SECOND PHASE MATRIX OF DIFFERENTIAL SYSTEMS $Y'' + P(x)\,Y = 0$

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Summary. The concept of the second phase matrix of differential systems $Y'' + P(x)\,Y = 0$ is introduced, where $P(x)$ is a symmetric positive definite $n \times n$ matrix. Basic properties of this matrix and relationships between the first and the second phase matrix of the same differential system are investigated.

Keywords: Selfadjoint differential systems, first phase matrix, second phase matrix, trigonometric matrices, associated system.

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1. INTRODUCTION

Let $u(x)$, $v(x)$ be linearly independent solutions of a scalar differential equation of the second order

$$y'' + p(x)\,y = 0\,, \tag{1.1}$$

where $p(x)$ is a positive continuous real function. This function is called the carrier of equation (1.1). It is known, see [3], that there exist real functions $\alpha(x)$, $\beta(x)$ with $\alpha'(x) \neq 0$, $\beta'(x) \neq 0$, and a real constant k such that

$$\begin{aligned} u(x) &= k\,.\,(|\alpha'(x)|)^{-1/2} \sin \alpha(x)\,, \\ v(x) &= k\,.\,(|\alpha'(x)|)^{-1/2} \cos \alpha(x)\,, \end{aligned} \tag{1.2}$$

$$\begin{aligned} u'(x) &= k\,.\,(|\beta'(x)|\; p(x))^{-1/2} \sin \beta(x)\,, \\ v'(x) &= k\,.\,(|\beta'(x)|\; p(x))^{-1/2} \cos \beta(x)\,. \end{aligned} \tag{1.3}$$

These functions were introduced by O. Borůvka, see [3], and they are called the first phase function and the second phase function of (1.1), respectively.

Introducing the concept of the first phase matrix of the matrix differential system

$$Y'' + P(x)\,Y = 0\,, \tag{1.4}$$

where $P(x)$ is a symmetric continuous $n \times n$ matrix, it was shown in [4] that the formulae (1.2) can be extended to the matrix system (1.4).

In this paper we shall introduce the concept of the second phase matrix of system (1.4), and by means of this concept we shall extend the relations (1.3) to system (1.4). We shall also study relationships between the first and the second phase matrices of (1.4) and some related topics. Throughout the paper the matrix $P(x)$ is supposed to be positive definite.

2. NOTATION AND PRELIMINARY RESULTS

The following notation is used. $C^m(I)$ denotes the space of m-times continuously differentiable real functions on an interval I. If $A(x)$ is a matrix of functions, we write $A(x) \in C^m(I)$ if each entry of $A(x)$ belongs to $C^m(I)$. If A is a matrix of any dimension, A^T and A^* denote the transpose and the conjugate transpose of A, E and 0 denote the identity and the zero matrix of any dimension, respectively. The system (1.4) is investigated on an interval I of an arbitrary kind.

First we recall some facts concerning the properties of the phase functions of equation (1.1). Let $p(x) \in C^2(I)$ and let $\alpha(x)$, $\beta(x)$ be a first and a second phase functions of (1.1). Then

$$(2.1) \qquad \begin{aligned} p(x) &= ((|\alpha'(x)|)^{-1/2})'' (|\alpha'(x)|)^{1/2} + \alpha'^2(x), \\ \hat{p}(x) &= ((|\beta'(x)|)^{-1/2})'' (|\beta'(x)|)^{1/2} + \beta'^2(x), \end{aligned}$$

where

$$(2.2) \qquad \hat{p}(x) = p(x) + ((p(x))^{-1/2})'' (p(x))^{1/2}$$

is the carrier of the so called associated equation

$$(2.3) \qquad y'' + \hat{p}(x)\, y = 0 .$$

If $y(x)$ is a solution of (1.1) then $y'(x)$ is the solution of $(p^{-1}(x)\, y')' + y = 0$ and $(p(x))^{-1/2}\, y'(x)$ is the solution of the associated equation (2.3). From (1.2) and (1.3) it follows that the first and the second phase functions of (1.1) can be defined as continuous functions satisfying

$$\operatorname{tg} \alpha(x) = \frac{u(x)}{v(x)}, \quad \operatorname{tg} \beta(x) = \frac{u'(x)}{v'(x)}.$$

In this case we say that the phase functions $\alpha(x)$, $\beta(x)$ are determined by the pair of linearly independent solutions $u(x)$, $v(x)$. If $\alpha(x)$, $\beta(x)$ are determined by the same pair of solutions $u(x)$, $v(x)$ then there exists an integer k such that $k\pi < \alpha(x) - \beta(x) < (k + 1)\,\pi$.

