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Ivan Havel

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ON THE COMPLETENESS-NUMBER OF A FINITE GRAPH II.

IVAN HAVEL, Praha

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In this paper we investigate in more details properties of a relation "the edges e_1 and e_2 are quasineighbours" introduced in [3]. Further, we deal with the correspondence of maximal number k_G of independent edges (in the sense that no two of them are quasineighbours) to the completeness-number $\omega(G)$ of the graph G. In general case there is only proved that $k_G \leqq \omega(G)$; graphs having the completeness-number up to 5 are trated fully.

We shall deal with finite non-directed connected graphs without loops. The completeness-number $\omega(G)$ of G is the smallest cardinality of the system $\mathscr{G}$ formed by complete subgraphs of G covering G, i.e. all the vertices and edges of G (cf. definition in [1], p. 19). In [3] the method was described, which makes possible to bring the problem of determination $\omega(G)$ to that of determination $\chi(G')$, where $\chi(G')$ denotes the chromatic number of the graph G' constructed in a special standard way to a given G. In this construction the notion of quasineighbouring edges plays an important role.

Definition 1. Two different edges e_1, e_2 of the graph $G = \langle V, E \rangle$ are called *quasineighbours* if both of them belong to a certain complete subgraph of G.

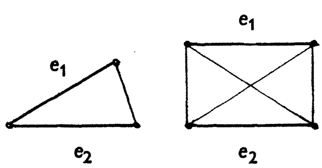

Fig. 1.

Note 1. It is obvious from this definition that e_1, e_2 are not quasineighbours if and only if there are vertices $v_1, v_2 \in V$, v_i incident with e_i $(i = 1, 2)$, $(v_1, v_2) \notin E$. Two possible cases of quasineighbouring edges are shown in Fig. 1. The construction of G' is carried out as follows: the edges of G correspond uniquely to the vertices of G', the vertices v'_1, v'_2 of G' are adjacent if and only if the corresponding edges in G are not quasineighbours. Then the equality $\omega(G) = \chi(G')$ holds (cf. theorem in [3]).

Many authors have investigated the connection between a chromatic number of a graph and an order of its complete subgraphs. An arbitrary circuit G of the length $\geqq 5$ and odd is an example of the graph without triangles with $\chi(G) = 3$.

There are well-known constructions of graphs without triangles with large enough chromatic number (cf. [2]).

Let us investigate in this connection the mapping described above (and in details in [3]) assigning to a given G the graph G'. For graphs which in considered mapping represent images of others, is the situation simpler. Graphs G up to $\chi(G) = 4$ have to contain a complete subgraph of an order equal to $\chi(G)$. But a graph G may be plotted in such a way that $\chi(G') = 5$ and G' does not contain a complete pentagon. If we take into account that $\chi(G') = \omega(G)$ and that to a complete subgraph of an order k in G' there corresponds a k-tuple of mutually not quasineighbouring edges, it is obvious, that the problem may be restated as follows:

What is the relation between a completeness-number $\omega(G)$ of a graph G and a maximum number k_G of independent edges of the same graph (independent means that no two of them are quasineighbours) like?

Theorem 1. 1) *If $G = \langle V, E\rangle$ is an arbitrary graph, then $k_G \leqq \omega(G)$.*

2) *If $\omega(G) \leqq 4$, then $k_G = \omega(G)$.*

3) *There is a graph G such that $\omega(G) = 5$, $k_G = 4$.*

Proof of 1): Suppose $\omega(G) = l$, let $A_1, \ldots, A_l$ be complete subgraphs of G chosen in such a way that the system $\{A_1, \ldots, A_l\}$ covers G. Let $e_1, \ldots, e_{k_G}$ be independent edges. If $k_G > l$, then there are i, j', j'', $(1 \leqq i \leqq l,\ 1 \leqq j' < j'' \leqq k_G)$ such that both $e_{j'}$ and $e_{j''}$ belong to A_i. It is a contradiction, since A_i is complete and $e_{j'}$, $e_{j''}$ are not quasineighbours.

Before proving 2), we shall adopt the following notational convention: we shall write

$$G \sim \{A_1, \ldots, A_l\}$$

in order to express that $\omega(G) = l$, A_i are maximum complete subgraphs of G and the system $\{A_1, \ldots, A_l\}$ covers G. The word "maximum" is used in the sense of inclusion, i.e. no other vertex may be added to A_i (such an addition would make A_i incomplete). The set of vertices (resp. edges) of A_i will be denoted by V_i (resp. E_i), hence $A_i = \langle V_i, E_i\rangle$.

