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NONLINEAR BOUNDARY VALUE PROBLEM FOR A SYSTEM OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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INTRODUCTION

We shall investigate a system of nonlinear differential equations with a special type of stable nonlinear boundary conditions. We suppose that by means of this system we can define a monotone operator T with a potential J. We seek a weak solution in the set M, a subset of a Sobolev space V. This set M is not a subspace because we consider nonlinear boundary conditions. The variational formulation of a weak solution is obtained in terms of the derivative of the functional J along the curves passing through the point of its minimum, and the test functions are taken from the set M_u, which is the manifold tangent to the set M at the point u. The proof of existence is not too difficult, but to the author's knowledge no similar theorem is known for a monotone operator with nonlinear boundary conditions. For this formulation it seems to be essential that the corresponding Sobolev space V be an algebra and that the identical imbedding V to C be completely continuous, which implies that every bounded closed part of the set M is weakly compact. This is the reason why we restrict ourselves to ordinary differential equations.

The main result is Theorem 4.1 in Section 4, where we give a sufficient condition for "local" uniqueness of the weak solution. This condition is a relation between the monotonicity of the operator T and the "curvature" of the set M expressed in terms of the second derivatives of the functions involved in the boundary conditions. As a limit case, for linear boundary conditions, the "curvature" is equal to zero and we get the global uniqueness without any additional condition.

1. DEFINITION OF THE WEAK SOLUTION

We start with some notation.

Let $\langle a, b\rangle$ be a closed bounded interval in R_1. We denote by $W^{1,2}(\langle a, b\rangle)$ the Sobolev space of functions, square integrable together with their first derivatives; $W^{1,2}(\langle a, b\rangle)$ is well known to be a Hilbert space – see [1].

Let $C(\langle a, b\rangle)$ be the space of functions defined and continuous on $\langle a, b\rangle$, $C^1(\langle a, b\rangle)$ and $C^2(\langle a, b\rangle)$ the space of functions defined and continuous together with their first or first and second derivatives on $\langle a, b\rangle$, respectively.

Further we denote

$$\nabla u = \left(\frac{\mathrm{d}u_1}{\mathrm{d}x}, \dots, \frac{\mathrm{d}u_n}{\mathrm{d}x}\right),$$

$$H = [L_2(\langle a, b\rangle)]^n, \quad V = [W^{1,2}(\langle a, b\rangle)]^n, \quad C = [C(\langle a, b\rangle)]^n$$

with the scalar product and the norms

$$(u, v) = \int_a^b \sum_{i=1}^n u_i(x)\, v_i(x)\, \mathrm{d}x\,,$$

$$\|u\|_H^2 = (u, u) \quad \text{for} \quad u, v \in H\,,$$

$$\|u\|_V^2 = \|u\|_H^2 + \|\nabla u\|_H^2 \quad \text{for} \quad u \in V,$$

$$\|u\|_C = \max_{i=1,\dots,n} \big(\max_{x\in\langle a,b\rangle} (|u_i(x)|)\big)$$

For $v \in V^*$ we denote the value of v at a point $u \in V$ by $\langle v, u\rangle$; if $v \in H$, we can write (v, u) instead of $\langle v, u\rangle$ – see [2],

$$\|v\|_* = \sup_{u\in V,\, \|u\|_V = 1} |\langle v, u\rangle|$$

Let us have positive integers m, n, $1 \leqq m \leqq n - 1$; $\alpha, \beta, \gamma, \delta \in R_n$, $\alpha_i = \beta_i = 0$ for $i = 1, \dots, m$, and real functions $h = (h_1(x), \dots, h_n(x))$, $h\colon \langle a, b\rangle \to R_n$, $f_i\colon R_m \to R_1$ for $i = m + 1, \dots, n$; $F = F(x, \xi, \eta)\colon \langle a, b\rangle \times R_n \times R_n \to R_1$.

We shall solve problem (1), (2):

$$\text{(1)} \quad a_i(x, u(x), \nabla u(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(\tilde{a}_i(x, u(x), \nabla u(x))) = h_i(x) \quad \text{for} \quad i = 1, \dots, n\,;$$

$$x \in \langle a, b\rangle\,,$$

with boundary conditions

$$\text{(2)} \qquad u_i(a) = f_i(u_1(a), \dots, u_m(a)) + \alpha_i\,,$$

$$u_i(b) = f_i(u_1(b), \dots, u_m(b)) + \beta_i \quad \text{for} \quad i = m + 1, \dots, n$$

and

$$a_i(a, u(a), \nabla u(a)) + \gamma_i + \sum_{j=m+1}^n (\tilde{a}_j(a, u(a), \nabla u(a)) + \gamma_j) \frac{\partial f_i}{\partial \xi_j}(u_1(a), \dots, u_m(a)) = 0$$

$$a_i(b, u(b), \nabla u(b)) + \delta_i + \sum_{j=m+1}^n (\tilde{a}_j(b, u(b), \nabla u(b)) + \delta_j) \frac{\partial f_i}{\partial \xi_j}(u_1(b), \dots, u_m(b)) = 0$$

for $i = 1, \ldots, m$, where

$$a_i(x, \xi, \eta) = \frac{\partial F}{\partial \xi_i}(x, \xi, \eta), \quad \tilde{a}_i(x, \xi, \eta) = \frac{\partial F}{\partial \eta_i}(x, \xi, \eta)$$

for $i = 1, \ldots, n$; $x \in \langle a, b \rangle$; $\xi, \eta \in R_n$.

