrt{p(s)}}\left\{-\binom{n}{2}\mu_0^n\, y_0^{n-2}(s)\, \varphi_i^2(s) + \right.$$

$$+ \sum_{\beta=0}^{n-1} \mu_0^\beta \left[\binom{\beta}{2} y_0^{\beta-2}(s)\, \varphi_i^2(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu) \prod_{j=1}^{\nu} y_0^{\beta+1}(t_j)\, dt_j + \right.$$

$$+ \beta(n - \beta)\, y_0^{\beta-1}(s)\, \varphi_i(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu)\, y_0^\beta(t_1)\, \varphi_i(t_1)\, dt_1 \prod_{j=2}^{\nu} y_0^{\beta+1}(t_j)\, dt_j +$$

$$+ \tfrac{1}{2}\beta(n - \beta)\, y_0^\beta(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu)\, y_0^{\beta-1}(t_1)\, \varphi_i^2(t_1)\, dt_1 \prod_{j=2}^{\nu} y_0^{\beta+1}(t_j)\, dt_j +$$

$$+ \tfrac{1}{2}(n - \beta)(n - 2\beta - 1)\, y_0^\beta(s) \int_{\mathscr{A}} \dots \int_{\mathscr{A}} L_\beta(st_1 \dots t_\nu)\, y_0^\beta(t_1)\, \varphi_i(t_1)\, y_0^\beta(t_2)\, \varphi_i(t_2)\, dt_1\, dt_2\, .$$

$$\left.\left. \cdot \prod_{j=3}^{\nu} y_0^{\beta+1}(t_j)\, dt_j \right]\right\}, \quad i = \overline{1, p}$$

and generally

$$v_{l_1 \dots l_p l}(s) = h(s;\, y_0,\, v_{10\dots 0}, \dots, v_{0\dots 01}, \dots, v_{l_1 - 1 l_2 \dots l_p l}, \dots, v_{l_1 l_2 \dots l_p l - 1})\,,$$
$$l_1 + \dots + l_p + l \geqq 2\,.$$

The convergence and uniqueness of the series (7) where the functions $v_{l_1 \dots l_p l}(s)$ are given in (19), can be proved analogically as in the case of equation (1) (seealso[3]). On the basis of (18) and (7) we have the system of equations determining C_i $(i = \overline{1, p})$

(20) $$\sum_{l_1 + \dots + l_p = 2}^{\infty} C_1^{l_1} \dots C_p^{l_p} D_{l_1 \dots l_p 0}^{(i)} + \sum_{l_1 + \dots + l_p = 0}^{\infty} C_1^{l_1} \dots C_p^{l_p} \sum_{l=1}^{\infty} \lambda^l D_{l_1 \dots l_p l}^{(i)} = 0\,, \quad i = \overline{1, p}$$

where

$$D_{l_1 \dots l_p l}^{(i)} = \int_{\mathscr{A}} v_{l_1 \dots l_p l}(t)\, z_i(t) \sqrt{(p(t))}\, dt\,.$$

For some details concerning the derivation of system (20) and its discussion for some special cases see [3]. From this system it is not obvious, in general, that there do not exist small solutions $C_i(\lambda)$ ($i = \overline{1, p}$), i.e. $C_i(\lambda) \to 0$ for $\lambda \to 0$ and, consequently, $v(s)$ of the same properties. Hence the set of eigenvalues of equation (14) is for $p > 1$ generally innumerable.

Assume now that number 1 is a simple eigenvalue of the kernel $K(s, t)$ with the corresponding eigenfunction $z_1(s) = y_0(s) \sqrt{p(s)}$. As there follows from (7) for $p = 1$, we can look for the solution of (17) in the form

$$v(s) = \sum_{m+l=1}^{\infty} C^m \lambda^l \, v_{ml}(s) . \tag{21}$$

For the functions $v_{ml}(s)$

$$v_{10}(s) = y_0(s) , \quad v_{01}(s) = \frac{-y_0(s)}{\mu_0} , \tag{22}$$

$$v_{i0}(s) = 0 , \quad i = \overline{2, \infty} ,$$

$$v_{ij}(s) = h(s; y_0) , \quad i, j = \overline{1, \infty}$$

is valid. C can be determined from the equation

$$\sum_{m=0}^{\infty} C^m \sum_{l=1}^{\infty} \lambda^{l-1} D_{ml} = 0 \tag{23}$$

where

$$D_{ml} = \int_{\mathscr{A}} v_{ml}(t) \, y_0(t) \, p(t) \, \mathrm{d}t .$$

As $D_{01} = -1/\mu_0$ is different from zero there exists no small solution of equation (23) as well as none of equation (17). The set of eigenvalues of (14) is in this case countably infinite.

