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ON THE STABILITY OF CHAOTIC FUNCTIONS

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Summary. Extremaly chaotic functions and chaotic functions with a very small scrambled sets are studied. Stability of these types of functions with respect to small perturbations is investigated.

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Recently, some papers have appeared showing that there are extremely chaotic functions, cf. [2], [5], [7], [8], and also papers exhibiting chaotic functions with a very small chaos (see e.g. [1] and [6]). The main aim of this note is to show that both the above quoted types of chaotic functions are unstable with respect to small perturbations.

Recall that a continuous selfmapping f of a compact real interval I is chaotic provided there is an uncountable scrambled set $S \subset I$ such that for any two different points $x, y \in S$ and any periodic point p of f

(1) $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$,

(2) $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$,

(3) $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0$

(cf. [4]), where f^n denotes the n-th iterate of f.

In [2], [5], [8], examples of chaotic functions with scrambled sets of positive Lebesgue measure are given. Now we show that there are also chaotic functions having scrambled sets which are large from the topological point of view.

Theorem 1. *Let $f: [0, 1] \to [0, 1]$ be defined by $f(x) = 1 - |2x - 1|$. Then, under the continuum hypothesis, f has a scrambled set S with the following properties: S is a second Baire category set in every subinterval J of $[0, 1]$ and the outer Lebesgue measure of S is* 1.

Proof of Theorem 1 is a modification of the proof of a theorem from [7]. We use the following lemma (see Lemmas 2–4 from [7]).

Lemma. *There is a G_δ subset $A \subset [0, 1]$ with the following properties:*

1. *For any $x \in A$ and any periodic point p of f, (3) is true.*

2. *For every $x \in A$ there is a G_δ subset $A(x)$ of A with $\mu(A(x)) = 1$ and such that for any $y \in A(x)$, (1) and (2) are true (μ denotes the Lebesgue measure).*

Proof of Theorem 1. Let Ω be the first uncountable ordinal. Let $\{P_\alpha\}_{\alpha<\Omega}$ be a sequence of all nowhere dense perfect subsets of $[0, 1]$ of positive Lebesgue measure, and let $\{Q_\alpha\}_{\alpha<\Omega}$ be a sequence of all G_δ sets of the second Baire category in $[0, 1]$. We use transfinite induction to construct S. Choose $x_0 \in A \cap P_0$, $y_0 \in A(x_0) \cap Q_0$. Next, assume that $\{x_\alpha\}_{\alpha<\beta}$ and $\{y_\alpha\}_{\alpha<\beta}$ are defined and take

$$x_\beta \in \bigcap_{\alpha<\beta} (A(x_\alpha) \cap A(y_\alpha)) \cap P_\beta$$

and

$$y_\beta \in \bigcap_{\alpha<\beta} (A(x_\alpha) \cap (\bigcap_{\alpha<\beta} A(y_\alpha))) \cap Q_\beta .$$

Now consider $S = \bigcap_{\alpha<\Omega} \{x_\alpha\} \cup \{y_\alpha\}$. If $x, y \in S$, $x \neq y$, then either $x \in A(y)$ or $y \in A(x)$ and hence, by Lemma, S is a scrambled set of f. Since S intersects every perfect subset of $[0, 1]$ of positive Lebesgue measure and each G_δ set of the 2-nd Baire category in $[0, 1]$, the set S has full outer Lebesgue measure and is a 2-nd Baire category in every subinterval of $[0, 1]$.

Remark 1. In addition, it is easy to see that for every $x \in S$ and every periodic point p of f, the extremal conditions (3) and (4) from [7] are satisfied.

Remark 2. The function f from the theorem has no scrambled set S of the 2-nd category with the Baire property.

Assume there is a scrambled set S of the 2-nd category with the Baire property. Let J be such an interval that $J \cap S$ is residual in J. Take $m > 1$ such that $2^m . \mu(J) > 2$. Then for some $i \leqq 2^{m+1}$, $[i/2^{m+1}, (i + 2)/2^{m+1}] \subset J$. Put $I_0 = [i/2^{m+1}, (i + 1)/2^{m+1}]$, $I_1 = [(i + 1)/2^{m+1}, (i + 2)/2^{m+1}]$. It is easy to see that $f^{m+1}(I_0) = f^{m+1}(I_1) = [0, 1]$, and since f^{m+1} restricted to I_j, $j = 0, 1$, is linear, both $S_0 = f^{m+1}(S \cap I_0)$ and $S_1 = f^{m+1}(S \cap I_1)$ are residual in $[0, 1]$. Consequently, there is a point $z \in S_0 \cap S_1$, i.e., for suitable $x \in S \cap I_0$, $y \in S \cap I_1$, we have $f^j(x) = f^j(y)$ for $j > m$, contrary to (1).