Now we can define

(i) a functional $I: V \to R_1$ by $I(u) = \int_a^b F(x, u(x), \nabla u(x))\,dx$,

(ii) a linear functional d on C by $d(u) = \sum_{i=1}^{n} (\delta_i u_i(b) - \gamma_i u_i(a))$.

Let the following assumptions be fulfilled:

(1.1) $h \in H$;

(1.2) $F \in C^2(\langle a, b \rangle \times R_n \times R_n)$;

(1.3) $I(u)$ is a continuous functional on V which has the Frechet derivative S, $S: V \to \to V^*$, and there exists a constant $L > 0$ such that

(i) $|I(u)| \leqq L(1 + \|u\|^2)$,

(ii) $\|S(u)\|_* \leqq L(1 + \|u\|)$ for $u \in V$;

(1.4) there exists a constant $K > 0$ such that the operator S is strictly monotone on V with the constant K, i.e.

$$\langle S(u) - S(\tilde{u}), u - \tilde{u} \rangle \geqq K\|u - \tilde{u}\|^2 \quad \text{for} \quad u, \tilde{u} \in V;$$

(1.5) $f_i \in C^2(R_m)$ for $i = m + 1, \ldots, n$;

(1.6) there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$|f_i(\xi)| \leqq \lambda_1(1 + \sum_{j=1}^{m} |\xi_j|); \quad \left|\frac{\partial f_i}{\partial \xi_j}(\xi)\right| \leqq \lambda_1; \quad \left|\frac{\partial^2 f_i}{\partial \xi_j\, \partial \xi_k}(\xi)\right| \leqq \lambda_2$$

for $i = m + 1, \ldots, n$; $j, k = 1, \ldots, m$; $\xi \in R_m$.

Remark 1.1. It is easy to show that

$$\langle S(u), v \rangle = \int_a^b \sum_{i=1}^{n} \left[\frac{\partial F}{\partial \xi_i}(x, u(x), \nabla u(x))\, v_i + \frac{\partial F}{\partial \eta_i}(x, u(x), \nabla u(x))\, \frac{dv_i}{dx}\right] dx$$

for $u, v \in V$.

Remark 1.2. A sufficient condition for the validity of (1.3), (1.4) is, for instance, the existence of real constants c_1, c_2 such that

(i) $\max\left(|F(x,0,0)|, \left|\frac{\partial F}{\partial \xi_i}(x,0,0)\right|, \left|\frac{\partial F}{\partial \eta_i}(x,0,0)\right|\right) \leqq c_1$ for $i = 1, \ldots, n$; $x \in \langle a, b\rangle$,

(ii) $\max\left(\left|\frac{\partial^2 F}{\partial \xi_i \, \partial \xi_j}(x,\xi,\eta)\right|, \left|\frac{\partial^2 F}{\partial \xi_i \, \partial \eta_j}(x,\xi,\eta)\right|, \left|\frac{\partial^2 F}{\partial \eta_i \, \partial \eta_j}(x,\xi,\eta)\right|\right) \leqq c_1$ for $i, j = 1, \ldots, n$; $x \in \langle a, b\rangle$, $\xi, \eta \in R_n$,

(iii)
$$\sum_{i=1}^{n}\left[\left(\frac{\partial F}{\partial \xi_i}(x,\xi,\eta) - \frac{\partial F}{\partial \xi_i}(x,\bar{\xi},\bar{\eta})\right)(\xi_i - \bar{\xi}_i) + \right.$$
$$\left. + \left(\frac{\partial F}{\partial \eta_i}(x,\xi,\eta) - \frac{\partial F}{\partial \eta_i}(x,\bar{\xi},\bar{\eta})\right)(\eta_i - \bar{\eta}_i)\right] \geqq$$
$$\geqq c_2 \sum_{i=1}^{n}\left[(\xi_i - \bar{\xi}_i)^2 + (\eta_i - \bar{\eta}_i)^2\right] \text{ for } \xi, \eta, \bar{\xi}, \bar{\eta} \in R_n;\ x \in \langle a, b\rangle.$$

Remark 1.3. It is obvious that (h, u) defines a continuous functional on V for $h \in H$.

Lemma 1.1. *The identical imbedding of $W^{1,2}(\langle a, b\rangle)$ into $C(\langle a, b\rangle)$ or of V into C is completely continuous and $\|u\|_C \leqq c_{a,b}\|u\|$, where $c_{a,b} = \max\left(\sqrt{(b-a)}, 1/\sqrt{(b-a)}\right)$.*

Proof. See [4].

Remark 1.4. As an easy consequence of this lemma we obtain that $d(u)$ defines a continuous functional on V.

Lemma 1.2. *Let $u, v \in V$. Then $uv \in V$ and $\|uv\| \leqq c_{a,b}\|u\| \, \|v\|$.*

Proof. Without loss of generality we prove Lemma 1.2 for $u, v \in W^{1,2}(\langle a. b\rangle)$. We have

$$\|uv\|^2_{W^{1,2}(\langle a,b\rangle)} = \int_a^b \left[|uv|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}x}(uv)\right|^2\right] \mathrm{d}x \leqq \int_a^b \left[|u|^2 \,.\, |v|^2 + \right.$$
$$\left. + \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 . \, |v|^2 + 2\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| . \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right| . \, |u| \,.\, |v| + \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right|^2 . \, |u|^2\right] \mathrm{d}x\,.$$

Using Lemma 1.1 we can estimate, for instance,

$$\int_a^b \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| . \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right| . \, |u| \,.\, |v| \, \mathrm{d}x \leqq \max_{x \in \langle a,b\rangle} |u(x)| \,.\, \max_{x \in \langle a,b\rangle} |v(x)| \,.\left(\int_a^b \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x\right)^{1/2} .$$
$$. \left(\int_a^b \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right|^2 \mathrm{d}x\right)^{1/2} \leqq c^2_{a,b} \, \|u\|^2_{W^{1,2}(\langle a,b\rangle)} \, \|v\|^2_{W^{1,2}(\langle a,b\rangle)} \,.$$

The other terms are estimated similarly.

