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SIMPLE CONTINUITY AND CLIQUISHNESS

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Summary. An example is given which shows that a real simply continuous function is not always cliquish. On the other hand, it is shown that any real function defined on a locally second countable topological space which is simply continuous is cliquish.

Keywords: Simply continuous function, cliquish function.

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In 1969 N. Biswas [1] introduced the concept of simple continuity and investigated some of its properties. The notion of cliquishness was introduced by W. W. Bledsoe [2] for the functions of a real variable taking the values in a metric space. It was generalized by H. P. Thielman [5] for the functions defined on a topological space. In 1973 A. Neubrunnová in her paper [4] showed that there exists a cliquish function which is not simply continuous. She also gave two sufficient conditions for simply continuous functions to be cliquish.

The purpose of the present paper is to investigate the interrelation among the simple continuity and the cliquishness.

Listed below are definitions and theorems that will be used in the paper.

Definition 1. (See [1; Definition 1].) Let X be a topological space. A subset A of X is said to be *simply open* if there exist two subsets B and C of X where B is open and C is nowhere dense in X, such that $B \cup C \subset A \subset \mathrm{Cl}(B \cup C)$.

Remark 1. A subset A of a topological space X is simply open if and only if $\mathrm{Fr}\, A$ (where $\mathrm{Fr}\, A = \mathrm{Cl}\, A - \mathrm{Int}\, A$) is nowhere dense in X (see [1; Remark 1]).

Definition 2. (See [1; Definition 3].) Let X, Y be two topological spaces. A function $f: X \to Y$ is called *simply continuous* if for every open subset G of Y the set $f^{-1}(G)$ is simply open in X.

Definition 3. (See [5].) Let X be a topological space and Y a metric space with a metric d. A function $f: X \to Y$ is said to be *cliquish at a point* $x_0 \in X$ if for each $\varepsilon > 0$ and each neighbourhood $U(x_0)$ of the point x_0 (in X) there exists an open set

$U \subset U(x_0)$, $U \neq \emptyset$ such that for each two points $x_1, x_2 \in U$ the inequality $d(f(x_1), f(x_2)) < \varepsilon$ holds.

A function $f: X \to Y$ is said to be *cliquish* (*on* X) if it is cliquish at each point $x \in X$.

Theorem A. (See [4; Theorem 2.1].) *Let X be a topological space of the second category at each of its points and let Y be a separable metric space. Then a function $f: X \to Y$ which is simply continuous is also cliquish.*

Theorem B. (See [4; Theorem 2.2].) *Let X be a topological space and Y a totally bounded metric space. Then any function $f: X \to Y$ which is simply continuous on X is cliquish on X.*

Lemma 1. (See [3; Theorem 1].) *Let X be a topological space and Y a metric space. Let $f: X \to Y$ be a function. Denote by A_f the set of all points at which the function f is cliquish. Then A_f is closed in X.*

A simply continuous real function is not always cliquish, as the following example shows.

Example 1. Let X be the set of all positive integers and $\mathscr{F}$ an ultrafilter in X, which contains no finite set. Let X be assigned the topology $\tau = \mathscr{F} \cup \{\emptyset\}$. Let Y be the set of all real numbers with the Euclidean metric and let $f: X \to Y$ be defined by $f(x) = x$ for each $x \in X$. Since each subset of X is simply open, the function f is simply continuous. Since each nonempty open subset of X is infinite, the set A_f is empty.

Remark 2. Since the Riemann function is cliquish but not simply continuous (see [4]), Example 1 implies that the simple continuity and the cliquishess are two independent notions.

Definition 4. A *π-base* for a topological space X is a family $\mathscr{A}$ of open subsets of X such that every nonempty open subset of X contains some nonempty $A \in \mathscr{A}$.

Notation. Let X be a topological space. Denote by $\mathscr{B}(X)$ the family of all open subsets of X which have (as topological spaces with the hereditary topology from X) a countable π-base.

Theorem 1. *Let X be a topological space such that $\mathscr{B}(X)$ is a π-base for X. Let Y be a metric space (with a metric d) such that every bounded subset of Y is totally bounded. Then any function $f: X \to Y$ which is simply continuous on X is cliquish on X.*

Proof. By contradiction. Suppose that there is a simply continuous function $f: X \to Y$ such that $A_f \neq X$. By Theorem B we obtain that the set Y is not bounded.

