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ISODYNAMIC SYSTEMS IN EUCLIDEAN SPACES AND AN n-DIMENSIONAL ANALOGUE OF A THEOREM BY POMPEIU

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INTRODUCTION

Isodynamic tetrahedrons and more generally, isodynamic n-simplexes have been studied in [1], [2].

We shall investigate here isodynamic systems of points in Euclidean spaces, i.e. unordered systems of points $A_1, \ldots, A_m$ such that for some positive numbers $c_1, \ldots, c_m$,

$$\varrho(A_i, A_k) = c_i c_k \tag{1}$$

for all $i, k = 1, \ldots, m, i \neq k$. By ϱ we mean throughout the whole paper the Euclidean distance. In particular, we shall be interested in maximal isodynamic systems in an n-dimensional Euclidean space and their properties.

PRELIMINARIES

Under an $(n-1)$-sphere we understand here and in the sequel a hypersphere in an n-dimensional Euclidean space; a generalized $(n-1)$-sphere is either an $(n-1)$-sphere or an $(n-1)$-dimensional linear space. We say that, in an n-space, a generalized $(n-1)$-sphere K_1 bisects the $(n-1)$-sphere K_2 with centre A_2 and radius r_2 iff either $r_1^2 = \varrho^2(A_1, A_2) + r_2^2$ in the case K_1 is an $(n-1)$-sphere with centre A_1 and radius r_1, or if K_1 contains A_2 in the case K_1 is a hyperplane. If K_1 and K_2 are two $(n-1)$-spheres in E_n with centres A_i and radii r_i $(i = 1, 2)$ then we call (K_1, K_2)-harmonic $(n-1)$-sphere the set $\{X;\ \varrho(X, A_1)/\varrho(X, A_2) = r_1/r_2\}$. It is thus the (generalized) sphere of Apollonius of the points A_1 and A_2 with the ratio r_1/r_2.

As usual, the power of a point X in E_n with respect to an $(n-1)$-sphere K (in E_n) with centre A and radius r is $\varrho^2(X, A) - r^2$. If K is a hyperplane, we shall agree that any point of K has any real number as its power with respect to K, and no point outside K has a defined power with respect to K.

Two points X, Y in E_n are inverse with respect to a generalized $(n-1)$-sphere K iff either they are symmetric with respect to K if K is a hyperplane or, in the other case, if they lie on one ray starting in the centre A of K and $\varrho(X, A) \cdot \varrho(Y, A) = r^2$, r being the radius of K.

RESULTS

It will be useful to assign to an isodynamic system satisfying (1), a new set of numbers $t_1, \dots, t_m$ by

$$t_i = c_i^2, \quad i = 1, \dots, m.$$

We shall call these numbers t_i radii of the isodynamic system corresponding to the points A_i. The following theorem is easy to prove.

Theorem 1. *Let $A_1, \dots, A_m$, $m \geqq 3$, be points in a Euclidean space which form an isodynamic system with the corresponding radii $t_1, \dots, t_m$, i.e.*

$$\varrho^2(A_i, A_k) = t_i t_k, \quad i \neq k, \quad i, k = 1, \dots, m. \tag{2}$$

Then the radii t_i are uniquely determined by (2).

Another trivial observation is formulated in the following

Theorem 2. *Any subsystem of an isodynamic system of points is isodynamic as well.*

Theorem 3. *A system $\{A_1, \dots, A_m\}$ of points is isodynamic iff any subsystem with four points is isodynamic.*

Proof. The "only if" part following from Thm. 2, assume that any subsystem with four points is isodynamic.

