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c-CONTINUITY AND CLOSED GRAPHS

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A type of generalized continuity, called c-continuity, was introduced by K. R. Gentry and H. B. Hoyle, (see [3]). It seems to be of interest that for fairly general topological spaces this notion coincides with that of the function with a closed graph. To show this fact is the main purpose of this note. It seems to be useful to formulate and prove our results for multifunctions. This approach is motivated by the intensive study of the problems of multifunctions and their graphs (see e.g. [5] for some references).

In what follows X, Y denote topological spaces. If necessary, further assumptions on X and Y will be employed. A multifunction F is a mapping defined on X with values in the power set of Y. We write $F: X \to Y$ instead of $F: X \to 2^Y$. A single-valued mapping $f: X \to Y$ may be interpreted as a multifunction assigning to $x \in X$ the one-point set $\{f(x)\}$. We write simply $f(x)$ instead of $\{f(x)\}$. For a multifunction $F: X \to Y$ we suppose $F(x) \neq \emptyset$ for any $x \in X$.

It is a well known fact (see [9], [10] and many others) that the notion of a closed graph of F is closely related to the well known notion of the upper semi-continuity of F. In this connection it is useful to introduce a more general notion of upper c-semi-continuity ($u.c.s.c.$).

A multifunction $F: X \to Y$ is said to be upper c-semi-continuous($u.c.s.c.$) at $p \in X$ if for any open V containing $F(p)$ and such that $Y - V$ is compact, there exists a neighbourhood U of p such that $F(x) \subset V$ for any $x \in U$. If F is u $c.s.c.$ at any $p \in X$, then F is said to be upper c-semi-continuous.

If we omit the condition that $X - V$ is compact we obtain the usual notion of the upper semi-continuity ($u.s.c.$) as is usually defined (see e.g. [6] p. 393).

In the case of a single-valued function the upper c-semi-continuity coincides with the notion of c-continuity as defined in [3]. Evidently the upper semi-continuity of a function $F: X \to Y$ implies the upper c-semi-continuity. The converse is not true. An example a single-valued c-continuous function which is not continuous may be taken. Such an example (see [3]) may be the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/x$ if $x \neq 0, f(0) = 0$.

We do not mention some interesting properties of upper c-semi-continuous multifunctions which are extensions or generalizations of the properties a discussed in [3], but restrict ourselves to the relations of *u.c.s.c.* to the closed graphs. Recall that the graph $G(F)$ of a multifunction $F: X \to Y$ is the set

$$G(F) = \{(x, y): y \in F(x)\} .$$

It is said to be closed if the set $G(F)$ is closed in $X \times Y$.

1. RELATIONS OF UPPER c-SEMI-CONTINUITY TO CLOSED GRAPHS

Theorem 1. *Let X be a topological space, Y a Hausdorff and locally compact topological space. If $F: X \to Y$ is a closed-valued upper c-semi-continuous multifunction then the graph $G(F)$ is closed.*

Proof. We prove that $X \times Y - G(F)$ is open. Let $(p, q) \in X \times Y - G(F)$. We have $q \notin F(p)$. Since $F(p)$ is closed and Y is a locally compact Hausdorff space there exists ([1] Theorem 6.2 p. 238) a neighbourhood W of q such that cl W is compact and

$$F(p) \cap \text{cl } W = \emptyset .$$

Put

$$V = Y - \text{cl } W .$$

We have $V \supset F(p)$. The upper c-semi-continuity of F at p implies that there exists a neighbourhood U of p such that $F(x) \subset V$ for any $x \in U$. Hence for any $x \in U$ and $y \in F(x)$ we have $y \notin W$. Thus $U \times W$ is a neighbourhood of (p, q) such that $U \times W \cap G(F) = \emptyset$.

So

$$U \times W \subset X \times Y - G(F) ,$$

proving that $X \times Y - G(F)$ is open.

The following corollary for the single-valued c-continuous functions (see [2], [4] is straightforward.