Now, let $Q(x)$ be a symmetric positive definite $n \times n$ matrix and consider the $2n$-dimensional matrix differential system of the first order

$$(2.4) \qquad S' = Q(x)\, C, \quad C' = -Q(x)\, S$$

with the initial condition $S(a) = M$, $C(a) = N$, where M, N are constant $n \times n$ matrices satisfying $M^T M + N^T N = E$, $M^T N = N^T M$.

Then the following identities hold:

$$\begin{aligned} &S^T(x)\,S(x) + C^T(x)\,C(x) = E\,, \quad S^T(x)\,C(x) = C^T(x)\,S(x)\,, \\ &S(x)\,S^T(x) + C(x)\,C^T(x) = E\,; \quad S(x)\,C^T(x) = C(x)\,S^T(x)\,, \end{aligned} \tag{2.5}$$

see [1]. For a more detailed study of these so called trigonometric matrices see [7], [8], [9]. At all points where the matrix $C(x)$ is nonsingular let us define the matrix $T(x) = C^{-1}(x)\,S(x)$. To emphasize that $\{S(x), C(x)\}$ is a solution of (2.4) with the matrix $Q(x)$, we shall sometimes write $S(x, Q)$, $C(x, Q)$ instead of $S(x)$, $C(x)$ and also $T(x, Q)$ instead of $T(x)$.

At the end of this section we shall recall some properties of solutions of (1.4). Let $U(x)$, $V(x)$ be two solutions of (1.4). Then $U^{T\prime}(x)\,V(x) - U^T(x)\,V'(x) = K$, where K is a constant $n \times n$ matrix. A solution $U(x)$ of (1.1) is said to be isotropic whenever $U^{T\prime}(x)\,U(x) - U^T(x)\,U'(x) = 0$.

Let $U(x)$, $V(x)$ be isotropic solutions of (1.4) for which

$$U^{T\prime}(x)\,V(x) - U^T(x)\,V'(x) = E\,. \tag{2.6}$$

There exists an $n \times n$ matrix $A(x) \in C^3(I)$, $A'(x)$ positive definite, such that $U(x)$ and $V(x)$ can be expressed in the form

$$U(x) = R(x)\,S(x)\,, \quad V(x) = R(x)\,C(x)\,,$$

where $\{S(x), C(x)\}$ is a solution of

$$S' = A'(x)\,C\,, \quad C' = -A'(x)\,S$$

for which (2.5) holds, and $R(x)$ is a nonsingular $n \times n$ matrix satisfying

$$R(x)\,R^T(x) = U(x)\,U^T(x) + V(x)\,V^T(x)\,, \tag{2.7$_1$}$$

$$(R^T(x)\,R(x))^{-1} = A'(x)\,, \tag{2.7$_2$}$$

$$R^{T\prime}(x)\,R(x) - R^T(x)\,R'(x) = 0 \tag{2.7$_3$}$$

and

$$P(x) = -R''(x)\,R^{-1}(x) + (R(x)\,R^T(x))^{-2}\,,$$

see [4].

The matrix $A(x)$ is called the *first phase matrix* of (1.4). Two matrices $A_1(x)$, $A_2(x) \in C^3(I)$ such that $A_1'(x)$ and $A_2'(x)$ are positive definite, are the first phase matrices of the same system (1.4) if and only if there exist constant $n \times n$ matrices K, L, M, N for which

$$K^T L - L^T K = 0\,, \quad M^T N - N^T M = 0\,, \quad K^T N - L^T M = E\,, \tag{2.8}$$

such that at all points where the following expression is defined we have

$$T(x, A_2') = (M + T(x, A_1')\,N)^{-1}\,(K + T(x, A_1')\,L)\,, \tag{2.9}$$

see [5].

3. SECOND PHASE MATRIX

In this section we shall introduce the concept of the second phase matrix of (1.4), and we shall establish some properties of this matrix.