To prove that the given system is isodynamic, we shall use induction with respect to m. For $m \leqq 4$, the assertion is clearly true. Suppose that $m > 5$ and theassertion holds for any system with $m-1$ points. Thus $A_1, \dots, A_{m-1}$ is isodynamic with (uniquely determined) radii $t_1, \dots, t_{m-1}$. Let $\bar{t}_m$ $\bar{t}_1, \bar{t}_2, \bar{t}_3$ be radii of the isodynamic subsystem $\{A_m, A_1, A_2, A_3\}$. Since the radii of $\{A_1, A_2, A_3\}$ are uniquely determined by Thm 1, we have $\bar{t}_i = t_i$ $i = 1, 2, 3$. Similarly, if $k > 3$, let $\hat{t}_m, \hat{t}_1, \hat{t}_2, \hat{t}_k$ be radii of the subsystem $\{A_m, A_1, A_2, A_k\}$. Then $\hat{t}_m = \bar{t}_m$, $\hat{t}_1 = t_1$, $\hat{t}_2 = t_2$, $\hat{t}_k = t_k$ so that

$$\varrho^2(A_i, A_k) = t_i t_k \quad (i \neq k)$$

is satisfied for all $i, k = 1, \dots, m$ if $t_m = \bar{t}_m$. This completes the proof.

Remark. It is easily seen that a quadruple $\{A_1, A_2, A_3, A_4\}$ of points is isodynamic iff these points are mutually distinct and

$$\varrho(A_1, A_2)\,\varrho(A_3, A_4) = \varrho(A_1, A_3)\,\varrho(A_2, A_4) = \varrho(A_1, A_4)\,\varrho(A_2, A_3).$$

To investigate existence of isodynamic systems, we recall the following theorem essentially due to MENGER [3] which is a point analogue of the well known theorem that the Gram matrix of a vector system is positive semidefinite and conversely.

Theorem 4. *Let m be a positive integer. The m^2 real numbers $e_{ij} = e_{ji}$, $i, j = 1, \ldots, m$, are squares of distances of some m points $A_1, \ldots, A_m$ in a Euclidean space:*

$$\varrho^2(A_i, A_j) = e_{ij},$$

iff $e_{ii} = 0$, $i = 1, \ldots, m$, and for any real m-tuple $(x_1, \ldots, x_m)$ satisfying

$$\sum_{i=1}^{m} x_i = 0, \tag{3}$$

the inequality

$$\sum_{i,j=1}^{m} e_{ij} x_i x_j \leqq 0 \tag{4}$$

holds.

If this is the case, the points $A_1, \ldots, A_m$ are linearly independent iff the only m-tuple satisfying (3) for which equality in (4) is attained, is the zero m-tuple. More generally, all linear dependence relations among the points $A_1, \ldots, A_m$ are exactly those relations

$$\sum_{i=1}^{m} y_i A_i = 0$$

satisfying

$$\sum_{i=1}^{m} y_i = 0,$$

for which

$$\sum_{i,j=1}^{m} e_{ij} y_i y_j = 0.$$

Now we are able to state the existence theorem on isodynamic systems.

Theorem 5. *Let $m \geqq 2$, let $t_1, \ldots, t_m$ be positive numbers. A necessary and sufficient condition that there exist in a Euclidean n-dimensional (and not $(n-1)$-dimensional) space m points $A_1, \ldots, A_m$ the mutual distances $\varrho(A_i, A_j)$ of which satisfy*

$$\varrho^2(A_i, A_j) = t_i t_j \quad (i \neq j,\ i, j = 1, \ldots, m) \tag{5}$$

is : either

(i) $n = m-1$ *and*

$$(m-1)\sum_{k=1}^{m} \frac{1}{t_k^2} < \left(\sum_{k=1}^{m} \frac{1}{t_k}\right)^2, \tag{6}$$

or

(ii) $n = m - 2$ *and*

$$(m-1)\sum_{k=1}^{m}\frac{1}{t_k^2} = \left(\sum_{k=1}^{m}\frac{1}{t_k}\right)^2. \tag{7}$$

In the second case, the only relation among the points $A_1, \ldots, A_m$ *is*

$$\left(\sum_{k=1}^{m}\frac{1}{t_k}\right)^{-1}\sum_{k=1}^{m}\frac{1}{t_k}A_k - \left(\sum_{k=1}^{m}\frac{1}{t_k^2}\right)^{-1}\sum_{k=1}^{m}\frac{1}{t_k^2}A_k = 0\,. \tag{8}$$