Remark 1.5. It is well known that there exists an equivalent norm $\|.\|_e$ in V such that the space V with the norm $\|.\|_e$ is a Banach algebra, i.e., $\|uv\|_e \leqq \|u\|_e . \|v\|_e$ for $u, v \in V$.

Lemma 1.3. *Let* (1.5), (1.6) *be fulfilled. Then* $f_i(u(x))$, $\partial f_i/\partial \xi_j\,(u(x)) \in W^{1,2}(\langle a, b\rangle)$ *for* $u \in V$; $i = m+1, \ldots, n$; $j = 1, \ldots, m$.

Proof. The assumptions (1.5) yield that $f_i(.)$, $\partial f_i/\partial \xi_j\,(.)$ fulfil the Carathéodory conditions and we get that $f_i(u(.))$, $\partial f_i/\partial \xi_j\,(u(.))$ are measurable. The assumption (1.6) implies

$$\|f_i(u)\|^2_{W^{1,2}(\langle a,b\rangle)} = \int_a^b \left[|f_i(u(x))|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}x}(f_i(u(x)))\right|^2\right] \mathrm{d}x \leqq$$

$$\leqq c \int_a^b \left[\lambda_1^2 \Big(1 + \sum_{j=1}^m |u_j(x)|^2\Big) + \left|\sum_{i=m+1}^n \frac{\partial f_i}{\partial \xi_j}(u(x)) \frac{\mathrm{d}u_j}{\mathrm{d}x}\right|^2\right] \mathrm{d}x \leqq$$

$$\leqq c\lambda_1^2 \int_a^b \left(1 + \sum_{j=1}^n \left(|u_j(x)|^2 + \left|\frac{\mathrm{d}u_j}{\mathrm{d}x}\right|^2\right)\right) \mathrm{d}x \leqq c\lambda_1^2((b-a) + \|u\|^2) < +\infty .$$

Similar estimates hold for the derivatives, which completes the proof.

Let us denote

$$M = \{u \in V \mid u_i(a) = f_i(u(a)) + \alpha_i ,$$
$$u_i(b) = f_i(u(b)) + \beta_i \quad \text{for} \quad i = m+1, \ldots, n\} ,$$

and for $u \in V$,

$$M_u = \Big\{v \in V \mid v_i(a) = \sum_{j=1}^m \frac{\partial f_i}{\partial \xi_j}(u(a))\, v_j(a) ,$$
$$v_i(b) = \sum_{j=1}^m \frac{\partial f_i}{\partial \xi_j}(u(b))\, v_j(b) \quad \text{for} \quad i = m+1, \ldots. n\Big\} .$$

Remark 1.6. The values $u_i(a)$ and $u_i(b)$ are taken in the sense of Lemma 1.1.

Now we can introduce

Definition 1.1. We say that $u \in V$ is a *weak solution* of the problem (1), (2) iff

(i) $u \in M$,
(ii) $\langle S(u), v\rangle - (h, v) - d(v) = 0$ for all $v \in M_u$.

Remark 1.7. It can be easily checked by integrating by parts that for $u \in \in [C^2(\langle a, b\rangle)]^n$ the weak solution and the classical one coincide.

We prove that the functional J attains its minimum on the set M. Then we define a set N_u of curves in M passing through this point of minimum. It can be easily checked that $v = \partial w/\partial t\,(t, .)|_{t=0} \in M_u$ for every $w(t, .) \in N_u$. Conversely, we have to prove that for every $v \in M_u$ there exists $w \in N_u$ such that $v = \partial w/\partial t\,(t, .)|_{t=0}$. Then we can use Euler's necessary condition for the existence of the minimum of $J(w(., x))$ and prove the existence of a weak solution.

Let us denote

$$J(u) = I(u) - (h, u) - d(u) \quad \text{for} \quad u \in V, \quad h \in H\,.$$

Remark 2.1. It is obvious that $J(.)$ is a continuous functional on V as $I(.)$ is and that the Frechet derivative T of J exists, $\langle T(u), v\rangle = \langle S(u), v\rangle - (h, v) - d(v)$ for $u, v \in V$.

Remark 2.2. In what follows we shall denote subsequences of a sequence $\{{}^n u\}$ again by $\{{}^n u\}$.

Lemma 2.1. *Let* (1.5), (1.6) *be fulfilled. Then*

(i) *M is not empty,*
(ii) *$U_c = \{u \in M|\ \|u\| \leqq c\}$ is weakly compact for $c > 0$.*

Proof.

(i) For $x \in \langle a, b\rangle$ we put

$u_i(x) = 0$ for $i = 1, \ldots, m$;

$$u_i(x) = f_i(0) + \frac{x - a}{b - a}\,\beta_i + \frac{b - x}{b - a}\,\alpha_i \quad \text{for} \quad i = m + 1, \ldots, n.$$

It is clear that $u \in M$.