Corollary 1. *Let X, Y be topological spaces, Y a locally compact Hausdorff space. Let $f: X \to Y$ be a single-valued c-continuous function. Then the graph $G(f)$ is closed.*

It is well known that for *u.s.c.* multifunctions and for continuous single-valued functions results analogous to Theorem 1 and Corollary 1 are valid without the assumptions of local compactness ([7], Theorem 3, [9], Theorem 1.8, [5], Theorem 3.12). Nevertheless, the local compactness in Theorem 1 and Corollary 1 is essential as the following example shows.

Example 1. Put

$$X = \langle 0, 1) - \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\}$$

with the usual topology. The underlying set of the topological space Y will contain all elements $y \in \langle 0, \infty)$ such that y is not an integer and moreover the set Z of all nonnegative integers. To define a topology on Y we determine the base of neighbourhoods for any element belonging to Y. If $y \in Y$ $y \neq Z$ then the base at y will consist of all open intervals $(y - \varepsilon \;\; y + \varepsilon)$ where $\varepsilon > 0$ is such that no integer belongs to $(y - \varepsilon \;\; y + \varepsilon)$. The base at the element Z will be the collection of all sets $G \cup \{Z\}$, $G \subset (-\infty \;\; \infty) - Z$ where $G \cup Z$ is open (in the usual sense) in $\langle 0 \;\; \infty)$.

Define a single-valued function $f: X \to Y$ as follows:

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x \neq 0\,, \\ b & \text{if} \quad x = 0\,, \end{cases}$$

where $b \neq Z$ is a fixed number.

The usual continuity of f at $x \neq 0$ is obvious. Hence f is also c-continuous at $x \neq 0$, $x \in X$. We prove the c-continuity at $x = 0$. Take an open set V (in Y). From the compactness of $Y - V$ one can easily see that the set $(Y - V) - \{Z\}$ is a bounded set (in the usual sense) on the real line. So there exists a positive number k such that $y \leqq k$ for any $y \in (Y - V) - \{Z\}$. If we choose $\delta = 1/k$, we have $f(x) > k$ for any $x \in (0, \delta)$, hence $f(x) \notin (Y - V) - \{Z\}$. Thus $f(x) \in V$ for any $x \in \langle 0, \delta)$ and evidently for any $x \in \langle 0, \delta)$. The c-continuity at $x = 0$ is proved.

Now take the point $(0, Z) \in X \times Y$. Let U, V be any neighbourhoods of 0 and Z in X and Y, respectively. For a suitable $\delta_0 > 0$ we have $(0, \delta_0) \cap X \subset U$. By the definition of V we have that $V \cup Z$ is open in $\langle 0, \infty)$. Thus a suitable $n \in Z$ and $\eta > 0$ may be chosen such that $1/(n + \eta) \in \langle 0, \delta_0) \cap X$, $(n, n + \eta) \subset V$.

So the point $(1/(n + \eta), n + \eta)$ belongs to $G(f) \cap (U \times V)$, hence $(0, Z) \in \operatorname{cl} G(f)$. Since $(0, Z) \notin G(f)$, the graph is not closed.

The next result (Theorem 2) is in fact known (see [5] Theorem 3.15) even in a slightly more general form. What is different here is the formulation and the relation to the upper c-semi-continuous functions which is explicitly stated. We give a proof independent of the other results.

Theorem 2. *Let X, Y be topological spaces. Let $F: X \to Y$ be a multifunction with a closed graph. Then F is u.c.s.c.*

Proof. Suppose F is not *u.c.s.c.* at a point $p \in X$. Then there exists a set V containing $F(p)$ such that $Y - V$ is compact and for any neighbourhood U of $F(p)$ there exists $x_u \in U$ with $F(x_u) \cap (Y - V) \neq \emptyset$. So there exists a net $\{x_s\}$ $(s \in S)$ convergent to p such that $F(x_s) \cap (Y - V) \neq \emptyset$. Take $y_s \in F(x_s) \cap (Y - V)$. Since $y_s (s \in S)$ belongs to the compact set $Y - V$, there exists an accumulation point $q \in (Y - V)$ of the net $\{y_s\}$ $(s \in S)$. Let $U \times W$ be a neighbourhood of the point (p, q) such that U, W are neighbourhoods of p and q, respectively. Take s_0 such that

$x_s \in U$ whenever $s \geqq s_0$. Since q is an accumulation point of $\{y_s\}$ $(s \in S)$, there exists s_1 such that $s_1 > s_0$ and $y_{s_1} \in W$. Since $y_{s_1} \in F(x_{s_1})$ we have $(x_{s_1}, y_{s_1}) \in G(F)$. Thus $(x_{s_1}, y_{s_1}) \in (U \times W) \cap G(F)$. So $(p, q) \in \operatorname{cl} G(F)$. Since $G(F)$ is closed we have $(p, q) \in G(F)$. Thus $q \in F(p) \subset V$, which is a contradiction.