Proof. Let first $A_1, \ldots, A_m$ satisfy (5). We shall show that then

$$(m-1)\sum_{k=1}^{m}\frac{1}{t_k^2} \leqq \left(\sum_{k=1}^{m}\frac{1}{t_k}\right)^2. \tag{9}$$

By Thm. 4,

$$\sum_{1\leqq i<k\leqq m} t_i t_k x_i x_k \leqq 0\,,$$

whenever $x_1, \ldots, x_m$ satisfy $\sum_{i=1}^{m} x_i = 0$. Especially, the numbers $y_1, \ldots, y_m$ where

$$y_i = \frac{1}{t_i}\sum_{k=1}^{m}\frac{1}{t_k^2} - \frac{1}{t_i^2}\sum_{k=1}^{m}\frac{1}{t_k}\,, \quad i = 1, \ldots, m \tag{10}$$

satisfy $\sum y_k = 0$; therefore,

$$2\sum_{1\leqq i<k\leqq m} t_i t_k y_i y_k \leqq 0\,. \tag{11}$$

The left hand side is equal to

$$\left(\sum t_i y_i\right)^2 - \sum t_i^2 y_i^2 = \left(m\sum\frac{1}{t_i^2} - \left(\sum\frac{1}{t_i}\right)^2\right)^2 - \left(m\left(\sum\frac{1}{t_i^2}\right)^2 - \right.$$

$$\left. - 2\left(\sum\frac{1}{t_i^2}\right)\left(\sum\frac{1}{t_i}\right)^2 + \left(\sum\frac{1}{t_i^2}\right)\left(\sum\frac{1}{t_i}\right)^2\right) = \left(m\sum\frac{1}{t_i^2} - \left(\sum\frac{1}{t_i}\right)^2\right)\cdot$$

$$\cdot\left((m-1)\sum\frac{1}{t_i^2} - \left(\sum\frac{1}{t_i}\right)^2\right).$$

The first factor is by the Schwarz inequality nonnegative, and positive if not all the t_i's are equal. If the t_i's are equal, (9) is satisfied. If not, the first factor is positive and (9) is satisfied by (11).

Observe that (7) implies that for $e_{ij} = \varrho(A_i, A_j) = t_i t_j$ $(i \neq j)$ and $e_{ii} = 0$,

$$\sum_{i,j=1}^{m} e_{ij} y_i y_j = 0\,.$$

The numbers (10) are easily seen not to be all equal to zero. By Thm. 4, (7) implies that the points $A_1, \ldots, A_m$ are linearly dependent, i.e.

$$n \leqq m - 2, \tag{12}$$

and moreover, (8) holds.

This means that if $A_1, \ldots, A_m$ are linearly independent then (6) is satisfied.

Let us show now that conversely, (9) implies that there exist, in a Euclidean space, points $A_1, \ldots, A_m$ satisfying (5) and even that (6) implies that they are linearly independent.

We shall use Thm. 4 again. Let $x_1, \ldots, x_m$ be real numbers satisfying $\sum x_i = 0$. Assume first (6). Then

$$m \sum \frac{1}{t_i^2} - \left(\sum \frac{1}{t_i}\right)^2 < \frac{1}{m-1}\left(\sum \frac{1}{t_i}\right)^2$$

and we can write for $e_{ii} = 0$, $e_{ik} = e_{ki} = t_i t_k$ $(i \neq k)$:

$$\sum_{i,j=1}^{m} e_{ij} x_i x_j = \frac{1}{m}\left((m-1)\left(\sum t_i x_i\right)^2 - \left(m \sum t_i^2 x_i^2 - \left(\sum t_i x_i\right)^2\right)\right) =$$

$$= \frac{m-1}{m\left(\sum \frac{1}{t_i}\right)^2}\left(-\frac{1}{m-1}\left(\sum \frac{1}{t_i}\right)^2 \left(m \sum t_i^2 x_i^2 - \left(\sum t_i x_i\right)^2\right) + \right.$$