Theorem 1. *Let $U(x)$, $V(x)$ be isotropic solutions of* (1.4) *for which* (2.6) *holds. Then there exist a symmetric positive definite $n \times n$ matrix $Q(x)$ and a nonsingular $n \times n$ matrix $H(x)$ satisfying*

$$H(x)\,H^{\mathrm{T}}(x) = U'(x)\,U^{\mathrm{T}\prime}(x) + V'(x)\,V^{\mathrm{T}\prime}(x)\,, \tag{3.1$_1$}$$

$$(H^{\mathrm{T}}(x)\,P^{-1}(x)\,H(x))^{-1} = Q(x)\,, \tag{3.1$_2$}$$

$$H^{\mathrm{T}\prime}(x)\,P^{-1}(x)\,H(x) - H^{\mathrm{T}}(x)\,P^{-1}(x)\,H'(x) = 0\,, \tag{3.1$_3$}$$

such that

$$U'(x) = H(x)\,S(x)\,,\quad V'(x) = H(x)\,C(x)\,, \tag{3.2}$$

where $\{S(x), C(x)\}$ is a solution of (2.4) *satisfying* (2.5).

Proof. From the fact that $U(x)$, $V(x)$ are isotropic and (2.6) holds we conclude

$$\begin{aligned} U'(x)\,V^{\mathrm{T}}(x) - V'(x)\,U^{\mathrm{T}}(x) &= E\,,\\ V'(x)\,U^{\mathrm{T}\prime}(x) - U'(x)\,V^{\mathrm{T}\prime}(x) &= 0\,,\\ U(x)\,V^{\mathrm{T}}(x) - V(x)\,U^{\mathrm{T}}(x) &= 0\,, \end{aligned} \tag{3.3}$$

see [4]. Denote $F(x) = U'(x)\,U^{\mathrm{T}\prime}(x) + V'(x)\,V^{\mathrm{T}\prime}(x)$. As $Y_1(x) = U'(x)$, $Y_2(x) = V'(x)$ are solutions of the system

$$(P^{-1}(x)\,Y')' + Y = 0\,, \tag{3.4}$$

for which $Y_i^{\mathrm{T}\prime}(x)\,P^{-1}(x)\,Y_i(x) = Y_i^{\mathrm{T}}(x)\,P^{-1}(x)\,Y_i'(x)$, $i = 1, 2$, and $Y_1^{\mathrm{T}\prime}(x)\,P^{-1}(x)\,.\,Y_2(x) - Y_1^{\mathrm{T}}(x)\,P^{-1}(x)\,Y_2'(x) = -E$, it can be proved by the same method as in [4] that the matrix $F(x)$ is positive definite. Let $D(x)$ be the symmetric positive definite $n \times n$ matrix for which $D^2(x) = F(x)$. Denote $K(x) = D'(x)\,P^{-1}(x)\,D(x) - D(x)\,P^{-1}(x)\,D'(x)$, $L(x) = K(x)\,(D(x)\,P^{-1}(x)\,D(x))^{-1} - (D(x)\,P^{-1}(x)\,.\,D(x))^{-1}\,K(x)$. Then the matrices $L(x)$ and $K(x)$ are obviously symmetric and antisymmetric, respectively, i.e. $L(x) = L^{\mathrm{T}}(x)$ and $K^{\mathrm{T}}(x) = -K(x)$. If $M(x)$ is the solution of the matrix system

$$(D(x)\,P^{-1}(x)\,D(x))^{-1}\,M + M(D(x)\,P^{-1}(x)\,D(x))^{-1} = L(x)$$

then $M(x)$ is symmetric, see [2]. Let $T(x)$ be the fundamental matrix of

$$T' = \tfrac{1}{2}(D(x)\,P^{-1}(x)\,D(x))^{-1}\,(K(x) + M(x))\,T$$

for which $T(a) = E$, $a \in I$. As $(DP^{-1}D)^{-1}\,(K + M) + [(DP^{-1}D)^{-1}\,(K + M)]^{\mathrm{T}} = (DP^{-1}D)^{-1}\,K + (DP^{-1}D)^{-1}\,M + (-K + M)\,(DP^{-1}D)^{-1} = (DP^{-1}D)^{-1}\,M + M(DP^{-1}D)^{-1} - L = 0$, the matrix $T(x)$ is orthogonal on I (i.e. $T^{-1}(x) = T^{\mathrm{T}}(x)$).

