Svatopluk Poljak; Daniel Turzík

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ON A FACET OF THE BALANCED SUBGRAPH POLYTOPE

SVATOPLUK POLJAK, DANIEL TURZÍK, Praha

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Summary. A class of graphs called wreaths is introduced. It is shown that the wreaths induce some new facets of the bipartite subgraph polytope studied in [2]. Similar results hold also for signed graphs. The relation between associated polytopes is discussed.

Keywords: Signed graph, balanced subgraph, facet inequality, wreath.

AMS Classification: 90C27, 05C99.

1. INTRODUCTION

A *signed* graph is an unoriented graph each edge of which is signed by + or − sign. The edges are called positive or negative according to their signs. A signed graph is called *balanced* if it does not contain a cycle with an odd number of negative edges. A simple equivalent condition for a graph to be balanced is the existence of a partition of vertices into two classes so that the positive edges have both their endpoints in the same class and the negative edges have their endpoints in different classes. The signed graphs were introduced in [3].

If $G = (V, E)$ is a signed graph and $F \subset E$ is a subset of edges, then the $0-1$ vector $x^F \in \mathbb{R}^E$ with $x^F_e = 1$ if $e \in F$ and $x^F_e = 0$ if $e \notin F$ is called the *incidence* vector of F. The convex hull $P_{BL}(G)$ of the incidence vectors of the edge sets of all balanced subgraphs is called the *balanced subgraph polytope*, i.e.

$$P_{BL}(G) = \operatorname{conv}\{x^F \in \mathbb{R}^E: (V, F) \text{ is a balanced subgraph of } G\}\,.$$

An inequality $ax \leqq \beta$ is called a *facet* of $P_{BL}(G)$ if it is a valid inequality for all $x \in P_{BL}(G)$ and $P_{BL}(G) \cap \{x: ax = \beta\}$ is a maximal proper face of $P_{BL}(G)$. We assume that the reader is familiar with the basic concepts of the polyhedral approach to the combinatorial problems, see e.g. [1].

The balanced subgraph polytope P_{BL} is closely related to the bipartite subgraph polytope P_B, which is the convex hull of the incidence vectors of the bipartite subgraphs of a (non signed) graph. We discuss the relation between P_B and P_{BL} in the next section.

If $G = (V, E)$ is a graph (a signed graph) and $e \in E$, then the inequalities $0 \leqq x_e$ and $x_e \leqq 1$ are facets of $P_B(G)$ (of $P_{BL}(G)$). These facets are called trivial. Some nontrivial facets and their constructions are given in [2]. The aim of this paper is to show some new facets of P_{BL} and at the same time some new facets of P_B. In spite of the fact that all results can be stated for ordinary graphs we use signed graphs because the formulation of the results and their proofs is simpler.

2. RESULTS

Let n, r be integers satisfying $n > 2r > 3$. We introduce three types of signed graphs $W_{n,r}$, $W_{n+,r}$, and $W_{n,r+}$ which are defined on the same underlying graph and differ only in signs of edges. Each of these signed graphs is called a *wreath*. The vertex set of the underlying graph is $V = \{0, 1, \ldots, n-1\}$, the edge set E is the union of two sets E_0 and E_I where

$$E_0 = \{(i, i+1) \colon i = 0, 1, \ldots, n-1\}, \quad \text{and}$$
$$E_I = \{(i, i+r) \colon i = 0, 1, \ldots, n-1\}.$$

The numbers of vertices are always taken modulo n. We call E_0 and E_I the set of the *outer* and *inner* edges, respectively.

The signs of edges are defined in the individual cases as follows:

$W_{n,r}$ has all edges negative,

$W_{n+,r}$ has positive outer and negative inner edges, and

$W_{n,r+}$ has negative outer and positive inner edges.

Fig. 1 shows a wreath $W_{9,2}$.

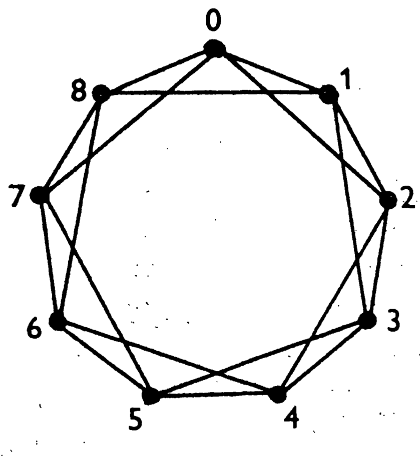

Fig. 1

Let us note that E_I forms a cycle in a wreath iff $g.c.d.(n, r) = 1$.