Corollary 2. *Let X, Y be topological spaces. Let $f: X \to Y$ be a single-valued function with a closed graph. Then f is c-continuous.*

The above mentioned more general form of Theorem 2 concerns the functions with subclosed graphs. A graph of F is said to be subclosed if for each $x \in X$ and a net $\{x_s\}$ in $X - \{x\}$ converging to x, and a net $\{y_s\}$ where $y_s \in F(x_s)$ converging to $y \in Y$, we have $y \in F(x)$. One can verify that the proof of Theorem 2 works also for multifunctions with subclosed graphs.

If we have a multifunction with values in a compact Hausdorff space then the definition of the upper c-semi-continuity immediately yields that the last coincides with the upper semi-continuity. Thus we have the following well known corollaries.

Corollary 3. *Let X be a topological space, Y a compact Hausdorff space. Then any multifunction $F: X \to Y$ with a closed graph is upper semi-continuous.*

Corollary 4. ([7] Theorem 4). *Under the same assumptions on X, Y as in Corollary* 3 *any single-valued function $f: X \to Y$ with a closed graph is continuous.*

The following result is a combination of Theorems 1 and 2 and gives a sufficient condition for the equivalence of the upper c-semi-continuity and the closedness of the graph of a multifunction.

Theorem 3. *Let X be a topological space, Y a locally compact Hausdorff topological space. Then a closed-valued multifunction $F: X \to Y$ is u.c.s.c. if and only of F has a closed graph.*

Corollary 5. *Under the same conditions on X and Y, a single-valued function $f: X \to Y$ is c-continuous if and only if it has a closed graph.*

2. REMARKS ON c^*-CONTINUITY

The idea of c-continuity provides a motivation for introducing a type of continuity which arises if we substitute the compactness by the countable compactness. Again the multifunctions will be discussed and we restrict ourselves mostly to results related to closed graphs.

A multifunction $F: X \to Y$ is said to be upper c^*-semi-continuous (*u.c*.s.c.*) at $p \in X$ if for any open $V \subset Y$ such that $F(p) \subset V$ and $Y - V$ is countably compact, there exists a neighbourhood U of p such that $F(x) \subset V$ for any $x \in U$. It is said to be upper c^*-semi-continuous if it is *u.c*.s.c.* at any $p \in X$.

In the case of a single-valued function we say c^*-continuity instead of upper c^*-semi-continuity.

Evidently $F: X \to Y$ is *u.c*.s.c.* if and only if for any open $V \subset Y$ with $Y - V$ countably compact the set $\{x: F(x) \subset V\}$ is open in X.

The proof of the following proposition follows from the known fact (see [1] p. 230, Theorem 3.6 (2)) that a closed subspace of a countably compact space is countably compact.

Proposition 1. *If $F: X \to Y$ is a multifunction and the space Y is a countably compact Hausdorff space then the u.c*.s. continuity and the u.s. continuity coincide.*

Proposition 2. *Let $F: X \to Y$ be a multifunction. Then the following implications hold: F is u.s.c. $\Rightarrow$ F is u.c*.s.c. $\Rightarrow$ F is u.c.s.c. None of the implications may be reversed.*

Proof. The validity of the implications is a straightforward consequence of the corresponding definitions. The simple example of the single-valued function $f: R \to R$, $f(x) = 1/x$, if $x \neq 0$, $f(0) = 0$ shows that the first implication may not be reversed. As to the second implication we give the following example.