If we set

$$H(x) = D(x)\,T(x)\,,$$

we can verify by a direct computation that $H(x)\,H^{\mathrm{T}}(x) = U'(x)\,U^{\mathrm{T}\prime}(x) + V'(x)\,V^{\mathrm{T}\prime}(x)$ and $H^{\mathrm{T}\prime}(x)\,P^{-1}(x)\,H(x) - H^{\mathrm{T}}(x)\,P^{-1}(x)\,H'(x) = 0$. Now, let $Q(x) = (H^{\mathrm{T}}(x)\,P^{-1}(x)\,.$ $.\,H(x))^{-1}$, $S(x) = H^{-1}(x)\,U'(x)$, $C(x) = H^{-1}(x)\,V'(x)$. Then using $(3.1)_1$ and (3.3) we have $S(x)\,S^{\mathrm{T}}(x) + C(x)\,C^{\mathrm{T}}(x) = E$, $S(x)\,C^{\mathrm{T}}(x) = C(x)\,S^{\mathrm{T}}(x)$ and this implies $S^{\mathrm{T}}(x)\,S(x) + C^{\mathrm{T}}(x)\,C(x) = E$, $C^{\mathrm{T}}(x)\,S(x) = S^{\mathrm{T}}(x)\,C(x)$, see [1]. Further, $(3.1)_3$ yields $HH^{\mathrm{T}}P^{-1}H'H^{\mathrm{T}} = HH^{\mathrm{T}\prime}P^{-1}HH^{\mathrm{T}}$ and using (2.6) and (3.3) we can directly verify that $HH^{\mathrm{T}}(UU^{\mathrm{T}\prime} + VV^{\mathrm{T}\prime}) = (U'U^{\mathrm{T}} + V'V^{\mathrm{T}})\,HH^{\mathrm{T}}$. Denote $X_1 = P(UU^{\mathrm{T}\prime} +$ $+ VV^{\mathrm{T}\prime})$, $Y_1 = (U'U^{\mathrm{T}} + V'V^{\mathrm{T}})\,P$, $X_2 = H'H^{\mathrm{T}}$, $Y_2 = HH^{\mathrm{T}\prime}$. Then

$$\begin{aligned} HH^{\mathrm{T}}P^{-1}X_1 - Y_1P^{-1}HH^{\mathrm{T}} &= 0\,, \quad X_1 + Y_1 = -(HH^{\mathrm{T}})'\,, \\ HH^{\mathrm{T}}P^{-1}X_2 - Y_2P^{-1}HH^{\mathrm{T}} &= 0\,, \quad X_2 + Y_2 = (HH^{\mathrm{T}})'\,, \end{aligned}$$

hence

$$(3.5)_1 \qquad \begin{aligned} HH^{\mathrm{T}}P^{-1}X_1 + X_1P^{-1}HH^{\mathrm{T}} &= -(HH^{\mathrm{T}})'\,P^{-1}HH^{\mathrm{T}}\,, \\ HH^{\mathrm{T}}P^{-1}Y_1 + Y_1P^{-1}HH^{\mathrm{T}} &= -HH^{\mathrm{T}}P^{-1}(HH^{\mathrm{T}})'\,, \end{aligned}$$

$$(3.5)_2 \qquad \begin{aligned} HH^{\mathrm{T}}P^{-1}X_2 + X_2P^{-1}HH^{\mathrm{T}} &= (HH^{\mathrm{T}})'\,P^{-1}HH^{\mathrm{T}}\,, \\ HH^{\mathrm{T}}P^{-1}Y_2 + Y_2P^{-1}HH^{\mathrm{T}} &= HH^{\mathrm{T}}P^{-1}(HH^{\mathrm{T}})'\,. \end{aligned}$$

As both systems $(3.5)_1$ and $(3.5)_2$ have unique solutions, we have $X_1 = -X_2$ and $Y_1 = -Y_2$, i.e.

$$(3.6) \qquad \begin{aligned} P(x)\,(U(x)\,U^{\mathrm{T}\prime}(x) + V(x)\,V^{\mathrm{T}\prime}(x)) &= -H'(x)\,H^{\mathrm{T}}(x)\,, \\ (U'(x)\,U^{\mathrm{T}}(x) + V'(x)\,V^{\mathrm{T}}(x))\,P(x) &= -H(x)\,H^{\mathrm{T}\prime}(x)\,. \end{aligned}$$