$$\left. + \left(m \sum x_i - \sum \frac{1}{t_i} \sum t_i x_i\right)^2\right) < \frac{m-1}{m\left(\sum \frac{1}{t_i}\right)^2}\left(\left(m \sum x_i - \sum \frac{1}{t_i} \sum t_i x_i\right)^2 - \right.$$

$$\left. - \left(m \sum \frac{1}{t_i^2} - \left(\sum \frac{1}{t_i}\right)^2\right)\left(m \sum t_i^2 x_i^2 - \left(\sum t_i x_i\right)^2\right)\right) =$$

$$= \frac{m-1}{m\left(\sum \frac{1}{t_i}\right)^2}\left(\left(\sum_{i<j}\left(\frac{1}{t_i} - \frac{1}{t_j}\right)(t_i x_i - t_j x_j)\right)^2 - \right.$$

$$\left. - \left(\sum_{i<j}\left(\frac{1}{t_i} - \frac{1}{t_j}\right)^2\right) \sum_{i<j} (t_i x_i - t_j x_j)^2\right) \leqq 0$$

by the Schwarz inequality. By Thm. 4, this implies the existence of linearly independent points $A_1, \ldots, A_m$ in a Euclidean space which satisfy (5). If only (9) is assumed, a similar chain of inequalities as above yields $\sum_{i,j=1}^{m} e_{ij} x_i x_j \leqq 0$ and by Thm. 4, m points $A_1, \ldots, A_m$ satisfying (5) also exist but are not necessarily linearly independent.

It remains to show that if (7) is fulfilled then $n = m - 2$. By (12), it suffices to disprove that $n < m - 2$. Suppose $n < m - 2$. Then some $n + 1$ points, say $A_1, \ldots, A_{n+1}$ of the points $A_1, \ldots, A_m$ are linearly independent and the points $A_1, \ldots, A_{n+3}$ also satisfy (5). Consequently, for each k, $1 \leqq k \leqq n + 3$, the relation corresponding to (7) holds, i.e.

$$(n + 1)\sum_{\substack{i=1\\i\neq k}}^{n+3} \frac{1}{t_i^2} = \left(\sum_{\substack{i=1\\i\neq k}}^{n+3} \frac{1}{t_i}\right)^2, \tag{13}$$

since the points $A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_{n+3}$ are linearly dependent. Also

$$(n + 2)\sum_{i=1}^{n+3} \frac{1}{t_i^2} = \left(\sum_{i=1}^{n+3} \frac{1}{t_i}\right)^2 \tag{14}$$

by the same reason.

However, (13) can be rewritten in the form

$$\sum_{\substack{1\leqq i<j\leqq n+3\\i\neq k\neq j}} \left(\frac{1}{t_i} - \frac{1}{t_j}\right)^2 = \sum_{\substack{i=1\\i\neq k}}^{n+3} \frac{1}{t_i^2}, \tag{15}$$

(14) in the form

$$\sum_{1\leqq i<j\leqq n+3} \left(\frac{1}{t_i} - \frac{1}{t_j}\right)^2 = \sum_{i=1}^{n+3} \frac{1}{t_i^2}. \tag{16}$$

Subtracting (15) from (16), we obtain

$$\sum_{i=1}^{n+3} \left(\frac{1}{t_i} - \frac{1}{t_k}\right)^2 = \frac{1}{t_k^2}, \quad k = 1, \ldots, n + 3.$$

Therefore, by summing up these equalities,

$$2 \sum_{1\leqq i<j\leqq n+3} \left(\frac{1}{t_i} - \frac{1}{t_j}\right)^2 = \sum_{k=1}^{n+3} \frac{1}{t_k^2},$$

a contradiction with (16). The proof is complete.

This theorem enables us to call complete such an isodynamic system which consists of $m \geqq 3$ points and is contained in an $(m - 2)$-dimensional Euclidean space.