Example 2. Let X be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1. Let Y be the set of all ordinal numbers less than ω_1. Both the sets are supposed to be equipped with the order topology. Define a single-valued function $f: X \to Y$ such that

$$f(x) = \begin{cases} x & \text{if} \quad x \neq \omega_1 , \\ 0 & \text{if} \quad x = \omega_1 . \end{cases}$$

The function f is c-continuous. This folows from the fact that if we take any open set $V \subset Y$ with $Y - V$ compact then necessarily there exists an ordinal number $\eta < \omega_1$ such that for any $y \in Y - V$ we have $y \leqq \eta$. Thus for any $x > \eta$ we have $f(x) \in V$. This proves the c-continuity at the point ω_1. The c-continuity at any $x \neq \omega_1$ is obvipous. The space Y is a countably compact Hausdorff space (see e.g. [1] p. 228, Ex. 1). So by Proposition 1, the c^*-continuity of f coincides with its continuity. Thus f is not c^*-continuous because it is not continuous at $x = \omega_1$.

The following result is a straightforward consequence of Theorem 1 and Proposition 2.

Theorem 4. *Let X, Y be topological spaces, Y a locally compact Hausdorff space. Let $F: X \to Y$ be a closed-valued u.c*.s.c. function. Then the graph $G(F)$ is closed.*

Corollary 6. *Under the same assumptions on X, Y as in Theorem 4 a single-valued function $f: X \to Y$ which is c^*-continuous has a closed graph.*

In the case that X is a first countable space we can prove an analogue of Theorem 2 also for upper c^*-semi-continuous multifunctions.

Theorem 5. *Let X be first countable, Y an arbitrary topological space. Let $F: X \to Y$ be a multifunction with a closed graph. Then F is u.c*.s.c.*

Proof. The reasoning of the proof is quite analogous to that in Theorem 2. Suppose F is not *u.c*.s.c.* at a point $p \in X$. Then there exists an open set $V \subset Y$ containing $F(p)$ such that $Y - V$ is countably compact and for any neighbourhood U of p there exists x such that $F(x) \cap (Y - V) \neq \emptyset$. Let $\{U_n\}_{n=1}^{\infty}$ be a descending base of neighbourhoods of p. Choose $x_n \in U_n$ such that $F(x_n) \cap (Y - V) \neq 0$ for $n = 1, 2, \ldots$. Let $y_n \in F(x_n) \cap (Y - V)$ be arbitrarily chosen. Since $(Y - V)$ is countably compact there exists a limit point $q \in (Y - V)$ of the sequence $\{y_n\}_{n=1}^{\infty}$. By an argument quite analogous to that in the proof of Theorem 2 one finds that $q \in F(p) \subset V$ and obtains a contradiction.

Corollary 7. *Under the same assumptions on X, Y as in Theorem 5 a single-valued function $f: X \to Y$ with a closed graph is c^*-continuous.*

The assuption of first countability of X is essential in Theorem 5 and Corollary 7. To show this it is sufficient to consider Example 2 where f has a closed graph but is not c^*-continuous.

The next result also follows immediately if we use Proposition 2 and Theorem 5.

Theorem 6. *Let X be first countable, Y a countably compact Hausdorff space. Let $F: X \to Y$ be a multifunction with a closed graph. Then F is u.s.c.*

As a corolary of Theorem 6 we obtain the known result for single-valued functions.

Corollary 8 ([8] Theorem 2). *Let X be first countable, Y a countably compact Hausdorff space. Let $f: X \to Y$ be a single-valued function with a closed graph. Then f is continuous.*

As a consequence of some of our results one can obtain conditions under which upper c^*-semi-continuity, upper c-semi-continuity and the closedness of the graph are equivalent. Then we obtain the corresponding equivalence for c-continuity, c^*-continuity and the closedness of the graph for the case of a single-valued function.

Theorem 7. *Let X be first countable, Y a locally compact Hausdorff space. The following statements are equivalent for a closed-valued multifunction $F: X \to Y$:*

(a) *F is upper c^*semi-continuous;*
(b) *The graph $G(F)$ is closed;*
(c) *F is upper c-semi-continuous.*

Corollary 9. *Under the same assumptions on X, Y as in Thoerem 7 the following statements are equivalent for a single-valued function $f: X \to Y$:*

(a) f is *c^*-continuous;*
(b) *The graph $G(f)$ is closed;*
(c) *f is c-continuous.*

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