In the sequel we shall use the following notation: C is the class of all continuous $[0, 1] \to [0, 1]$ functions and $F \subset C$ is the class of chaotic functions (i.e. functions possessing a scrambled set). Moreover, define the following three subclasses F_1, F_2, F_3 of F:

$f \in F_1$ iff f has a scrambled set of positive outer Lebesgue measure,
$f \in F_2$ iff f has a scrambled set of positive Lebesgue measure,
$f \in F_3$ iff f has scrambled set of the 2-nd Baire category in $[0, 1]$.

Theorem 2. *The sets F_2, F_3 (and hence also F_1) are dense in C.*

Proof. We show that in any neighbourhood of a continuous function there is a function from F_2, F_3. Let $g: [0, 1] \to [0, 1]$ be continuous, $\varepsilon > 0$. By the continuity g has a fixed point $p \in [0, 1]$, and there is a $\delta > 0$, $\delta < \varepsilon$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. At least one of the intervals $[p - \delta, p]$, $[p, p + \delta]$ is a subinterval of $[0, 1]$. Assume it is $[p - \delta, p]$. In the other case the proof is similar. Let $f_3: [0, 1] \to [0, 1]$ be the function from Theorem 1, $f_2: [0, 1] \to [0, 1]$ a chaotic function with a scrambled set of positive measure. Let $h: [0, 1] \to [p - \delta, p]$ be defined by $h(x) = \delta(x - 1) + p$. Denote $\varphi_i = h \circ f_i \circ h^{-1}$, $i = 2, 3$, where h^{-1} is the inverse to h. For $x \in [p - \delta, p]$ we have

$$|g(x) - \varphi_i(x)| \leqq |g(x) - p| + |p - \varphi_i(x)| < 2\varepsilon .$$

Hence there is a function $g_i \in C$, $g_i(x) = \varphi_i(x)$ for $x \in [p - \delta, p]$ and $\|g_i - g\| < 2\varepsilon$, $i = 2, 3$. It is easy to see that $g_i \in F_i$, $i = 2, 3$, q.e.d.

Theorem 3. *The set $F \setminus (F_2 \cup F_3)$ is dense in C.*

Proof. Take $f \in C$, $\varepsilon > 0$. By [3] there is a function $g \in C$, g with a 3-cycle $x_1 \to x_2 \to x_3 \to x_1$ and hence chaotic (see [4]), $\|f - g\| < \varepsilon$. Since g is uniformly continuous there is $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Let $0 = y_0 < y_1 < \ldots < y_n = 1$ be such a division of $[0, 1]$ that $|y_{i-1} - y_i| < \delta$ for $i = 1, \ldots, n$ and $x_1, x_2, x_3 \in \{y_i\}_{i=1}^n$. Denote $I_i = [y_{i-1}, y_i]$. Let $h_i: I_i \to I_i$, $i = 1, \ldots, n$, be a Cantor-type function which is continuous, non decreasing, and there is a nowhere dense perfect set $A_i \subset I_i$ with $\mu(A_i) = 0$ and such that h_i is constant on every interval J contiguous to A_i. Put $A = \bigcup_{i=1}^{n} A_i$ and let $h: [0, 1] \to [0, 1]$ be defined by $h(x) = h_i(x)$ for $x \in I_i$. Let $f^*(x) = g(h(x))$ for $x \in [0, 1]$. Since every y_i is a fixed point of h, we have $|x - h(x)| < \delta$ for every x, and consequently

$$\|f - f^*\| \leqq \|f - g\| + \|g - f^*\| < 2\varepsilon .$$

Clearly f^* has a 3-cycle. On the other hand, every scrambled set S of f^* contains only a denumerable set of points lying outside of A, every interval contiguous to A contains at most one point from S, and consequently, S is nowhere dense and $\mu(S) = 0$, q.e.d.

Remark 3. In connection with the above results recall the following problem presented by J. Smítal at the First Czechoslovak Summer School on Dynamical systems (June 1984): Is any of the sets F_1, F_2, F_3 or $F \setminus (F_1 \cup F_3)$ a first Baire category set?

Remark 4. In connection with Theorem 1 the following problem seems to be interesting: Does there exist a chaotic function $[0, 1] \to [0, 1]$ possessing a scrambled set which is residual in a certain subinterval $J \subset [0, 1]$? Note that in [5], [6], [7] the chaotic functions have only scrambled sets of the 1-st category in $[0, 1]$.

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Súhrn

O STABILITE CHAOTICKÝCH FUNKCIÍ

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V práci sa uvažujú extrémne chaotické funkcie a chaotické funkcie s veľmi malou chaotickou množinou. Skúma sa stabilita týchto funkcií vzhľadom na malé perturbácie.

Резюме

ОБ УСТОЙЧИВОСТИ ХАОТИЧЕСКИХ ФУНКЦИЙ

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В работе исследуются экстремально хаотические функции и функции с очень малым хаотическим множеством. Исследуется устойчивость этих типов функций относительно малых возмущений.

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