Theorem 6. (i) *In a Euclidean n-dimensional space, $n \geqq 1$, the maximum number of points in an isodynamic system is $n + 2$.*

(ii) *A linearly independent isodynamic system with $m \geqq 3$ points is contained in exactly two complete isodynamic systems in the same space, with the only exception that the points $A_1, \ldots, A_m$ form vertices of a regular $(m - 1)$-simplex; in this case, there is only one complete isodynamic system in the same space in which the given system is contained. The additional point is the center of the simplex.*

(iii) *For any* $m \geqq 3$, *there exist complete isodynamic systems with* m *points.*

(iv) *Any complete isodynamic system* $\mathfrak{S}_1$ *with* $m \geqq 3$ *points in* E_{m-2} *is contained in a complete isodynamic system* $\mathfrak{S}_2$ *with* $m + 1$ *points in* E_{m-1} *(containing* E_{m-2}*).* $\mathfrak{S}_2$ *is determined in* E_{m-1} *uniquely up to congruence leaving all points of* E_{m-2} *invariant. The radius of the* $(m + 1)$*-th point is*

$$t_{m+1} = \left(\frac{1}{m - 1}\sum_{i=1}^{m} \frac{1}{t_i}\right)^{-1}$$

where $t_1, \ldots, t_m$ *are the radii of the points of* $\mathfrak{S}_1$.

(v) *Any isodynamic system which contains a complete isodynamic subsystem is complete.*

(vi) *A complete isodynamic system* $\mathfrak{S}$ *contains a minimal complete isodynamic subsystem, i.e. a complete isodynamic subsystem which is contained in every complete isodynamic subsystem of* $\mathfrak{S}$. *This minimal subsystem contains exactly those points of* $\mathfrak{S}$ *whose coefficient in the (up to a factor unique) relation among the points in* $\mathfrak{S}$ *is different from zero.*

(vii) *If* $\{A_1, \ldots, A_n\}$ *is a complete isodynamic system and* $\{A_1, \ldots, A_k\}$ *its minimal complete isodynamic subsystem then* $A_{k+1}, \ldots, A_m$ *are vertices of a regular simplex.*

(viii) *Any three different points in a line form a complete isodynamic system.*

Proof. (i) is a consequence of Thm. 5. To prove (ii), let $\{A_1, \ldots, A_m\}$ $(m \geqq 3)$ be a linearly independent isodynamic system so that (5) and (6) holds. Assume this system to be contained in a complete isodynamic system $\{A_1, \ldots, A_{m+1}\}$ (by (i), not more than $m + 1$ points exist). By Thm. 1, the corresponding $m + 1$ radii are unique and the first m coincide with t_i, $i = 1, \ldots, m$. Let t_{m+1} be the $(m + 1)$-th. Then, an analogous relation to (7) holds:

$$m\sum_{k=1}^{m+1} \frac{1}{t_k^2} = \left(\sum_{k=1}^{m+1} \frac{1}{t_k}\right)^2$$

so that

$$(m - 1)\frac{1}{t_{m+1}^2} - 2\frac{1}{t_{m+1}}\sum_{i=1}^{m} \frac{1}{t_i} + m\sum_{i=1}^{m} \frac{1}{t_i^2} - \left(\sum_{i=1}^{m} \frac{1}{t_i}\right)^2 = 0\,.$$

The discriminant of this quadratic equation for $1/t_{m+1}$ is easily computed to be positive by (6).

If the t_i's are not all equal, the absolute member of the equation is positive by the Schwarz inequality and the two positive roots yield two distinct complete isodynamic systems.

If all the t_i's are equal, $t_i = t$, $i = 1, \ldots, m$, i.e. if the given system is the set of the vertices of a regular $(m - 1)$-simplex (with all edges having the same length), one root of the equation is zero and there is only one positive root

$$t_{m+1} = \frac{m - 1}{2m}\,t\,.$$

(iii) follows e.g. from the preceding case of the vertices and center of the regular simplex.