Now, using (3.6) we can verify that

$$(3.7) \qquad S'(x)\,S^{\mathrm{T}}(x) + C'(x)\,C^{\mathrm{T}}(x) = 0\,,$$

and direct computation gives

$$(3.8) \qquad S'(x)\,C^{\mathrm{T}}(x) - C'(x)\,S^{\mathrm{T}}(x) = Q(x)\,.$$

Multiplication of (3.7) and (3.8) from the right by $S(x)$ and $C(x)$, respectively, and addition of these equations gives $S'(x) = Q(x)\,C(x)$. Similarly we obtain $C'(x) =$ $= -Q(x)\,S(x)$. It remains to verify that if $U(x)$ and $V(x)$ are expressed by (3.2) then (2.6) really holds. $U^{\mathrm{T}\prime}V - U^{\mathrm{T}}V' = -U^{\mathrm{T}\prime}P^{-1}V'' + U^{\mathrm{T}\prime\prime}P^{-1}V' = -S^{\mathrm{T}}H^{\mathrm{T}}P^{-1}(HC)' + (S^{\mathrm{T}}H^{\mathrm{T}})'$ $P^{-1}HC = -S^{\mathrm{T}}(H^{\mathrm{T}}P^{-1}H' - H^{\mathrm{T}\prime}P^{-1}H)\,C + S^{\mathrm{T}}H^{\mathrm{T}}P^{-1}HQS + C^{\mathrm{T}}QH^{\mathrm{T}}P^{-1}HC = E$. This computation shows that the relation $(3.1)_3$ is essential and cannot be removed.

Remark 1. Consider a more general selfadjoint system of the second order

$$(3.9) \qquad (F(x)\,Y')' + P(x)\,Y = 0\,,$$

where $F(x)$ is a symmetric positive definite $n \times n$ matrix. In [6] the following transformation of systems (3.9) was investigated. Let $R(x) \in C^1(I)$ be a nonsingular $n \times n$ matrix for which

$$R^{T\prime}(x)\, F(x)\, R(x) - R^T(x)\, F(x)\, R'(x) = 0\,.$$

Then the transformation $U(x) = R^{-1}(x)\, Y(x)$ transforms system (3.9) into the system

$$(F_1(x)\, U')' + P_1(x)\, U = 0\,,$$

where

$$F_1(x) = R^T(x)\, F(x)\, R(x)\,, \tag{3.10}$$
$$P_1(x) = R^T(x)\,[(F(x)\, R'(x))' + P(x)\, R(x)]\,.$$

Theorem 1 shows that the transformation $S(x) = H^{-1}(x)\, Z(x)$ transforms the system $(P^{-1}(x)\, Z')' + Z = 0$ into the system $(Q^{-1}(x)\, S')' + Q(x)\, S = 0$.

If $n = 1$ it is easy to see that in this case $Q(x) = \beta'(x)$, where $\beta(x)$ is a second phase function of this scalar equation. Consequently, if $a \in I$, we call the matrix $B(x) = \int_a^x Q(s)\, \mathrm{d}s$ the *second phase matrix* of (1.4) determined by the pair of solutions $U(x)$, $V(x)$. It follows from $(3.1)_2$ that $B(x) \in C^1(I)$.

4. THE ASSOCIATED SYSTEM

In this section we shall suppose that $P(x) \in C^2(I)$. Hence, according to $(3.1)_2$, we have $B(x) \in C^3(I)$ for any second phase matrix of (1.4).

Denote by $G(x)$ a nonsingular $n \times n$ matrix for which

$$G^T(x)\, P^{-1}(x)\, G(x) = E\,, \tag{4.1}$$
$$G^{T\prime}(x)\, P^{-1}(x)\, G(x) = G^T(x)\, P^{-1}(x)\, G'(x)\,.$$

Note that if $G(x)$ is a matrix satisfying (4.1) (the existence of such a matrix was proved in [6]) then $G(x)\, G_0$, where G_0 is a constant orthogonal $n \times n$ matrix, also satisfies (4.1). We can prove similarly as in [10, Lemma 3.1] that $G(x) \in C^2(I)$. Let

$$\hat{P}(x) = G^T(x)\, G(x) - G^T(x)\, (G^{T-1}(x))''\,. \tag{4.2}$$

The system

$$Y'' + \hat{P}(x)\, Y = 0 \tag{4.3}$$

is called the associated system to (1.4). In the case $n = 1$, $G(x) = (P(x))^{-1/2}$ and hence the associated system (4.3) is identical with the asociated equation (2.3).