Lemma 1. *Suppose $G = \langle V, E\rangle$, $G \sim \{A_1, \ldots, A_l\}$, $A_i = \langle V_i, E_i\rangle$ $(i = 1, \ldots, l)$, $1 \leqq j' < j'' \leqq l$. Then there are two different vertices $u \in V_{j'}$, $v \in V_{j''}$, which are not adjacent $((u, v) \notin E)$.*

Proof. If each vertex of $A_{j'}$ had been connected with all the vertices of $A_{j''}$ different from it, it would have been possible to cover G by less than l complete subgraphs. One would have removed $A_{j'}$, $A_{j''}$ from the system $\{A_1, \ldots, A_l\}$ and add a complete subgraph induced by the set of vertices $V_{j'} \cup V_{j''}$, i.e. the graph $\langle V_{j'} \cup \cup V_{j''}, E \cap ((V_{j'} \cup V_{j''}) \times (V_{j'} \cup V_{j''}))\rangle$.

Proof of 2). If $\omega(G) = 2$ and $G \sim \{A_1, A_2\}$, one has to choose $u \in V_1$, $v \in V_2$ such that $(u, v) \notin E$ (it is possible according to the lemma 1). Further, one must choose arbitrarily $u' \in V_1$, $u' \neq u$ and $v' \in V_2$, $v' \neq v$. The edges (u, u'), (v, v') are not quasineighbours.

Let $\omega(G) = 3$, $G \sim \{A_1, A_2, A_3\}$. We use the lemma three times and choose $u, u' \in V_1$, $v, v' \in V_2$, $x, x' \in V_3$ in such a way, that $(u, v) \notin E$, $(u', x) \notin E$, $(v', x') \notin E$. If accidentally $u' = u$, we choose for u' an arbitrary element of V_1 fulfilling the condition $u' \neq u$. (Similarly in the cases $v' = v$, $x' = x$.) No two edges out of the 3-tuple (u, u'), (v, v'), (x, x') are quasineighbours.

Let $\omega(G) = 4$, $G \sim \{A_1, A_2, A_3, A_4\}$. Let us suppose this covering to be fixed in the sequel.

Definition 2. A vertex $v \in V$ is 1-vertex of the graph G (with respect to the covering $\{A_1, \ldots, A_l\}$ if it belongs exactly to one of the sets $V_1, \ldots, V_l$. We define an m-vertex in a similar way (for any $m \leqq l$).

Lemma 2. *If $\omega(G) = 4$ and $G \sim \{A_1, \ldots, A_4\}$, then each V_i $(i = 1, \ldots, 4)$ contains at least one 1-vertex or 2-vertex.*

Proof. Let us suppose that e.g. A_1 contains neither 1-vertex nor 2-vertex. All the vertices of A_1 are either 3-vertices or 4-vertices. In this case, however, every edge $e \in E$ covered by the subgraph A_1 is also covered by some of the subgraphs A_2, A_3, A_4. A_1 may be omitted and $\omega(G) < 4$, which is a contradiction.

Note 2. The assumption $\omega(G) = 4$ in the lemma is essential. In Fig. 2 there is given such a graph G that $\omega(G) = 5$; the assertion of the lemma is not valid for G.

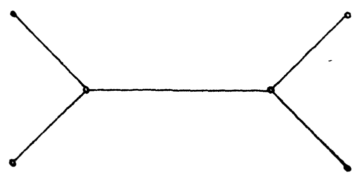

Fig. 2.

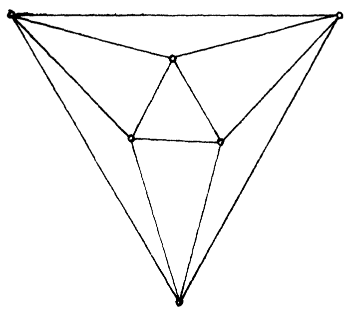

Fig. 3.

Continuation of the proof 2):

Case 1. Each A_i $(i = 1, \ldots, 4)$ contains a 1-vertex. Let $u \in V_1$, $v \in V_2$, $x \in V_3$, $y \in V_4$ be these 1-vertices. No two of them are adjacent. If some two of them had been connected by an edge, this edge would have been covered by some of subgraphs $A_1, \ldots, A_4$ and would have belonged together with its end-vertices to this subgraph. This would have been a contradiction to the fact that the latter ones are 1-vertices belonging A_i, A_j, $i \neq j$. We choose $u' \in V_1$, $v' \in V_2$, $x' \in V_3$, $y' \in V_4$ so that $u' \neq u$, $v' \neq v$, $x' \neq x$, $y' \neq y$. The edges (u, u'), (v, v'), (x, x'), (y, y') are independent.

Case 2. Assumptions of the case 1 are not fulfilled. Let us suppose that e.g. A_1 contains no 1-vertex. (Fig. 3 shows a graph G, $\omega(G) = 4$; the covering system being chosen in any way, G does not contain 1-vertices.)

According to the lemma 2 there is a 2-vertex u in A_1. $u \in V_1$ and also $u \in V_i$ for some $i \neq 1$. We can assume that $u \in V_2$, therefore $u \in V_1 \cap V_2 - (V_3 \cup V_4)$.