Put $U = X - A_f$. Then by Lemma 1 the set U is open. Choose $y_0 \in Y$. Denote by N the set of all positive integers. Put

$$A_n = \{y \in Y: d(y, y_0) \geqq n\} \quad \text{for each} \quad n \in N .$$

Let $n \in N$. We will show that $f^{-1}(A_n)$ is dense in U. By contradiction, suppose that there is a nonempty open set $G \subset U$ such that $G \cap f^{-1}(A_n) = \emptyset$. Then $f(G) \subset Y - A_n$. Let $g: G \to (Y - A_n)$ be defined as $g(x) = f(x)$ for each $x \in G$. Since g is simply continuous and $Y - A_n$ is totally bounded, by Theorem B the function g is cliquish on G. Thus f is cliquish on G, which contradicts the inclusion $G \subset U = X - A_f$.

Let $B \in \mathscr{B}(X)$ be a nonempty open subset of X such that $B \subset U$. Let $\{G_n\}_{n=1}^{\infty}$ be a countable π-base for B. We will show that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points (in X) such that

$$(1) \qquad x_n \in G_{[(n+1)/2]} \quad \text{for each} \quad n \in N ,$$

$$(2) \qquad d(f(x_i), f(x_j)) \geqq 2 \quad \text{for} \quad i, j \in N , \quad i \neq j$$

(where $[r]$ denotes the integer part of r).

Choose $x_1 \in G_1$. Suppose $x_1, \ldots, x_k$ have been constructed. Choose $m \in N$ such that

$$m > \max \{d(y_0, f(x_i)); \ i = 1, 2, \ldots, k\} + 2 .$$

Since the set $f^{-1}(A_m)$ is dense in U, there exists a point $x_{k+1} \in f^{-1}(A_m) \cap G_{[(k+2)/2]}$. Hence for each $i \in \{1, 2, \ldots, k\}$ we obtain $d(y_0, f(x_i)) + 2 < m \leqq d(y_0, f(x_{k+1})) \leqq d(y_0, f(x_i)) + d(f(x_i), f(x_{k+1}))$. Thus $d(f(x_i), f(x_{k+1})) > 2$. Put

$$B_n = \{y \in Y; \ d(y, f(x_n)) < 1\} \quad \text{for each} \quad n \in N ,$$

$$E = \bigcup_{k=1}^{\infty} B_{2k} \quad \text{and} \quad F = \bigcup_{k=1}^{\infty} B_{2k-1} .$$

Then by (2) we have $E \cap F = \emptyset$. Since for each $k \in N$ we get $x_{2k} \in G_k$, the set $f^{-1}(E)$ is dense in B. Since

$$(\operatorname{Int} f^{-1}(F) \cap B) \cap f^{-1}(E) \subset f^{-1}(F) \cap f^{-1}(E) = \emptyset ,$$

we obtain

$$(3) \qquad \operatorname{Int} f^{-1}(F) \cap B = \emptyset .$$

Since for each $k \in N$ we get $x_{2k-1} \in G_k$, the set $f^{-1}(F)$ is dense in B. Thus $B \subset \operatorname{Cl} f^{-1}(F)$. Therefore (3) yields

$$(4) \qquad B \subset \operatorname{Fr} f^{-1}(F) .$$

Since f is simply continuous and F is open in Y, the set $f^{-1}(F)$ is simply open in X. By Remark 1 the $\operatorname{Fr} f^{-1}(F)$ is nowhere dense in X. Thus by (4) the set B is nowhere dense in X, a contradiction.

Corollary. *Let X be locally second countable topological space. Let Y be a metric space such that every bounded subset of Y is totally bounded. Then any function $f: X \to Y$ which is simply continuous on X is cliquish on X.*

Remark 3. The assumption "$\mathscr{B}(X)$ is a π-base" in Theorem 1 cannot be omitted, as Example 1 shows.

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Souhrn

JEDNODUCHÁ SPOJITOSŤ A KĽUKATOSŤ

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V tejto práci je zostrojený príklad, ktorý ukazuje, že reálna jednoducho spojitá funkcia nemusí byť kľukatá. Okrem toho je tu dokázané, že každá reálna jednoducho spojitá funkcia definovaná na topologickom priestore, ktorý spĺňa lokálne druhú axiómu spočítateľnosti, je kľukatá.

Резюме

ПРОСТАЯ НЕПРЕРЫВНОСТЬ И ИЗВИЛИСТНОСТЬ

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В работе показано, что вещественная просто непрерывная функция не обязана быть извилистой. С другой стороны, если область определения вещественной просто непрерывной функции f локально удовлетворяет второй аксиоме счетности, то функция f извилиста.

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