(ii) Let $c > 0$, c arbitrary but fixed. Let $\{{}^n u\} \subset U_c$. From the reflexivity of V we get that there exist $\tilde{u} \in V$ and a subsequence $\{{}^n u\}$ such that ${}^n u \rightharpoonup \tilde{u}$ in V, $\|\tilde{u}\| \leqq c$. Using Lemma 1.1 we get that U_c is compact in C, i.e. we can again choose a subsequence such that ${}^n u \to \tilde{u}$ in C. Then for $i = m + 1, \ldots, n$ the inequalities

$|\tilde{u}_i(a) - f_i(\tilde{u}(a)) - \alpha_i| \leqq |\tilde{u}_i(a) - {}^n u_i(a) - f_i(\tilde{u}(a)) + f_i({}^n u(a))| \leqq |\tilde{u}_i(a) -$

$- {}^n u_i(a)| + |f_i(\tilde{u}(a)) - f_i({}^n u(a))|$ hold.

With regard to (1.5) we obtain that the term on the right hand side of the last inequality tends to zero for $n \to \infty$. Analogously for the point b, i.e. $\tilde{u} \in M$.

Lemma 2.2. *Let us suppose that* (1.1)–(1.6) *hold. Then*

(i) *the functional $J: V \to R_1$ is weakly lower semicontinuous, i.e.*

$$J(u) \leqq \liminf_{n\to\infty} J(^n u) \quad \textit{for} \quad \{^n u\} \subset V, \quad ^n u \rightharpoonup u \quad \textit{in} \quad V,$$

(ii) *the functional J is weakly coercive, i.e.* $\lim\limits_{\|u\|\to\infty} J(u) = +\infty$.

Proof. Fix arbitrary $u, \tilde{u} \in V$. With the help of (1.4) we get

$$\langle T(u) - T(\tilde{u}), u - \tilde{u}\rangle = \langle S(u) - S(\tilde{u}), u - \tilde{u}\rangle \geqq K\|u - \tilde{u}\|^2, \quad \text{i.e.},$$

the operator T is strictly monotone. The assertions (i), (ii) are standard consequences of this fact. For the proof see [2], [5].

Lemma 2.3. *Let* (1.1)–(1.6) *hold. Then J attains its minimum on M.*

Proof. Let w be an arbitrary element from the set M. The coerciveness of the functional J implies the existence of a constant $c > 0$ such that

$$\|u\| \geqq c \Rightarrow J(u) > J(w) + 1.$$

Write

$$U_c = \{u \in M \mid \|u\| \leqq c\}.$$

If the minimum exists, it clearly cannot lie outside the set U_c. Therefore we have

$$\inf_{v\in M} J(v) = \inf_{v\in U_c} J(v).$$

Putting

$$\inf_{v\in Uc} J(u) = m,$$

we can find a minimizing sequence $\{^n u\} \subset U_c$, $J(^n u) \to m$. Lemma 2.2 implies that there exist $\tilde{u} \in U_c$ and a subsequence $\{^n u\}$, $^n u \rightharpoonup \tilde{u}$.

Since J is weakly lower semicontinuous, we obtain

$$m = \lim_{n\to\infty} J(^n u) = \liminf_{n\to\infty} J(^n u) \geqq J(\tilde{u}),$$

therefore $J(\tilde{u}) = m$ and J attains its minimum at the point $\tilde{u}$, which was to prove.

For $\varepsilon > 0$ we put

$$N_u = \{w \in [C^1(\langle -\varepsilon, \varepsilon\rangle) \to V] \mid w(t, x) \in M \quad \text{for} \quad t \in \langle -\varepsilon, \varepsilon\rangle,$$
$$w(0, x) = u(x) \quad \text{for} \quad x \in \langle a, b\rangle\}.$$

Lemma 2.4. *Let $u \in M$. Then for $v \in M_u$ there exists $w \in N_u$ such that* $\partial w_i/\partial t\,(t, x)|_{t=0} = v_i(x)$ *for* $i = 1, \ldots, n$; $x \in \langle a, b\rangle$.

Proof. We choose $v \in M_u$ arbitrary but fixed. For $t \in \langle -\varepsilon, \varepsilon\rangle$; $x \in \langle a, b\rangle$ put

$$w_i(t, x) = u_i(x) + t\, v_i(x) \quad \text{for} \quad i = 1, \ldots, m;$$

$$w_i(t, x) = f_i(w(t, x)) + u_i(x) - f_i(u(x)) +$$
$$+ t\left(v_i(x) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(x))\, v_j(x)\right) \quad \text{for} \quad i = m + 1, \dots, n\,.$$

Obviously $w \in V$ and

(i) $w_i(0, x) = u_i(x)$ for $i = 1, \dots, m$; $x \in \langle a, b\rangle$,

$w_i(0, x) = f_i(w(0, x)) + u_i(x) - f_i(u(x)) = u_i(x)$ for $i = m + 1, \dots, n$; $x \in \langle a, b\rangle$,

(ii) $\left.\dfrac{\partial w_i(t, x)}{\partial t}\right|_{t=0} = v_i(x)$ for $i = 1, \dots, m$; $x \in \langle a, b\rangle$,

$$\left.\frac{\partial w_i(t, x)}{\partial t}\right|_{t=0} = \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(w(0, x)) \left.\frac{\partial w_j(t, x)}{\partial t}\right|_{t=0} + v_i(x) -$$
$$- \sum_{j=1}^{n} \frac{\partial f_i}{\partial \xi_j}(u(x))\, v_j(x) = v_i(x) \text{ for } i = m + 1, \dots, n;\ x \in \langle a, b\rangle,$$

(iii) for $t \in \langle -\varepsilon, \varepsilon\rangle$, $i = m + 1, \dots, n$ we have

$$w_i(t, a) = f_i(w(t, a)) + u_i(a) - f_i(u(a)) +$$
$$+ t\left(v_i(a) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(a))\, v_j(a)\right) = f_i(w(t, a)) + \alpha_i.$$

Analogously for the point b.