Lemma 3. *Let $\omega(G) = l$, $G \sim \{A_1, \ldots, A_l\}$, $1 \leqq i \leqq l$, $u \in V$, $u \notin V_i$. Then there is $v \in V_i$ such that $(u, v) \notin E$.*

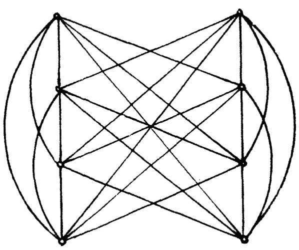

Fig. 4.

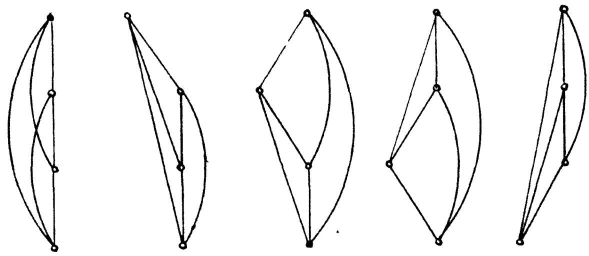

Fig. 5.

Proof. If u had been connected by some edges with all the vertices of A_i, it would have belonged to A_i, since A_i is a maximum complete subgraph of G.

Continuation of the main proof. $u \notin V_3$; there is $x \in V_3$, $(u, x) \notin E$. $u \in V_4$; there is $y \in V_4$, $(u, y) \notin E$. If $x \neq y$ and $(x, y) \notin E$, one has to choose x', y' so that $x' \in V_3$, $x' \neq x$, $y' \in V_4$, $y' \neq y$. If $x = y$, there must be found (according to the lemma 1) $x' \in V_3$, $y' \in V_4$ so that $(x', y') \notin E$. (Then $x \neq x'$, $x = y \neq y'$.) Finally in the case $x \neq y$, $(x, y) \in E$ one has to choose $x' \in V_3$, $y' \in V_4$ such that $(x', y') \notin E$ (lemma 1). If in this case $x = x'$ (resp. $y = y'$) – both of them cannot occur simultaneously – we choose for x' (resp. y') an arbitrary vertex of V_3 (resp. V_4) different from x (resp. y). At last we use once more the lemma 1 and find $u' \in V_1$, $v' \in V_2$ such that $(u', v') \notin E$.

The edges (u, u'), (u, v'), (x, x'), (y, y') are independent.

Proof of 3): Let us investigate the graph given in Fig. 4.

We shall prove that

1. $\omega(G) = 5$,
2. there is no 5-tuple of independent edges in G.

First, we shall present 5 complete subgraphs of G covering G; it will be done in Fig. 5.

We have proved $\omega(G) \leqq 5$. Let us notice that maximum complete subgraphs of G are those of order 4; every such a graph has 6 edges, G itself has 24 edges. If $\omega(G) = 4$, then 4 complete subgraphs of order 4 would have to be included in G, no two of them having an edge in common. Obviously it is impossible. Therefore $\omega(G) > 4$ and thus $\omega(G) = 5$.

Let us further investigate a covering of G by its complete subgraphs $A_1, \ldots, A_5$ from the Fig. 5. Each vertex will be assigned the numbers of graphs $A_1, \ldots, A_5$ that it belongs to. The obtained graph is given in Fig. 6.

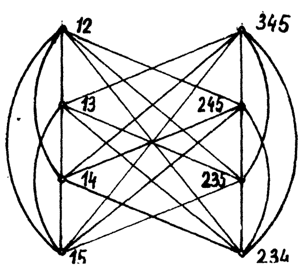

Fig. 6.

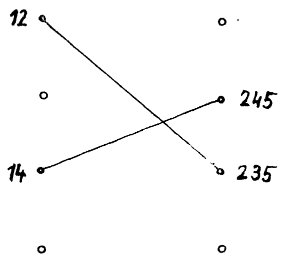

Fig. 7.

Suppose the 5-tuple of independent edges in G be chosen. Each of these edges belongs just to one of the graphs $A_1, \ldots, A_5$, no two of them belong to the same graph. Let us denote the one of edges, which belongs to A_i $(i = 1, \ldots, 5)$ by e_i. Obviously, for each $i = 2, \ldots, 5$ e_i must be incident with a vertex, indices of which are 1, i. We shall see that even the 4-tuple e_2, e_3, e_4, e_5 possesses a couple of quasineighbouring edges. Suppose e.g. e_2 be incident with a vertex, indices of which are 2, 3, 5. In this case, e_4 will be investigated. Suppose e_4 be incident e.g. with a vertex, indices of which are 2, 4, 5 (cf. Fig. 7).

e_5 and one of the edges e_2, e_4 are quasineighbours. We proceed quite similarly also in all the remaining possibilities.

Thus, each 5-tuple of edges in G contains at least two quasineighbours. However, one can easily find a 4-tuple of independent edges in G (cf. Fig. 8).