Theorem 1. *Let $n = kr + j$, $0 \leq j < r < n/2$ and let W be one of the following* *wreaths* with the specified parameters k, r and j:

(i) $W = W_{n,r}$, r even and $k + j$ odd,

(ii) $W = W_{n+,r}$, k odd,

(iii) $W = W_{n,r+}$, r and j odd.

If W is a subgraph of a signed graph G, then the inequality

$$\sum_{e \in E_1} x_e + j \sum_{e \in E_0} x_e \leqq (j + 1)\, n - jk - r \tag{1}$$

is valid for the polytope $P_{BL}(G)$, and it is a facet of $P_{BL}(G)$ if and only if $g.c.d.(n, r) = 1$.

Let us denote by rank G the maximum number of edges of a balanced subgraph of a signed graph G.

Corollary 1. *Let W be a wreath from Theorem* 1. *Then*

$$rank\ W = 2n - k - r + j - 1 \tag{2}$$

provided $k \geqq j > 0$.

Theorem 2. *Let* $n = kr + j$, $0 \leqq j < r < n/2$, *and let W be one of the following wreaths with specified parameters*:

(i) $W = W_{n,r}$, *r and $j + k$ even*,

(ii) $W = W_{n+,r}$, *k even*,

(iii) $W = W_{n,r+}$, *r odd and j even.*

Then rank $W = 2n - k - j$ *provided* $r \leqq k + j + 1$.

Remark. Corollary 1 and Theorem 2 provide rank formulas for some instances of wreaths. However, the approach of this paper does not seem to be applicable to a general wreath. We give a different method for establishing the rank formula in [4].

At the end of this section we give two propositions which show the connection between P_B and P_{BL}.

Proposition 1. *Let G be a signed graph. Let G' be the graph obtained from G by changing the signs of edges incident to some fixed vertex v of G. Then* $P_{BL}(G) = P_{BL}(G')$.

Proof. It follows immediately from the fact that the balanced subgraphs of G and G' are the same.

Proposition 2. *Let G be a signed graph, let*

$$\sum_{e \in E(G)} a_e x_e \leqq \beta \tag{3}$$

be an inequality and $e_0 = (u, v)$ *a positive edge of G. Let us define a signed graph G'*

and the inequality

(4) $$\sum_{e \in E(G')} a_e x_e \leqq \beta'$$

as follows: The graph G' is obtained from G by adding a new vertex w to $V(G)$ and by replacing the edge e_0 by two negative edges $e_1 = (u, w)$ and $e_2 = (w, v)$. Let $a_{e_1} = a_{e_2} = a_{e_0}$ and $\beta' = \beta + a_{e_0}$. Then the inequality (3) is a nontrivial facet of $P_{BL}(G)$ if and only if the inequality (4) is a nontrivial facet of $P_{BL}(G')$.

Proof. By a modification of the proof of Theorem 4.4 of [2] for signed graphs.

3. PROOFS

Proof of Theorem 1. We show that (1) is a valid inequality for $P_{BL}(G)$. To prove it we need to consider two sets $\mathscr{C}_1$ and $\mathscr{C}_2$ of cycles of W. The set $\mathscr{C}_1$ contains n cycles of length r, each of the form

$$\{(i, i+1), (i+1, i+2), \ldots, (i+r-1, i+r)\} \cup \{(i+r, i)\},\ i = 0, \ldots, n-1 .$$

The set $\mathscr{C}_2$ contains n cycles of length $k + j$, each of the form

$$\{(i, i+r), (i+r, i+2r), \ldots, (i+(k-1)r, i+kr)\} \cup$$
$$\cup \{(i+kr, i+kr+1), (i+kr+1, i+kr+2), \ldots, (i+n-1, i)\},$$
$$i = 0, \ldots, n-1 .$$

Let B be a maximal balanced subgraph of W (i.e., no further edge of W can be added to B). As it is more convenient for us to count the edges in the complement of B, denote $u^B = |E_1 - B|$ and $v^B = |E_0 - B|$. As the cycles from $\mathscr{C}_1$ and $\mathscr{C}_2$ are not balanced, u^B and v^B must satisfy