Remark 2. Unlike in the scalar case the associated system to (1.4) is not determined uniquely. If $\hat{P}(x)$ is the carrier of the associated system and G_0 is a constant orthogonal $n \times n$ matrix then $\hat{P}_0(x) = G_0^T\, \hat{P}(x)\, G_0$ is also the carrier of the associated system to (1.4), see [6].

Theorem 2. *An $n \times n$ matrix $Y(x)$ is the solution of* (1.4) *if and only if the matrix $U(x) = G^{-1}(x)\, Y'(x)$ is the solution of the associated system* (4.3).

Proof. If $Y(x)$ is a solution of (1.4) then $Z(x) = Y'(x)$ is the solution of (3.4) and hence $0 = (P^{-1}Z')' + Z = G^{T}[(P^{-1}Z')' + Z] = G^{T}(P^{-1}G'U + P^{-1}GU')' + G^{T}GU = (G^{T}P^{-1}G'U + G^{T}P^{-1}GU')' - G^{T\prime}P^{-1}G'U - G^{T\prime}P^{-1}GU' + G^{T}GU = G^{T\prime}P^{-1}G'U + G^{T}(P^{-1}G')'\,U + G^{T}P^{-1}G'U' + U'' - G^{T\prime}P^{-1}G'U - G^{T\prime}P^{-1}GU' + G^{T}GU = U'' + (G^{T}G + G^{T}(P^{-1}G')')\,U = U'' + (G^{T}G - G^{T}((G^{T-1})')')\,U = U'' + \hat{P}(x)\,U$, which was to be proved.

Theorem 3. *A matrix $B(x) \in C^{3}(I)$ is a second phase matrix of* (1.4) *if and only if it is a first phase matrix of* (4.3).

Proof. Let $B(x)$ be a second phase matrix of (1.4) determined by a pair of isotropic solutions $Y_1(x)$, $Y_2(x)$ for which $Y_1^{T\prime}(x)\, Y_2(x) - Y_1^{T}(x)\, Y_2'(x) = E$. According to Theorem 1 there exists a nonsingular $n \times n$ matrix $H(x)$ satisfying $(3.1)_2$, $(3.1)_3$ and such that $Y_1'(x) = H(x)\, S(x)$, $Y_2'(x) = H(x)\, C(x)$, where $\{S(x), C(x)\}$ is a solution of $S' = B'(x)\, C, C' = -B'(x)\, S$ satisfying (2.5). By Theorem 2, $V(x) = G^{-1}(x)\, Y_1'(x)$, $U(x) = G^{-1}(x)\, Y_2'(x)$ are isotropic solutions of (4.3) for which (2.6) holds. Set $R(x) = G^{-1}(x)\, H(x)$. Then $V(x) = R(x)\, S(x)$, $U(x) = R(x)\, C(x)$, $R(x)\, R^{T}(x) = U(x)\, U^{T}(x) + V(x)\, V^{T}(x), R^{T\prime}R - R^{T}R' = (H^{T}G^{T-1})'\, G^{-1}H - H^{T}G^{T-1}(G^{-1}H)' =$

$$= H^{T\prime}G^{T-1}G^{-1}H - H^{T}G^{T-1}G^{T\prime}G^{T-1}G^{-1}H - H^{T}G^{T-1}G^{-1}H' +$$
$$+ H^{T}G^{T-1}G^{-1}G'G^{-1}H = H^{T\prime}P^{-1}H - H^{T}P^{-1}H' - H^{T}G^{T-1}(G^{T\prime}P^{-1}G -$$

$- G^{T}P^{-1}G')\, G^{-1}H = 0$ and $(R^{T}R)^{-1} = (H^{T}G^{T-1}G^{-1}H)^{-1} = (H^{T}P^{-1}H)^{-1} = B'$. Consequently, by $(2.7)_{1-3}$, $B(x)$ is the first phase matrix of (1.4). As all arguments can be reversed, the proof is complete.

From Theorem 3 and (2.9) the following statement immediately follows.

Theorem 4. *Two matrices $B_i(x)$, $i = 1, 2$, for which $B_i'(x)$ are positive definite, are second phase matrices of the same differential system* (1.4) *if and only if there exist constant $n \times n$ matrices K, L, M, N satisfying* (2.8) *such that*

$$T(x, B_2) = (T(x, B_1)\, N + M)^{-1}\, (T(x, B_1)\, L + K)$$

at all points where this expression is defined.