This accomplishes the proof of 3) and the theorem 1.

Let now a graph $G = \langle V, E \rangle$ be given. Suppose $\omega(G) = l$, choose an arbitrary covering of G by its maximum complete subgraphs, e.g.

$$G \sim \{A_1, \ldots, A_l\} . \tag{1}$$

For each $u \in V$ let $I(u)$ denote the set of indices of u (with respect to (1)) defined as follows:

$$I(u) = \{i; u \in V_i\} .$$

(Notice that $A_i = \langle V_i, E_i \rangle$, $i = 1, \ldots, l$.) Let us define a graph $\tilde{G}$ (the definition depends on the covering (1)): the set of vertices is $\{I(u);\, u \in V\}$; the vertices $I(u)$, $I(v)$ are adjacent in $\tilde{G}$ if and only if $I(u) \cap I(v) \neq \emptyset$. When G is connected (and we suppose it), $\tilde{G}$ is also connected.

Example: Let us construct $\tilde{G}$ to G built of one complete pentagon (A_1) and two complete tetragons (A_2, A_3) (cf. Fig. 9):

Theorem 2. *For any graph G and its covering* (1)

$$\omega(G) = \omega(\tilde{G}), \quad k_G = k_{\tilde{G}}.$$

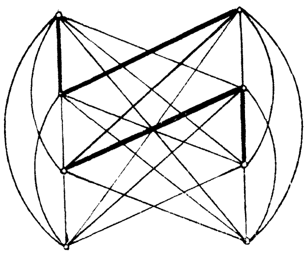

Fig. 8.

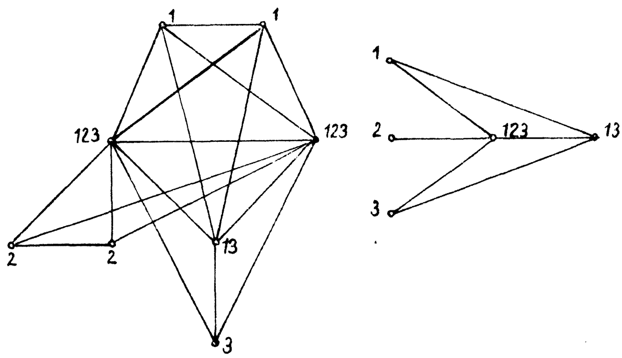

Fig. 9.

The proof immediately follows from the lemmas 5 and 6 given below.

Lemma 4. *Let $G = \langle V, E\rangle$, $u \in V$; G_1 will denote a graph obtained on removing the vertex u and all the incident edges. (G_1 does not have to be a connected graph.) Then*

$$\omega(G_1) \leqq \omega(G).$$

Proof. Let $G \sim \{A_1, \ldots, A_l\}$. We shall construct a system $\{A'_1, \ldots, A'_l\}$ of complete subgraphs of G_1 in a following manner: if $i \in I(u)$, remove from A_i the vertex u and all the incident edges – the result is A'_i. If $i \notin I(u)$, let $A'_i = A_i$. The system $\{A'_1, \ldots, A'_l\}$ covers G_1.

Lemma 5. *Let $G \sim \{A_1, \ldots, A_l\}$, let $u, v \in V$ such that $u \neq v$ and $I(u) = I(v)$. Suppose G_1 be the same as in lemma* 4. *Then*

$$\omega(G_1) = \omega(G).$$

Proof. For any $x \in V$, $x \neq u$ a following statement holds: $(v, x) \in E \Leftrightarrow (u, x) \in E$, since $(v, x) \in E \Leftrightarrow I(v) \cap I(x) \neq \emptyset \Leftrightarrow I(u) \cap I(x) \neq \emptyset \Leftrightarrow (u, x) \in E$. Assume

(2) $$G_1 \sim \{A'_1, \ldots, A'_{l'}\}.$$

We shall construct a system $\{A_1, \ldots, A_{l'}\}$ of complete subgraphs G in a following way: if $i \in I(v)$ (with respect to the covering (2)), the vertex u and all the edges joining it with the remaining vertices of A'_i will be added to A'_i – the obtained graph is A_i. If $i \notin I(v)$, let $A_i \doteq A'_i$. The system $\{A_1, \ldots, A_{l'}\}$ obviously covers G, hence $l \leqq l'$ and together with the lemma 4 it follows $l = l'$.