To prove (iv), assume $\mathfrak{S}_1$ consists of the points $A_1, \ldots, A_m$ with radii $t_1, \ldots, t_m$ so that

$$(m-1)\sum_{i=1}^{m} \frac{1}{t_i^2} = \left(\sum_{i=1}^{m} \frac{1}{t_i}\right)^2.$$

Assume $\mathfrak{S}_2$ arises from $\mathfrak{S}_1$ by adding a point A_{m+1} with radius t_{m+1} (the radii $t_1, \ldots, t_m$ coincide).

Then

$$m\left(\sum_{i=1}^{m} \frac{1}{t_i^2} + \frac{1}{t_{m+1}^2}\right) = \left(\sum_{i=1}^{m} \frac{1}{t_i} + \frac{1}{t_{m+1}}\right)^2$$

from which, the discriminant of the quadratic equation for $1/t_{m+1}$ being zero,

$$\frac{1}{t_{m+1}} = \frac{1}{m-1} \sum_{i=1}^{m} \frac{1}{t_i}.$$

Since the converse is also true, $\mathfrak{S}_2$ exists by Thm. 5. The distances $\varrho(A_i, A_{m+1})$ are thus uniquely determined which completes the proof of (iv).

(v) follows from the fact that the assumption implies the points of the system are linearly dependent so that case (ii) of Thm. 5 occurs.

To prove (vi), we shall also use the fact that an isodynamic system is complete iff its points are linearly dependent. Thus, if the essentially unique relation among the points of $\mathfrak{S}$ has non-zero coefficients corresponding to points A_j for $j \in J$, the subsystem $\{A_j\}_{j \in J}$ is complete and every complete subsystem contains this subsystem. Before proving (vii), we shall prove the following lemma:

Lemma. *Let k, n be integers, $2 \leqq k < n$. Let $x_1, \ldots, x_n$ be real numbers such that*

$$(k-1)\sum_{i=1}^{k} x_i^2 = \left(\sum_{i=1}^{k} x_i\right)^2.$$

Then

$$(n-1)\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2$$

iff

$$x_{k+1} = \ldots = x_n = \frac{1}{k-1} \sum_{i=1}^{k} x_i.$$

Proof. From the equality

$$\sum_{i=1}^{k+1} x_i^2 - \frac{1}{k}\left(\sum_{i=1}^{k+1} x_i\right)^2 = \sum_{i=1}^{k} x_i^2 - \frac{1}{k-1}\left(\sum_{i=1}^{k} x_i\right)^2 + \frac{k-1}{k}\left(x_{k+1} - \frac{1}{k-1}\sum_{i=1}^{k} x_i\right)^2$$

it follows that

$$\sum_{i=1}^{k+1} x_i^2 - \frac{1}{k}\left(\sum_{i=1}^{k+1} x_i\right)^2 \geqq \sum_{i=1}^{k} x_i^2 - \frac{1}{k-1}\left(\sum_{i=1}^{k} x_i\right)^2,$$

with equality iff

$$x_{k+1} = \frac{1}{k-1}\sum_{i=1}^{k} x_i.$$

Thus,

$$\sum_{i=1}^{n} x_i^2 - \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i\right)^2 \geqq \ldots \geqq \sum_{i=1}^{k} x_i^2 - \frac{1}{k-1}\left(\sum_{i=1}^{k} x_i\right)^2,$$

with equality of the first and last member iff

$$x_{k+1} = \frac{1}{k-1}\sum_{i=1}^{k} x_i, \quad x_{k+2} = \frac{1}{k}\sum_{i=1}^{k+1} x_i = \frac{1}{k-1}\sum_{i=1}^{k} x_i, \ldots,$$
$$x_n = \frac{1}{n-2}\sum_{i=1}^{n-1} x_i = \frac{1}{k-1}\sum_{i=1}^{k} x_i.$$

The lemma then follows.

To prove (vii), use the lemma for $n = m$, $x_i = 1/t_i$, $i = 1, \ldots, m$.

The assertion (viii) being trivial, the proof is complete.