(5) $$u^B + r \,.\, v^B \geqq n ,$$

(6) $$k \,.\, u^B + j \,.\, v^B \geqq n .$$

We claim that u^B and v^B satisfy also the inequality

(7) $$u^B + j \,.\, v^B \geqq kj + r .$$

If (7) does not hold, then

(8) $$u^B \leqq kj + r - j \,.\, v^B - 1 .$$

Substituting (8) into (5) we get

$$j(k-1)\, v^B \leqq k^2 j - k - j .$$

This inequality is false for $j = 0$, and it gives

(9) $$v^B \leqq k$$

for $j \neq 0$ as v^B is an integer. Similarly, substituting (8) into (6), we get

$$(r - j)\, v^B \geqq kr - kj - r + j + 1\,,$$

which yields

$$v^B \geqq k\,. \tag{10}$$

Thus, there is no integral solution of (5), (6) and (8) for $j = 0$, and for $j \neq 0$ the integral solutions u^B, v^B must satisfy $v^B = k$ due to (9) and (10). We show that there is no maximal balanced subgraph B like that. In the signed graphs $W_{n,r}$ and $W_{n,r+}$, the set E_0 of outer edges forms a cycle of negative edges. As B is maximal, the intersection $|B \cap E_0|$ must be even. However,

$$|B \cap E_0| = n - v^B = kr + j - k = (r - 1)\,k + j$$

which is odd for all instances of parameters in (i) and (iii). Let $W = W_{n+,r}$. As B is maximal and E_0 is a cycle of positive edges, the set-difference $|E_0 - B|$ must be even. But $|E_0 - B| = v^B = k$ which is odd by (ii). As we arrived at a contradiction in all cases, u^B and v^B satisfy (7). As the both coefficients at u^B and v^B in (7) are positive, the inequality (7) is valid for all not necessarily maximal balanced subgraphs of W. Using the substitution $u^B = n - \sum_{e \in E_I} x_e^B$ and $v^B = n - \sum_{e \in E_0} x_e^B$ in (7), we get that (1) is satisfied by all incidence vectors $x^B = (x_e^B)$ of balanced subgraphs B of W. This proves that (1) is valid for $P_{BL}(G)$.

2. Let $g.c.d.(n, r) = 1$. We show that (1) is a facet of $P_{BL}(G)$. In order to prove it we need sufficiently many balanced subgraphs of G satisfying (1) with the equality.

Let F_0 be the subgraph of W arising from W by deleting $k + 1$ edges $(n - 1, 0)$, $(r - 1, r), \ldots, (kr - 1, kr)$ from the set E_0, and $r - j$ edges $(kr - 1, r - j - 1)$, $(kr - 2, r - j - 2), \ldots, (kr - (r - j), 0)$ from the set E_I. Let H_0 be the subgraph of W arising from W by deleting $k - 1$ edges $(0, 1)$, $(r, r + 1), \ldots, ((k - 2)\, r, (k - 2)\, r + 1)$ from E_0, and $r + j$ edges $((k - 2)\, r + 1, (k - 1)\, r + 1)$, $((k - 2)\, r + 2, (k - 1)\, r + 2), \ldots, (n - r, 0)$ from the set E_I. Figures 2a and 2b show the graphs F_0 and H_0, subgraphs of $W_{9,2}$.

Further, let F_0', H_0' be defined by

$$F_0' = (F_0 - \{(0, 1), (0, r)\}) \cup \{(n - 1, 0), (0, n - r)\} \quad \text{and}$$
$$H_0' = (H_0 - \{(0, n - 1), (0, r)\}) \cup \{(0, 1), (0, n - r)\}\,.$$

(F_0' and H_0' are obtained from F_0 and H_0 by moving the vertex 0 into the opposite partition class.)

Finally, let F_k, F_k', H_k and H_k', $k = 0, 1, \ldots, n - 1$ be the graphs arising from F_0, F_0', H_0 and H_0', respectively, by their rotation, i.e.

$$(i + k, j + k) \in X_k \quad \text{iff} \quad (i, j) \in X_0\,,$$

where X stands for F, F', H and H'. One can check that all the subgraphs defined are balanced and their incidence vectors satisfy (1) with equality. Let

$$a \, . \, x \leqq \beta \tag{11}$$

be a facet of $P_{BL}(G)$ such that $a \, . \, x = \beta$ for all $x \in P_{BL}(G)$ satisfying (1) with equality. We claim that there are α, α' such that $a_e = \alpha$ for all $e \in E_I$, $a_e = \alpha'$ for all $e \in E_0$, and $a_e = 0$ for all $e \in E(G) - (E_0 \cup E_I)$. As x^{F_0}, $x^{F'_0}$, x^{H_0} and $x^{H'_0}$ satisfy (1), and hence also (11) with equality, we get