Lemma 6. *Let $G \sim \{A_1, \ldots, A_l\}$, $u \neq v$, $I(u) = I(v)$, let G_1 be the same as in lemma* 4. *Then $k_G = k_{G_1}$.*

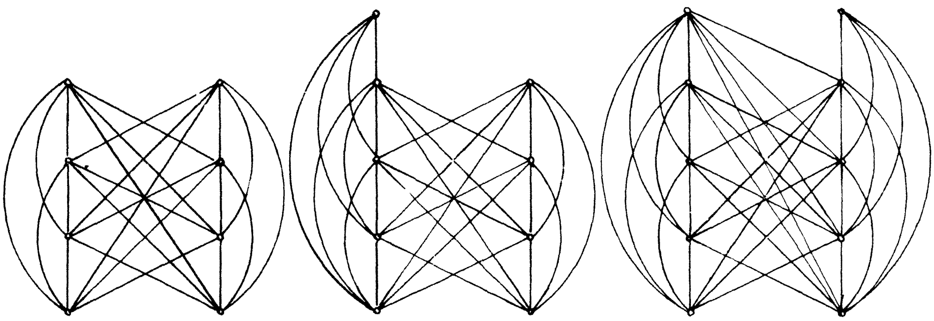

Fig. 10.

Proof. First, let us choose independent edges $e_1, \ldots, e_{k_G}$ in G. We shall construct a system $e'_1, \ldots, e'_{k_G}$ of independent edges in G_1. If the edge e_i $(i = 1, \ldots, k_G)$ is incident with the vertex u, i.e. if it is of the form (u, x) where $x \neq v$, let us put $e'_i = (v, x)$. If it is not the case, let $e'_i = e_i$. The only remaining possibility is $e_i = (u, v)$. In this case we find similarly as in the proof of 3) theorem 1, that $I(u) \cap I(v)$ has to be one element set, i.e. both u and v have to be 1-vertices. We put $e'_i = (v, v')$, where v' is such an arbitrary vertex that $I(v) \cap I(v') \neq \emptyset$. No two edges of $e'_1, \ldots, e'_{k_G}$ are quasineighbours. Thus, we have proved $k_{G_1} \geqq k_G$, the inversion is obvious.

Let us now consider only graphs having fixed completeness-number, say l. If G is such a graph, find arbitrarily its covering (1) and construct a graph $\tilde{G}$. There is only a finite number of such graphs $\tilde{G}$ (each of them has less than 2^l vertices). We shall use this fact in order to investigate in details graphs having the completeness-number 5. More exactly, we shall find necessary and sufficient conditions the graph G $(\omega(G) = 5)$ must fulfil in order to vanish the equality $k_G = 5$.

We shall see that all such graphs are in some sense similar to that given in Fig. 4.

Theorem 3. *Let $\omega(G) = 5$. If $\{1, 2, 3, 4, 5\}$ is a vertex of $\tilde{G}$ (constructed to an arbitrarily chosen covering $G \sim \{A_1, \ldots, A_5\}$), remove it from $\tilde{G}$ with all the incident edges, otherwise do not change $\tilde{G}$.*

The necessary and sufficient condition for the equality $k_G = 4$ is that the graph obtained is isomorphic to some of graphs given in Fig. 10.

We shall prove a sequence of lemmas.

Lemma 7. *Let $G \sim \{A_1, \ldots, A_5\}$, let G contain a pair of 2-vertices which are not adjacent. Then $k_G = 5$.*

Proof. Without loss of generality we may assume that

$$u \in V_1 \cap V_2 - (V_3 \cup V_4 \cup V_5)\,, \quad v \in V_3 \cap V_4 - (V_1 \cup V_2 \cup V_5)\,, \quad (u, v) \notin E\,.$$

We use the lemma 1 and find $u' \in V_1$, $x \in V_2$ so that $(u', x) \notin E$ (if accidentally $u' = u$, we choose for u' an arbitrary vertex different from u; this convention also holds sometimes in the sequel). Let $v' \in V_3$, $y \in V_4$ be chosen in such a way that $(v', y) \notin E$. Since $u \notin V_5$, $v \notin V_5$, there are $z, z' \in V_5$ (lemma 3), $(u, z) \notin E$, $(v, z') \notin E$. The edges (u, u'), (u, x), (v, v'), (v, y), (z, z') are independent.

Lemma 8. *Let $G \sim \{A_1, \ldots, A_5\}$, suppose each A_i $(i = 1, \ldots, 5)$ contains a* 1-vertex. *Then $k_G = 5$.*

Proof is obvious. One has to find arbitrarily $u' \in V_1$, $v' \in V_2, \ldots, z' \in V_5$ to given 1-vertices u, v, x, y, z and look at the edges $(u, u'), \ldots, (z, z')$.

Lemma 9. *Let $G \sim \{A_1, \ldots, A_5\}$, suppose $A_1, \ldots, A_4$ contain 1-vertices, A_5 contains a 2-vertex. Then $k_G = 5$.*

Proof. Let u, v, x, y be given 1-vertices $(u \in V_1, \ldots, y \in V_4)$, let $z \in V_1 \cap V_5$ be 2-vertex. $u \notin V_5$, therefore one can find such $z' \in V_5$ that $(u, z') \notin E$. Let us choose $u' \in V_1, \ldots, y' \in V_4$, $u' \neq u, \ldots, y' \neq y$. The edges $(u, u'), \ldots, (z, z')$ are independent.