5. RELATIONSHIPS BETWEEN FIRST AND SECOND PHASE MATRICES

Let $U(x)$, $V(x)$ be isotropic solutions of (1.4) for which (2.6) holds, and let $A(x)$, $B(x)$ be the first and the second phase matrix of (1.4), respectively, determined by this pair of solutions, i.e.

$$
\begin{aligned}
U(x) &= R(x)\, S(x, A'), \quad & V(x) &= R(x)\, C(x, A'), \\
U'(x) &= H(x)\, S(x, B'), \quad & V'(x) &= H(x)\, C(x, B'),
\end{aligned}
\tag{5.1}
$$

where $R(x)$ and $H(x)$ are determined by (2.7) and (3.1), respectively, and $\{S(x, A'), C(x, A')\}$, $\{S(x, B'), C(x, B')\}$ satisfy (2.5). Then (5.1) implies

$$H(x)\, S(x, B') = R'(x)\, S(x, A') + R(x)\, A'(x)\, C(x, A')\,,$$
$$H(x)\, C(x, B') = R'(x)\, C(x, A') - R(x)\, A'(x)\, S(x, A')\,.$$

Multiplication of the first and the second equation from the right by $C^{T}(x, A')$ and $-S^{T}(x, A')$, respectively, and addition of these equations gives

$$S(x, B')\, C^{T}(x, A') - C(x, B')\, S^{T}(x, A') = H^{-1}(x)\, R^{T-1}(x)\,. \tag{5.2}$$

Before we state the main result of this section we give one auxiliary statement.

Lemma 1. *Let $\{S_i(x), C_i(x)\}$, $i = 1, 2$, be solutions of*

$$\begin{aligned} S_i' &= Q_i(x)\, C_i\,, \quad S_i(a) = M_i\,,\\ C_i' &= -Q_i(x)\, S_i\,, \quad C_i(a) = N_i\,, \end{aligned} \tag{5.3}$$

where $Q_i(x)$ are symmetric positive definite $n \times n$ matrices and M_i, N_i are constant $n \times n$ matrices satisfying $M_i^T M_i + N_i^T N_i = E$, $M_i^T N_i = N_i^T M_i = 0$. If the matrix $S_1(x)\, C_2^T(x) - C_1(x)\, S_2^T(x)$ is nonsingular on I then for every $a \in I$ there exist a real $c \in [0, \pi/n)$ and an integer k such that

$$c + k\pi < \frac{1}{n}\int_a^x \operatorname{tr}\,(Q_1(s) - Q_2(s))\, ds < c + (k + 1)\,\pi$$

for every $x \in I$.

Proof. See [7, Theorem 4].

Theorem 5. *Let $A(x), B(x)$ be the first and the second phase matrices of (1.4) determined by the same pair of isotropic linearly independent solutions. Then there exist a real $c \in [0, \pi/n)$ and an integer k such that*

$$c + k\pi < \frac{1}{n} \operatorname{tr}\,(A(x) - B(x)) < c + (k + 1)\,\pi$$

for every $x \in I$.

Proof. Since the matrices $H(x)$, $R(x)$ are nonsingular, it is seen from (5.2) that the matrix $S(x, B')\, C^{T}(x, A') - C(x, B')\, S^{T}(x, A')$ is nonsingular on I and the statement follows from Lemma 1.

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Souhrn

DRUHÁ FÁZOVÁ MATICE DIFERENCIÁLNÍHO SYSTÉMU $Y'' + P(x)\,Y = 0$

ONDŘEJ DOŠLÝ

V práci je zaveden pojem druhé fázové matice diferenciálního systému $Y'' + P(x) = 0$, kde $P(x)$ je symetrická positivně definitní matice typu $n \times n$. Jsou vyšetřovány základní vlastnosti této matice a vztahy mezi první a druhou fázovou maticí téhož diferenciálního systému.

Резюме

ВТОРАЯ ФАЗОВАЯ МАТРИЦА ДИФФЕРЕНЦИАЛЬНОЙ СИСТЕМЫ $Y'' + P(x)\,Y = 0$

ONDŘEJ DOŠLÝ

В работе введено понятие второй фазовой матрицы дифференциальной системы $Y'' + P(x) = 0$, где $P(x)$—симметрическая положительно определенная матрица размера $n \times n$. Рассматриваются основные свойства этой матрицы и отношения между первой и второй фазовой матрицей одной и той же системы.

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