Remark. The two (or one) additional points in (iii) of Thm. 6 are the isodynamic centres [2] of the corresponding $(m-1)$-simplex.

In the following main theorem about complete isodynamic systems several characterizations are given.

Theorem 7. *Let $A_1, \ldots, A_{n+2}$ be different points in a Euclidean n-space E_n. Then the following conditions are equivalent:*

1° *$A_1, \ldots, A_{n+2}$ is a complete isodynamic system in E_n, i.e. there exist positive numbers $t_1, \ldots, t_{n+2}$ such that*

$$\varrho^2(A_i, A_k) = t_i t_k \quad \text{for all} \quad i, k = 1, \ldots, n+2, \quad i \neq k;$$

2° *there exists a system of $n+3$ real $(n-1)$-spheres $K_0, K_1, \ldots, K_{n+2}$ such that*

21° *K_i has centre in A_i for $i = 1, \ldots, n+2$ and bisects K_0,*

22° *for each pair i, j $(i \neq j)$, $i, j = 1, \ldots, n+2$, the (K_i, K_j)-harmonic $(n-1)$-sphere K_{ij} contains all points A_k for $i \neq k \neq j$;*

3° *there exists a system of $\binom{n+2}{2}$ generalized $(n-1)$-spheres $K_{ij}(=K_{ji})$, $i, j = 1, \ldots, n+2$, $i \neq j$, such that*

31° *A_i and A_j are inverse with respect to K_{ij},*

32° *K_{ij} contains all points A_k for $i \neq k \neq j$;*

33° *there exists a point having the same negative power with respect to all $(n-1)$-spheres K_{ij}.*

4° *there exists a point R in E_n and a point $B_0 \neq R$ in a Euclidean $(n+1)$-space containing E_n, on the line perpendicular to E_n in R such that the second intersection points B_i $(i = 1, \ldots, n+2)$ of the lines A_iB_0 with the n-sphere $K = \{X; \varrho(X, R) = \varrho(B_0, R)\}$ form vertices of a regular $(n+1)$-simplex.*

5° *there exists, in a Euclidean $(n+1)$-space E_{n+1} containing E_n, a regular $(n+1)$-simplex Σ such that $A_1, \ldots, A_{n+2}$ correspond to the vertices of Σ in an inversion in E_{n+1}.*

6° *there exists, in a Euclidean $(n+1)$-space $\hat{E}_{n+1}$ a regular $(n+1)$-simplex with vertices $B_1, \ldots, B_{n+1}$ and a point X (different from all the points B_i) on its circumscribed n-sphere such that, for some $k > 0$,*

$$\varrho(A_i, A_j) = \frac{k}{\hat{\varrho}(B_i, X)\,\hat{\varrho}(B_j, X)}$$

for all $i, j = 1, \ldots, n+2$, $i \neq j$.

7° *there exists, in a Euclidean $(n+1)$-space $\hat{E}_{n+1}$, a regular $(n+1)$-simplex with vertices $B_1, \ldots, B_{n+1}$ and a point X (different from all the points B_i) such that, for some $k > 0$*

$$\varrho(A_i, A_j) = \frac{k}{\hat{\varrho}(B_i, X)\,\hat{\varrho}(B_j, X)}$$

for all $i, j = 1, \ldots, n+2$, $i \neq j$.

Proof. We shall prove the implications $1° \Rightarrow 2° \Rightarrow 3° \Rightarrow 4° \Rightarrow 5° \Rightarrow 6° \Rightarrow 7° \Rightarrow 1°$.

Assume 1°. By (iv) of Thm. 6, the system $\{A_1, \ldots, A_{n+2}\}$ is contained in a complete isodynamic system, with the additional point A_{n+3}, of an $(n+1)$-dimensional space E_{n+1} containing E_n. Define for $i = 1, \ldots, n+2$, $K_i = \{X \in E_n;\ \varrho^2(X, A_i) = \varrho^2(A_{n+3}, A_i)\}$. If R is the orthogonal projection of the point A_{n+3} on E_n and $r = \varrho(R, A_{n+3})$ then $K_0 = \{X \in E_n;\ \varrho(X, R) = r\}$ satisfies $\varrho^2(A_i, R) = \varrho^2(A_i, A_{n+3}) - r^2$ which means that K_i bisects K_0. Moreover, let $i \neq j$. The (K_1, K_2) – harmonic $(n-1)$-sphere K_{ij} is easily checked to contain the points A_k for all k, $i \neq k \neq j$. Thus $1° \Rightarrow 2°$.