Lemma 10. *Let $G \sim \{A_1, \ldots, A_5\}$, suppose $A_1, \ldots, A_4$ contain 1-vertices, A_5 contains neither 1-vertex nor 2-vertex. Then $k_G = 5$.*

Proof. We can establish similarly as in the proof of lemma 2 that there must be $z, z' \in V_5$ such that $I(z) \cap I(z') = \{5\}$ i.e. the only index the vertices z and z' have in common is 5. Choose arbitrarily $u' \in V_1, \ldots, y' \in V_4$. The edges $(u, u'), \ldots, (z, z')$ are independent.

Lemma 11. *Let $G \sim \{A_1, \ldots, A_5\}$, suppose A_1, A_2, A_3 contain 1-vertices and the following condition (3) is fulfilled:*

(3) *if u, v are different 2-vertices of G, then $(u, v) \in E$ (any different 2-vertices of G are adjacent).*

Then $k_G = 5$.

Proof. Let $u \in V_1$, $v \in V_2$, $x \in V_3$ be given 1-vertices. We can find $y \in V_4$ and $z \in V_5$ (lemma 1) so that $(y, z) \notin E$. One of the vertices y, z is necessarily 2-vertex, the other a 3-vertex. (Any two 3-vertices are always adjacent when the completeness-number of a graph is 5; y, z cannot be 2-vertices, since it contradicts to (3)). Let e.g. $y \in$ $\in V_1 \cap V_4 - (V_2 \cup V_3 \cup V_5)$. Then $z \in V_2 \cap V_3 \cap V_5 - (V_1 \cup V_4)$.

Let us assume now that A_5 contains a vertex z', which belongs neither to A_2 nor to A_3. Furthermore: $u \in V_1 - (V_2 \cup V_3 \cup V_4 \cup V_5)$. So there is a vertex $y' \in V_4$, $(u, y') \notin E$. In G one can find a partial subgraph given in Fig. 11 (the full lines stand for the edges which are definitely present in G, the dotted ones for those which are not present in G; this convention holds for the following figures too).

The edges (u, y), (v, z), (x, z), (y, y'), (z, z') are independent.

Let us now assume that each vertex of A_5 is contained either in A_2 or in A_3. We shall find such a pair of vertices z'', $z''' \in V_5$ that $I(z'') \cap I(z''') = \{5\}$. At least one of them does not belong to A_4, let it be e.g. $z''(z'' \in V_5 - V_4)$. z'' is not a 1-vertex (A_5 does not contain any 1-vertices at all). z'' is not a 2-vertex. It belongs to A_5 and in the same time either to A_2 or A_3 and it could not be joined with the 2-vertex y (an edge joining of them must also belong to some A_i and it is impossible). Obviously z'' is neither 4-vertex nor 5-vertex. It is 3-vertex. There are 3 possibilities:

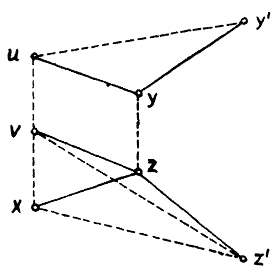

Fig. 11.

$$z'' \in V_1 \cap V_2 \cap V_5\,, \quad z'' \in V_1 \cap V_3 \cap V_5\,,$$
$$z'' \in V_2 \cap V_3 \cap V_5\,.$$

Actually, the last is impossible because of the following: $z''' \in V_5$, therefore either $z''' \in V_2$ or $z''' \in V_3$. In this case, however, $I(z'') \cap I(z''')$ is not only $\{5\}$.

First, let us investigate the case $z'' \in V_1 \cap V_2 \cap V_5$. $z'' \notin V_4 \Rightarrow$ there is $y' \in V_4$ so that $(y', z'') \notin E$. G contains a partial subgraph given in Fig. 12.

Any two vertices are adjacent if and only if they have at least one index in common. In the figure not all existing edges are shown (e.g. $(u, z'') \in E$). One can easily see a 5-tuple of independent edges in the figure.

The case $z'' \in V_1 \cap V_3 \cap V_5$ is symmetric. This accomplishes the proof of lemma 11.

Lemma 12. *Let $G \sim \{A_1, \ldots, A_5\}$, let A_1, A_2 contain 1-vertices, do not let A_3, A_4, A_5 contain 1-vertices. Suppose (3) is fulfilled. Then $k_G = 5$.*

Proof. Let u, v be 1-vertices of graphs A_1, A_2. At the beginning, let us assume that each of graphs A_3, A_4, A_5 contains a 2-vertex. The possible situations (when graphs are numbered in a proper way) are:

a) there are $x \in V_3 \cap V_4 - (V_1 \cup V_2 \cup V_5)$ $x' \in V_3 \cap V_5 - (V_1 \cup V_2 \cup V_4)$,

b) the condition a) is not fulfilled; in this case, however, there must be either a triple such of 2-vertices x', y', z' that $x' \in V_1 \cap V_3$, $y' \in V_1 \cap V_4$, $z' \in V_1 \cap V_5$ or a triple of such 2-vertices x'', y'', z'' that $x'' \in V_2 \cap V_3$, $y'' \in V_2 \cap V_4$, $z'' \in V_2 \cap V_5$.