Theorem 2.1. *Let* (1.2)–(1.6) *hold. Then for* $h \in H$, $\alpha, \beta, \gamma, \delta \in R_n$, $\alpha_i = \beta_i = 0$ *for* $i = m + 1, \dots, n$, *there exists a weak solution of the problem* (1), (2).

Proof. Let the functional J attain its minimum on M at the point $\tilde{u} \in M$. We choose $v \in M_{\tilde{u}}$ arbitrary but fixed. From Lemma 2.4 we get that for this $v \in M_{\tilde{u}}$ there exists $w \in N_{\tilde{u}}$ such that

$$\left.\frac{\partial w_i(t, x)}{\partial t}\right|_{t=0} = v_i(x) \quad \text{holds for} \quad i = 1, \dots, n\,; \quad x \in \langle a, b\rangle\,.$$

Euler's necessary condition yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} J(w(t, x))\Big|_{t=0} = \left\langle T(w(0, x)), \left.\frac{\partial w(t, x)}{\partial t}\right|_{t=0}\right\rangle =$$
$$= \langle T(\tilde{u}), v\rangle = \langle S(\tilde{u}), v\rangle - (h, v) - d(v)\,.$$

3. APRIORI ESTIMATE OF THE NORM OF THE WEAK SOLUTION

We get an apriori estimate using the weak lower semicontinuity and the weak coerciveness of the functional J. We cannot use the estimate $\inf J(v) \leqq |J(O)|$ on M because we do not know whether the set M includes a zero element, therefore we have to construct an element "similar to zero".

For the sake of brevity we put

$$D = \|d\|_* ,$$
$$E = \|h\|_H ,$$
$$B = \max_{i=m+1,\dots,n} (\max (|\alpha_i|, |\beta_i|)) ,$$
$$A_1 = \frac{1}{K}(L + E + D + 1),$$
$$A_2 = L + E + D + LA_1 ,$$
$$A_3^2 = (n - m)\left[(b - a)(\lambda_1 + 2B)^2 + \frac{4B^2}{b - a}\right],$$
$$A_4 = A_1(1 + A_2) + L(2 + A_3^2) + (E + D) A_3 ,$$

where the constants L, K, λ_1 were defined in (1.3), (1.4) and (1.6).

Remark 3.1. It is easy to prove by means of Lemma 1.1 that

$$D \leqq c_{a,b}\, 2n \max_{i=1,\dots,n} (\max (|\delta_i|, |\gamma_i|)) .$$

Now we can formulate

Theorem 3.1. *Let the assumptions* (1.1)–(1.6) *be fulfilled. Let u be a weak solution of the problem* (1), (2). *Then*

$$\|u\| \leqq \max (1, A_1, A_4) .$$

Proof. Let u be a weak solution, $\|u\| > \max (1, A_1)$.
First of all we define $w \in M$:

$$w_i(x) = 0 \quad \text{for} \quad i = 1, \dots, m ; \quad x \in \langle a, b\rangle ,$$

$$w_i(x) = f_i(0) + \frac{x - a}{b - a}\beta_i + \frac{b - x}{b - a}\alpha_i \quad \text{for} \quad i = m + 1, \dots, n ; \; x \in \langle a, b\rangle .$$

We shall prove the following inequalities (i)–(v):

(i) $$\frac{1}{\|u\|}\langle T(u), u\rangle \geqq 1 \quad \text{for} \quad \|u\| \geqq A_1 .$$

The assumptions (1.3), (1.4) and Remarks 1.3, 1.4 yield

$$\langle T(u), u\rangle = \langle S(u), u\rangle - (h, u) - d(u) =$$
$$= \langle S(u) - S(0), u\rangle + \langle S(0), u\rangle - (h, u) - d(u),$$

hence

$$\langle T(u), u\rangle \geqq K\|u\|^2 - (L + E + D)\|u\| \geqq \|u\| [K\|u\| - (L + E + D)], \quad \text{i.e.}$$

$$\frac{\langle T(u), u\rangle}{\|u\|} \geqq [K\|u\| - (L + E + D)] \geqq 1 \quad \text{for} \quad \|u\| \geqq A_1 .$$

(ii)
$$\sup_{u\in V, \|u\| \leqq A_1} \|T(u)\|_* \leqq A_2 .$$

Using 1.3 and the previous inequality we obtain

$$\sup_{u\in V, \|u\| \leqq A_1} \|T(u)\|_* = \sup_{u\in V, \|u\| \leqq A_1} \{ \sup_{v\in V, \|v\| \leqq 1} |\langle S(u), v\rangle - (h, v) - d(v)|\} \leqq$$
$$\leqq \sup_{u\in V, \|u\| \leqq A_1} \{ \sup_{v\in V, \|v\| \leqq 1} [(L(1 + \|u\|) + E + D)\|v\|]\} \leqq$$
$$\leqq L(1 + A_1) + E + D = A_2 .$$