4. THE SECOND ORDER SYMMETRIC ALGEBRAIC INTEGRAL EQUATION

Let us consider the special case of equation (14) for $n = 2$, i.e. the equation

$$\mu^2 \, y^2(s) - \int_{\mathscr{A}} \int_{\mathscr{A}} L(st_1t_2) \, y(t_1) \, y(t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2 = 0 . \tag{24}$$

As in this case

$$p(s) \equiv \mu_0^2 \, y_0(s) ,$$

let us assume that $y_0(s)$ is different from zero in $\mathscr{A}$ and positive. Further assume that number 1 is a simple eigenvalue of the kernel

$$\frac{1}{\mu_0^2 \sqrt{(y_0(s) \, y_0(t))}} \int_{\mathscr{A}} L(stt_2) \, y_0(t_2) \, \mathrm{d}t_2 .$$

The increment $v(s)$ of the solution $y_0(s)$ in a neighbourhood of the point μ_0 can be determined from the equation

$$(25)\quad v(s) = C\, y_0(s) + \frac{1}{2\mu_0^2 \sqrt{y_0(s)}} (I + E) \frac{1}{\sqrt{y_0(s)}} \Big[-(\lambda^2 + 2\lambda\mu_0)\,(v(s) + y_0(s))^2 - \\ - \mu_0^2 v^2(s) + \int_{\mathscr{A}} \int_{\mathscr{A}} L(st_1 t_2)\, v(t_1)\, v(t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2 \Big].$$

Looking for the solution of (25) in the form (21) we can prove easily that $v_{ml}(s) = v_{ml}\, y_0(s)$ (v_{ml} are constants). Hence, let us look for the solution of (25) in the form of the series

$$(26)\qquad v(s) = y_0(s) \Big(\sum_{m=1}^{\infty} C^m v_{m0} + \sum_{l=1}^{\infty} \lambda^l \sum_{m=0}^{\infty} C^m v_{ml} \Big).$$

Substituting (26) into (25) and equating coefficients at the same powers of λ, we obtain

$$(27)\qquad \sum_{m=1}^{\infty} C^m v_{m0} = C\,,$$

$$\sum_{m=0}^{\infty} C^m v_{ml} = \left(\frac{-1}{\mu_0} \right)^l (C + 1)^2\, P_{l-1}(C)\,, \quad l \geqq 1$$

where $P_{l-1}(C)$ is a polynomial of $(l - 1)$-st degree with respect to C. As

$$\mu_0^2 \int_{\mathscr{A}} y_0^3(t)\, \mathrm{d}t = 1$$

holds we get on the basis of (23) the equation for C

$$(28)\qquad (C + 1)^2 \left(-1 + \sum_{l=1}^{\infty} \left(\frac{\lambda}{\mu_0} \right)^l P_{l+1}(C) \right) = 0\,.$$

It is obvious that equation (28) has not small solutions $C(\lambda)$; the unique solution in certain neighbourhood of the point $\lambda = 0$ is $C = -1$, which gives $v(s) = -y_0(s)$ and thus $y(s) \equiv 0$ in this neighbourhood of μ_0. Thus the countable infinity of the eigenvalues of equation (24) follows again.

5. CONCLUSION

So we have proved the following

Theorem. *Let $L_\beta(st_1 \dots t_\nu)$ be real functions symmetric with respect to all the variables and continuous in the whole definition region. If every eigenvalue $\mu \neq 0$*

and the corresponding eigenfunction $y(s)$ continuous in $\mathscr{A}$ of equation (14) *satisfy the assumptions*

a) *the expression* (15) *is different from zero in* $\mathscr{A}$,

b) *number* 1 *is a simple eigenvalue of the kernel* $K(s, t)$ *defined in* (16),

then the set of the eigenvalues of (14) *is at most countably infinite.*

From the proof given in Sec. 3 there follows that the condition b) of the Theorem cannot be made generally weaker. The consideration from the theory of branching of solutions on which the proof is based is useful for the study of the set of the eigenvalues of a general non-symmetric homogeneous algebraic integral equation in contradistinction to the consideration used by W. Schmeidler and leads to interesting results which will be included in the next paper.

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