First, let us investigate the case a): $x \notin V_5 \Rightarrow$ there is such $z \in V_5$ that $(x, z) \notin E$ (it has to be $z \in V_1 \cap V_2 \cap V_5$). $x' \notin V_4 \Rightarrow$ there is $y \in V_4$, $(x', y) \notin E$. The situation is described in Fig. 13.

The edges (u, z), (x', z), (x, x'), (x, y), (v, y) are independent.

b) Both possibilities are symmetric, let us investigate the first one:

$$x' \notin V_5 \Rightarrow \text{ there is } z'' \in V_5\,, \quad (x'', z'') \notin E\,.$$
$$y' \notin V_3 \Rightarrow \text{ there is } x'' \in V_3\,, \quad (x'', y') \notin E\,.$$
$$z' \notin V_4 \Rightarrow \text{ there is } y'' \in V_4\,, \quad (y'', z') \notin E\,.$$

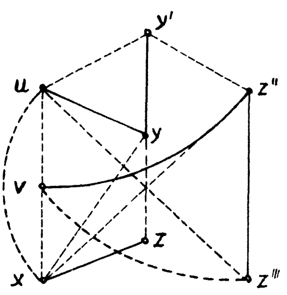

Fig. 12.

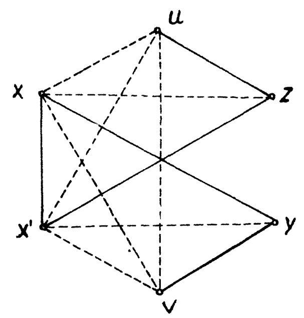

Fig. 13.

Notice that x'', y'', z'' have to be 3-vertices. G contains a partial subgraph given in Fig. 14.

The edges (u, x'), (x', x''), (v, x''), (y', y''), (z', z'') are independent. The investigation of the case "each of graphs A_3, A_4, A_5 contains a 2-vertex" is finished.

Notice that the last condition may be violated by one of graphs A_3, A_4, A_5 only. Suppose on the contrary that both A_3 and A_4 have neither 1-vertices nor 2-vertices. In this case for any pair $x \in V_3$, $y \in V_4$ $(x, y) \in E$ would be in contradiction with lemma 1.

Suppose, therefore, that A_1, A_2 have 1-vertices u, v, A_3 has neither 1-vertices nor 2-vertices, A_4, A_5 have 2-vertices. We shall find $x, x' \in V_3$ so that $I(x) \cap I(x') = \{3\}$. x and x' have to be 3-vertices. Three possibilities must be considered (the remaining three are obtained by interchange x and x'):

$$x \in V_1 \cap V_2 \cap V_3\,, \quad x' \in V_3 \cap V_4 \cap V_5\,,$$
$$x \in V_1 \cap V_3 \cap V_4\,, \quad x' \in V_2 \cap V_3 \cap V_5\,,$$
$$x \in V_1 \cap V_3 \cap V_5\,, \quad x' \in V_2 \cap V_3 \cap V_4\,.$$

We shall prove now that the 2nd and 3rd cases are impossible. Really, $x \in V_1 \cap \cap V_3 \cap V_4 \Rightarrow x \notin V_5 \Rightarrow$ there is $z \in V_5$, $(x, z) \notin E$; $x' \notin V_4 \Rightarrow$ there is such $y \in V_4$ that

$(x', y) \notin E$. y and z are 2-vertices: $y \in V_1 \cap V_4$, $z \in V_2 \cap V_5$ and they are not joined by any edge, which contradicts to (3). In a similar manner we can treat the 3rd case.

Let us investigate the 1st one: $x \notin V_4 \Rightarrow$ there is $y \in V_4 \cap V_5 - (V_1 \cup V_2 \cup V_3)$. Choose $y' \in V_4$ and $z' \in V_5$ so that $(y', z') \notin E$. A partial subgraph of G and a 5-tuple of independent edges are shown in Fig. 15.

We have proved entirely the lemma.

Lemma 13. *Let $G \sim \{A_1, \ldots, A_5\}$, do not let be fulfilled assumptions of any lemma 7–12. Then $\tilde{G}$ (after removing a vertex $\{1, 2, 3, 4, 5\}$ if it occurs in $\tilde{G}$) is isomorphic to some graph of those given in Fig. 10 and $k_G = 4$.*

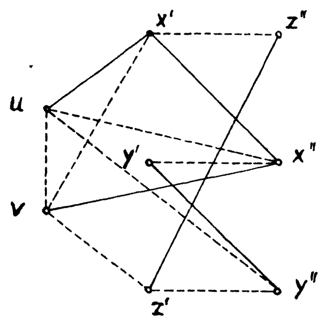

Fig. 14.