(iii)
$$\|w\| \leqq A_3 .$$

We estimate

$$\|w\|^2 = \int_a^b \sum_{i=m+1}^n \left[\left(f_i(0) + \frac{x-a}{b-a}\beta_i + \frac{b-x}{b-a}\alpha_i\right)^2 + \left(\frac{\beta_i}{b-a} - \frac{\alpha_i}{b-a}\right)^2\right] \mathrm{d}x \leqq$$
$$\leqq \sum_{i=m+1}^n \int_a^b \left[(\lambda_1 + 2B)^2 + \left(\frac{2B}{b-a}\right)^2\right] \mathrm{d}x \leqq$$
$$\leqq (n - m)\left[(b-a)(\lambda_1 + 2B)^2 + \frac{4B^2}{b-a}\right] = A_3^2 .$$

(iv)
$$J(u) \geqq -L - A_1 A_2 + \|u\| - A_1 .$$

Let us write

$$\varphi(t) = J(tu) .$$

By means of (1.3) we get

$$\varphi(t) = \varphi(0) + \int_0^1 \varphi'(s)\,\mathrm{d}s, \quad \text{i.e.}$$

$$J(u) = J(0) + \int_0^1 \langle T(tu), tu\rangle \frac{\mathrm{d}t}{t} = J(0) + \int_0^{\|u\|} \left\langle T\left(s\frac{u}{\|u\|}, s\frac{u}{\|u\|}\right\rangle \frac{\mathrm{d}s}{s} =$$
$$= J(0) + \int_0^{A_1} \left\langle T\left(s\frac{u}{\|u\|}, s\frac{u}{\|u\|}\right\rangle \frac{\mathrm{d}s}{s} + \int_{A_1}^{\|u\|} \left\langle T\left(s\frac{u}{\|u\|}, s\frac{u}{\|u\|}\right\rangle \frac{\mathrm{d}s}{s} \geqq$$

$$\geqq -L - A_1 A_2 + \|u\| - A_1 .$$

(v)
$$\|u\| \leqq A_4 .$$

For w defined above we get

$$\|u\| - L - A_1(A_2 + 1) \leqq J(u) = \inf_{v \in M} J(v) \leqq |J(w)| \leqq$$

$$\leqq L(1 + \|w\|^2) + E\|w\| + D\|w\| \leqq L(1 + A_3^2) + (E + D) A_3 , \quad i.e.$$

$$\|u\| \leqq L(2 + A_3^2) + A_1(1 + A_2) + (E + D) A_3 = A_4 ,$$

which was to prove.

4. UNIQUENESS OF THE WEAK SOLUTION

Let u, $\tilde{u}$ be two weak solutions. We can not use the estimate

$$0 = \langle T(u) - T(\tilde{u}), u - \tilde{u} \rangle \geqq \|u - \tilde{u}\|^2$$

as in the case when M is a linear manifold, because the space of the test functions depends on the weak solution and the equality is not valid. We have to construct functions $v \in M_u$, $\tilde{v} \in M_{\tilde{u}}$ such that

$$\langle T(u), v \rangle - \langle T(\tilde{u}), \tilde{v} \rangle \geqq \lambda \|u - \tilde{u}\|^2 ,$$

where λ depends on the distance of the sets M_u, $M_{\tilde{u}}$ and the "curvature" of the set M at the point $\tilde{u}$.

Let us denote

$$U_s(\tilde{u}) = \{u \in M | \ \|u - \tilde{u}\| \leqq s\} \quad \text{for} \quad s \in R_1 ,$$

$$\lambda(s, \tilde{u}) = \sup_{u \in U_s(\tilde{u})} \left\{ \max_{\substack{i=m+1,\ldots,n \\ j,k=1,\ldots,m}} \left(\left| \frac{\partial^2 f_i}{\partial \xi_j \, \partial \xi_k} (u(a)) \right| , \left| \frac{\partial^2 f_i}{\partial \xi_j \, \partial \xi_k} (u(b)) \right| \right) \right\} ,$$

$$A_5 = m c_{a,b} \sqrt{(n - m)} \sqrt{\left(4(b - a) + \frac{4}{b - a} \right)} ,$$

and the linear function

$$P(x) = A_5[Lx + 2(L + E + D)] .$$

Theorem 4.1. *Let* $(1.1)-(1.6)$ *be fulfilled and let* $\tilde{u} \in M$ *be a weak solution of the problem* (1), (2). *Let there exist* $s_0 > 0$ *such that*

(4.1)
$$\lambda(s_0, \tilde{u}) < \frac{K}{P(2\|\tilde{u}\| + s_0)} .$$

Then there exists exactly one weak solution in $U_{s_0}(\tilde{u})$.

Proof. Let $u, \tilde{u} \in V$ be two weak solutions, $\|u - \tilde{u}\| \leqq s_0$, $u \neq \tilde{u}$. For $x \in \langle a, b\rangle$; $i = 1, \ldots, n$ we define

$$v_i(x) = u_i(x) - \tilde{u}_i(x) + w_i(x)\,,$$
$$\tilde{v}_i(x) = u_i(x) - \tilde{u}_i(x) + \tilde{w}_i(x)\,,$$

where

$$w_i(x) = \tilde{w}_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m\,; \quad x \in \langle a, b\rangle\,,$$

$$w_i(x) = -\frac{b-x}{b-a}\Big(f_i(u(a)) - f_i(\tilde{u}(a)) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(a))\,(u_j(a) - \tilde{u}_j(a))\Big) -$$
$$- \frac{x-a}{b-a}\Big(f_i(u(b)) - f_i(\tilde{u}(b)) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(b))\,(u_j(b) - \tilde{u}_j(b))\Big)\,,$$

$$\tilde{w}_i(x) = -\frac{b-x}{b-a}\Big(f_i(u(a)) - f_i(\tilde{u}(a)) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(\tilde{u}(a))\,(u_j(a) - \tilde{u}_j(a))\Big) -$$
$$- \frac{x-a}{b-a}\Big(f_i(u(b)) - f_i(\tilde{u}(b)) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(\tilde{u}(b))\,(u_j(b) - \tilde{u}_j(b))\Big)$$

for $i = m + 1, \ldots, n$; $x \in \langle a, b\rangle$.