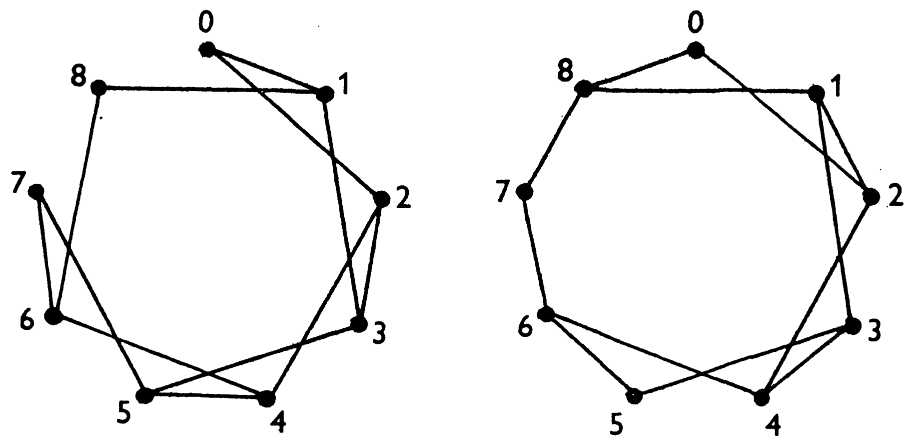

Fig. 2a, b

$$0 = a \, . \, x^{F_0} - a \, . \, x^{F'_0} = a_{(0,1)} + a_{(0,r)} - a_{(0,n-1)} - a_{(0,n-r)}$$

and

$$0 = a \, . \, x^{H_0} - a \, . \, x^{H'_0} = a_{(0,n-1)} + a_{(0,r)} - a_{(0,1)} - a_{(0,n-r)} \, ,$$

which gives

$$a_{(0,1)} = a_{(0,n-1)} \quad \text{and} \quad a_{(0,r)} = a_{(0,n-r)} \, .$$

As both E_0 and E_I are cycles, using graphs F_k and H_k we derive that $a_e = a_{e'}$ for all $e, e' \in E_I$, and $a_e = a_{e'}$ for all $e, e' \in E_0$. Using again the fact that x^{F_0} and x^{H_0} satisfy (11) with equality, and summing the outer and inner edges separately, we get

$$\alpha(n - k - 1) + \alpha'(n - r + j) = \beta \quad \text{and}$$
$$\alpha(n - k + 1) + \alpha'(n - r - j) = \beta \, .$$

This gives $\alpha' = j\alpha$.

If $e \in E(G) - (E_0 \cup E_I)$, then there is at least one k such that $F_k \cup \{e\}$ is a balanced subgraph of G. As x^{F_k} satisfies (11) with equality, we have $a_e = 0$. Thus, (1) is a positive multiple of (11) and hence a facet of $P_{BL}(G)$.

3. To complete the proof let us suppose that $g.c.d.(n, r) = d$, $d > 1$. We show that in this case (1) is not a facet of $P_{BL}(G)$. In particular, we show that (1) is the sum of d facet inequalities of $P_{BL}(G)$. Let W^i, $i = 0, 1, \ldots, d - 1$, be a subgraph of W

which contains all outer edges of W but only n/d inner edges of W of the form $(i, i + r), (i + r, i + 2r), \ldots, (i - r, i)$. Denote this set of edges by E^i_{I}. Let us note that the inner edges E^i_{I} of W^i form a cycle of length n/d. We show that

$$(12) \qquad d \cdot \sum_{e \in E^i_{\mathrm{I}}} x_e + j \cdot \sum_{e \in E_0} x_e \leqq (j + 1)\, r - jk - r$$