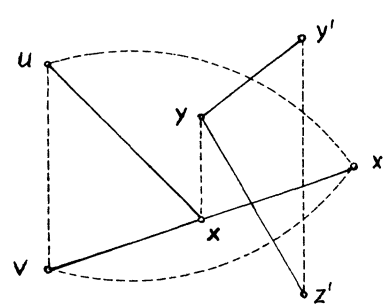

Fig. 15.

Proof. We shall consider the assumptions of any lemma be fulfilled also in the case that they can be fulfilled by an appropriate re-numbering of graphs $A_1, \ldots, A_5$.

Two cases must be investigated:

a) None of graphs $A_1, \ldots, A_5$ has 1-vertex. Let us assume that e.g. A_1 does not have 2-vertices either. We shall find $u, u' \in V_1$ so that $I(u) \cap I(u') = \{1\}$. Both u and u' must be 3-vertices. In the manner similar to that used at the end of the proof lemma 12 we shall find that there would be a couple of non-adjacent 2-vertices (and therefore assumptions of the lemma 7 would be fulfilled).

Therefore, each A_i $(i = 1, \ldots, 5)$ must contain a 2-vertex. Without loss of generality let us assume that G contains 2-vertices of indices $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$. G contains also vertices of indices $\{3, 4, 5\}$, $\{2, 4, 5\}$, $\{2, 3, 5\}$, $\{2, 3, 4\}$. Maybe, it contains (though it need not) the vertices of index $\{1, 2, 3, 4, 5\}$ too. There are no vertices of other indices in G. E.g. in G is no 4-vertex (it would have to be in G also a 1-vertex). Let us prove that G does not contain e.g. a 3-vertex $u \in V_1 \cap V_3 \cap V_5$. Suppose on the other hand that such a vertex is present in G. $u \notin V_2 \Rightarrow$ there is a 2-vertex $v \in V_2 \cap V_4$. v is not joined by any edge with the vertex of indices $\{1, 5\}$; surely, the last vertex is present in G. In order to finish the proof a graph $\tilde{G}$ must be constructed, i.e. if there are in G two vertices of the same indices, one of them has to be removed together with all the incident edges. On removing a vertex $\{1, 2, 3, 4, 5\}$ we obtain a graph isomorphic to the first of graphs given in Fig. 10.

b) Just one out of graphs $A_1, \ldots, A_5$ contains 1-vertices; let it be A_1 and the corresponding set of indices be $\{1\}$. We can easily ascertain (in the way similar to that used in case a)) that $A_2, \ldots, A_5$ have to contain 2-vertices. Let be 2-vertices of indices $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}$ in G. Then G must also contain 3-vertices of indices $\{3, 4, 5\}, \{2, 4, 5\}, \{2, 3, 5\}$ and $\{2, 3, 4\}$. Further, we do not exclude that G contains 4-vertices of indices $\{2, 3, 4, 5\}$ and 5-vertices $\{1, 2, 3, 4, 5\}$. The vertex of other indices cannot occur in G (one can easily prove it in the way similar to that used above). $\tilde{G}$ (or a graph obtained on removing the vertex $\{1, 2, 3, 4, 5\}$) is isomorphic to 2-nd or 3-rd of graphs given in Fig. 10. These graphs cannot contain a 5-tuple of independent edges.

There is another possibility of 2-vertices in G: G contains 2-vertices $\{2, 3\}, \{2, 4\}, \{2, 5\}$ (and analogously for $i = 3, 4, 5$ instead of 2). In the same time, G must contain 3-vertices of indices $\{1, 4, 5\}, \{1, 3, 5\}, \{1, 3, 4\}$. Further it may contain 4-vertices $\{2, 3, 4, 5\}$ and finally 5-vertices. Moreover, in G there may be also the vertices of indices $\{1, 2\}$ and $\{3, 4, 5\}$ which even must be in G if the completeness-number of G ought to be 5. $\tilde{G}$ (after removing $\{1, 2, 3, 4, 5\}$ and incident edges) is isomorphic to 2-nd graph in Fig. 10.

We have proved the case b) and the lemma 13.

The proof of the theorem 3 may be now easily deduced from the lemmas 7–13. If G is given ($\omega(G) = 5$), choose arbitrarily its covering $G \sim \{A_1, \ldots, A_5\}$. Then, $k_G = 4$ if and only if the assumptions of the lemma 13 is fulfilled. But it happens if and only if the graph $\tilde{G}$ (or that obtained on removing $\{1, 2, 3, 4, 5\}$) is isomorphic to some of graphs given in Fig. 10.

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Author's address: Žitná 25, Praha 1 (Matematický ústav ČSAV).