It is easy to prove that $v \in M$, $\tilde{v} \in M_{\tilde{u}}$.

From the Mean Value Theorem see e.g. [3] – and the assumptions (1.5), (1.6) we get that there exist real numbers $t_{i,j}$, $r_{i,j,k} \in (0, 1)^m$ for $i = 1, \ldots, m$; $j, k = m + 1, \ldots, n$ such that

$$\Big|f_i(u(a)) - f_i(\tilde{u}(a)) - \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(a))\,(u_j(a) - \tilde{u}_j(a))\Big| \leqq$$
$$\leqq \sum_{j=1}^{m} \Big|\frac{\partial f_i}{\partial \xi_j}(\tilde{u}(a) + t_{i,j}(u(a) - \tilde{u}(a))) - \frac{\partial f_i}{\partial \xi_j}(u(a))\Big|\,|u_j(a) - \tilde{u}_j(a)| \leqq$$
$$\leqq \sum_{j=1}^{m} \sum_{k=1}^{m} \Big|\frac{\partial^2 f_i}{\partial \xi_j\, \partial \xi_k}(u(a) + r_{i,j,k}(1 - t_{i,j})\,(u(a) - \tilde{u}(a)))\Big|\,.$$
$$.\, |u_j(a) - \tilde{u}_j(a)|\,.\,|u_k(a) - \tilde{u}_k(a)| \leqq m^2\,\lambda(s, \tilde{u})\,.\,c_{a,b}^2\|u - \tilde{u}\|^2\,.$$

Analogous estimates hold for the other terms in w, $\hat{w}$.

Now we can estimate

$$\|w\|^2 = \sum_{i=m+1}^{n} \int_a^b \Big\{\Big[-\frac{b-x}{b-a}\Big(f_i(u(a)) - f_i(\tilde{u}(a)) -$$
$$- \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}(u(a))\,(u_j(a) - \tilde{u}_j(a))\Big) - \frac{x-a}{b-a}\Big(f_i(u(b)) - f_i(\tilde{u}(b)) -$$

$$- \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j} (u(b)) (u_j(b) - \tilde{u}_j(b)))\Big]^2 + \Big[\frac{1}{b-a} (f_i(u(a)) - f_i(\tilde{u}(a)) -$$

$$- \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j} (u(a)) (u_j(a) - \tilde{u}_j(a))) - \frac{1}{b-a} (f_i(u(b)) - f_i(\tilde{u}(b)) -$$

$$- \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j} (u(b)) (u_j(b) - \tilde{u}_j(b)))\Big]^2\Big\} \, dx \leqq$$

$$\leqq (n-m) \, m^4 c_{a,b}^4 \, \lambda^2(s, \tilde{u}) \, \|u - \tilde{u}\|^4 \, . \int_a^b \Big[4 + \Big(\frac{2}{b-a}\Big)^2\Big] \, dx \leqq$$

$$\leqq (n-m) \, m^4 c_{a,b}^4 \Big[4(b-a) + \frac{4}{b-a}\Big] . \, \lambda^2(s, \tilde{u}) \, \|u - \tilde{u}\|^4 \, .$$

Analogous result holds for $\tilde{w}$, i.e.

$$\|w\| \leqq A_5 \, \lambda(s, \tilde{u}) \, \|u - \tilde{u}\|^2 \, .$$

From Definition 1.1 we obtain

$$0 = \langle S(u), v\rangle - \langle S(\tilde{u}), \tilde{v}\rangle = \langle T(u), u - \tilde{u} + w\rangle - (h, u - \tilde{u} + w) -$$

$$- d(u - \tilde{u} + w) - \langle T(\tilde{u}), u - \tilde{u} + \tilde{w}\rangle + (h, u - \tilde{u} + \tilde{w}) + d(u - \tilde{u} + \tilde{w}) =$$

$$= \langle T(u) - T(\tilde{u}), \, u - \tilde{u}\rangle + \langle T(u), w\rangle - \langle T(\tilde{u}), \tilde{w}\rangle - (h, w) +$$

$$+ (h, \tilde{w}) - d(w) + d(\tilde{w}) \geqq K\|u - \tilde{u}\|^2 - L(1 + \|u\|) . \|w\| -$$

$$- L(1 + \|\tilde{u}\|) . \|\tilde{w}\| - E\|w\| - E\|\tilde{w}\| - D\|w\| - D\|\tilde{w}\| \geqq K\|u - \tilde{u}\|^2 -$$

$$- A_5 \, \lambda(s_0, \tilde{u}) \, \|u - \tilde{u}\|^2 \, [L(2 + 2\|\tilde{u}\| + s_0) + 2E + 2D] =$$

$$= \|u - \tilde{u}\|^2 \, [K - \lambda(s_0, \tilde{u}) \, P(2\|\tilde{u}\| + s_0)] \, ,$$

which contradicts (4.1).