is a facet of W. Then the inequality (1) is the sum of d inequalities (12) for $i = 0, 1, \ldots, d - 1$. In the graph W^i there are n/d paths of length d formed by outer edges such that each inner vertex of this path has degree 2. By using Propositions 1 and 2 each of these paths can be replaced by a single edge with the positive sign if the number of negative edges in the path is even, and with the negative sign in the opposite case. Thus we obtain another wreath $\bar{W}$ with parameters $\bar{n} = n/d$, $\bar{r} = r/d$ and $\bar{j} = j/d$. Clearly, $\bar{n} = k \cdot \bar{r} + \bar{j}$. If the wreath W is of type (i) and d is even then $\bar{W}$ is of type (i). If W is of type (i) and d is odd then $\bar{W}$ is of type (ii). If W is of type (ii), then $\bar{W}$ is of type (ii), too. If W is of type (iii), then d is odd as $g.c.d.(n, r) = g.c.d.(j, r) = d$ and hence $\bar{W}$ is of type (iii). Denote by $\bar{E}_0$ and $\bar{E}_{\mathrm{I}}$ the set of the outer and inner edges of $\bar{W}$, respectively. Then by the second part of the proof,

$$\sum_{e \in \bar{E}_{\mathrm{I}}} x_e + \bar{j} \sum_{e \in \bar{E}_0} x_e \leqq (\bar{j} + 1) \cdot \bar{n} - \bar{j}k - \bar{r}$$

is a facet of $P_{BL}(\bar{W})$. By Proposition 2 this gives that (12) is a facet of $P_{BL}(W)$.

Proof of Corollary 1. Let B be a balanced subgraph of W. Multiplying inequality (7) by $(k - j)$, inequality (6) by $(j - 1)$, and summing them up, we get

$$j(k - 1)\,(u^B + v^B) \geqq j(k - 1)\,(k + 1 + r - j)\,.$$

This gives that rank $W \leqq 2n - k - r + j - 1$. The converse inequality is proved by the balanced subgraph F_0 whose number of edges is the right hand side of (2).

Proof of Theorem 2. Let W be one of the signed graphs with specified parameters k, r, and j. Let $\mathscr{C}_3$ be the set of n cycles of length $k + 1 + r - j$, each of the form

$$\{(i, i + r), (i + r, i + 2r), \ldots, (i + (k - 1)\, r, i + kr), (i + kr, i + (k + 1)\, r)\} \cup$$
$$\cup \{(i + (k + 1)\, r, i + (k + 1)\, r - 1)\,,$$
$$(i + (k + 1) - 1, i + (k + 1) - 2), \ldots, (i + 1, i)\}\,,$$

$i = 0, 1, \ldots, n - 1$. Let B be a balanced subgraph of W. As the cycles from $\mathscr{C}_3$ are not balanced, we have

$$(13) \qquad (k + 1)\, u^B + (r - j)\, v^B \geqq n\,.$$

Multiplying inequality (5) by $(k + 1 - r + j)$, inequality (13) by $(r - 1)$, and summing them up, we get

$$(kr + j)\,(u^B + v^B) \geqq (kr + j)\,(k + j)\,.$$

This gives rank $W \leqq 2n - k - j$. The opposite inequality is proved by constructing the maximal balanced subgraph B which arises when deleting the following edges from W:

$$(0, 1), (r, r + 1), \ldots, (kr, kr + 1) \text{ from } E_0, \quad \text{and}$$

$$(kr + 1, (k + 1) r + 1), (kr + 2, (k + 1) r + 2), \ldots, (n - r, 0) \quad \text{from} \quad E_1 .$$

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Souhrn

O STĚNĚ MNOHOSTĚNU BALANCOVANÉHO PODGRAFU

SVATOPLUK POLJAK, DANIEL TURZÍK

V práci je zaveden věnec $W_{n,r}$ jako graf, který je tvořen jedním cyklem délky n a všemi jeho r-chordami. Pro některé hodnoty n a r je nalezen maximální bipartitní podgraf grafu $W_{n,r}$. Jako aplikace jsou odvozeny nerovnosti popisující některé maximální stěny mnohostěnu bipartitních podgrafů a mnohostěnu balancovaných podgrafů.

Резюме

О ВЕНКЕ МНОГОГРАННИКА СБАЛАНСИРОВАННОГО ПОДГРАФА

SVATOPLUK POLJAK, DANIEL TURZÍK

В работе введен венок $W_{n,r}$ как граф состоящий из одного цикла длины n и всех его r-хорд. Для некоторых значений n и r описан максимальный двудольный подграф графа $W_{n,r}$ и в качестве следствия этого описания выведены неравенства для некоторых максимальных граней многогранника двудольных подграфов и многогранника сбалансированных подграфов.

Authors' addresses: S. Poljak, MFF UK, Malostranské nám. 25, 118 00 Praha 1, D. Turzík, VŠCHT, Suchbátarova 5, 166 28 Praha 6.