Corollary 4.1. *Let* (1.1)–(1.6) *hold. Let* $\lambda(s_0, 0) < K/P(2s_0)$, *where* $s_0 = A_4$, A_4 *having been defined in Section* 3. *Then there exists exactly one weak solution of the problem* (1), (2).

Proof is obvious.

Remark 4.1. According to Lemma 1.1 we have

$$\lim_{\|u-\tilde{u}\| \to 0} \lambda(\|u - \tilde{u}\|, \tilde{u}) = \lambda(0, \tilde{u}) \, ,$$

i.e., the uniqueness guaranteed in Theorem 4.1 depends on the local behaviour of the functions $f_i(\xi_1, \ldots, \xi_m)$ for $i = 1, \ldots, m$ at the point $\xi_i = \tilde{u}_i(a)$ or $\xi_i = \tilde{u}_i(b)$.

Remark 4.2. Let f_i be linear functions. Then $\lambda(s, \tilde{u}) = 0$, (4.1) is fulfilled and we have the global uniqueness of the weak solution.

Theorem 5.1. *Let* (1.1)–(1.6) *be fulfilled and let* u *be the weak solution. Let* (4.1) *hold and*

(5.1) $$\delta_i = \gamma_i = 0 \quad \text{for} \quad i = 1, \dots, n\,,$$

(5.2) *there exists* $\alpha_1 > 0$ *such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial \eta_i\, \partial \eta_j}(x, \xi, \eta)\, \eta_i \eta_j \geqq \alpha_1 \sum_{i=1}^{n} |\eta_i|^2$$

for $x \in \langle a, b \rangle$, $\xi, \eta \in R_n$,

(5.3) *there exists* $\alpha_2 > 0$ *such that*

$$\left| \frac{\partial^2 F}{\partial \eta_i\, \partial \eta_j}(x, \xi, \eta) \right| < \alpha_2 \quad \text{for all} \quad x \in \langle a, b \rangle\,, \quad \xi, \eta \in R_n\,.$$

Then $u \in [C^1(\langle a, b \rangle)]^n$.

Proof. Let us define a function P: $\langle a, b \rangle \to R_n$,

$$P_i(x) = \tilde{a}_i(x, u(x), \nabla u(x)) - \int_a^x [a_i(t, u(t), \nabla u(t)) - h(t)]\, \mathrm{d}t\,,$$

where a_i, $\tilde{a}_i$ were defined in Section 1.

Let

$$c_i = \frac{1}{b - a} \int_a^b P_i(x)\, \mathrm{d}x$$

and

$$v_i(x) = \int_a^x [P_i(t) - c_i]\, \mathrm{d}t\,.$$

The assumptions (1.1)–(1.3) yield that $v \in V$. It can be readily checked that $v_i(a) = v_i(b) = 0$ for $i = 1, \dots, n$, therefore $v \in M_u$ (M_u is a linear subspace).

Using Definition 1.1 and Green's theorem we can write

$$0 = \langle T(u), v \rangle - (h, v) = \int_a^b \sum_{i=1}^{n} \Big[a_i(x, u(x), \nabla u(x))\, v_i(x) +$$

$$+ \tilde{a}_i(x, u(x), \nabla u(x)) \frac{\mathrm{d}v_i}{\mathrm{d}x} - h_i(x)\, v_i(x) \Big] \mathrm{d}x =$$

$$= \int_a^b \sum_{i=1}^{n} \left[\tilde{a}_i(x, u(x), \nabla u(x)) - \int_a^x (a_i(t, u(t), \nabla u(t)) - h(t))\, \mathrm{d}t \right] \frac{\mathrm{d}v_i}{\mathrm{d}x}\, \mathrm{d}x =$$

$$= \int_a^b \sum_{i=1}^{n} P_i(x)\, (P_i(x) - c_i)\, \mathrm{d}x = \int_a^b \sum_{i=1}^{n} (P_i(x) - c_i)^2\, \mathrm{d}x\,,$$

hence $P_i(x) = c_i$ a.e. and

$$\tilde{a}_i(x, u(x), \nabla u(x)) = \int_a^x (a_i(t, u(t), \nabla u(t)) - h(t))\,dt + c_i \quad \text{a.e.}$$

Now we can define a function $G: \langle a, b\rangle \times R_n \to R_n$, $G_i(x, z) = \tilde{a}_i(x, u(x), z) - \int_a^x (a_i(t, u(t), \nabla u(t)) - h(t))\,dt - c_i$ for all $x \in \langle a, b\rangle$, $z \in R_n$.

Let us choose $x_0 \in \langle a, b\rangle$ arbitrary but fixed. The assumtpions (1.1)–(1.3) and (5.1)–(5.3) yield

(i) there exists $z_0 \in R_n$ such that $G(x_0, z_0) = 0$,

(ii) $(G)'_2(x_0, z_0, .)$ is a continuous isomorphism of R_n onto R_n,

(iii) $(G)'_2(x, z, y)$ is continuous as a mapping $R_1 \times R_n$ to $L(R_n, R_n)$.

The Implicit Function Theorem implies that there exists a neighbourhood U of the point x_0 and a function $z: U \to R_n$ such that $G(x, z(x)) = 0$ on U and z is a continuous function on U. The local uniqueness yields $z(x) = \nabla u(x)$ a.e., i.e. $u \in [C^1(\langle a, b\rangle)]^n$.

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