To prove that 2° implies 3°, it suffices to show that the (K_i, K_j) – harmonic spheres K_{ij} satisfy 31°, 32°, 33°. 31° follows from the harmonic property of A_i, A_j and the intersection points of the line A_iA_j with K_{ij}, 32° is immediate. To prove 33°, take R as the centre of K_0 in 2°. Since K_0 is bisected by K_i and K_j, it is bisected by K_{ij} (belonging to the pencil determined by K_i and K_j) as well. Thus R has the same negative power with respect to all K_{ij}'s which are nonlinear. According to our agreement, this is also true if some – but not all – of the K_{ij}'s are linear. However, all the K_{ij}'s cannot be linear since in this case the mutual distances of $n+2$ points A_i in E_n would be equal.

Assume 3°. Let E_{n+1} be any Euclidean $(n+1)$-space containing E_n. Let B_0 be a point on the line perpendicular to E_n passing through R, such that $\varrho^2(B_0, R) = -p$, p being the power of R with respect to all K_{ij}'s. Let $\hat{K}_{ij}$ $(i \neq j,\ i, j = 1, \ldots, n+2)$ be the generalized n-sphere in E_{n+1} with the same centre and radius as K_{ij} if K_{ij} is an $(n-1)$-sphere; if K_{ij} is linear, let $\hat{K}_{ij}$ be that n-dimensional linear space in E_{n+1} which contains K_{ij} and is orthogonal to E_n. It follows that $\hat{K}_{ij}$ contains the point B_0 for all $i, j = 1, \ldots, n+2$, $i \neq j$. Let K be the n-sphere with centre in R and radius $\varrho(R, B_0)$, let B_i $(i = 1, \ldots, n+2)$ be the second intersection point of the line A_iB with K. Denote by $\hat{K}$ the n-sphere with centre B_0 which bisects K. Using the well known properties of inversion, it follows that E_n corresponds to K in the inversion $\mathscr{I}$ with respect to $\hat{K}$; A_i corresponds to B_i in $\mathscr{I}$, $\hat{K}_{ij}$ corresponds to a hyperplane H_{ij}, $i, j = 1, \ldots, n+2$, $i \neq j$. Since $\hat{K}_{ij}$ is orthogonal to E_n, H_{ij} is orthogonal to K and thus contains R, as well as all the points B_k for $i \neq k \neq j$. A_i and A_j being inverse with respect to $\hat{K}_{ij}$, B_i and B_j are symmetric with respect to H_{ij} (since any sphere containing both B_i and B_j is orthogonal to H_{ij}, this being true for their transforms in $\mathscr{I}$). Consequently, $\varrho(B_i, B_k) = \varrho(B_j, B_k)$ for all i, j, k, $i \neq j \neq k \neq i$. It follows that the points B_i, $i = 1, \ldots, n+2$, form vertices of a regular $(n+1)$-simplex. The proof of $3° \Rightarrow 4°$ is complete.

The implication $4° \Rightarrow 5°$ is immediate since B_i and A_i correspond to each other in the inversion determined by the n-sphere $\hat{K}$ having the centre B_0 and bisecting K.

Assume 5°. Denote by $\mathscr{I}$ the inversion, by B_i $(i = 1, \ldots, n+2)$ the points in E_{n+1} corresponding to A_i in $\mathscr{I}$ so that B_i are vertices of a regular $(n+1)$-simplex Σ. Let X be the centre of the inversion $\mathscr{I}$. Thus $X \neq B_i$ for all $i = 1, \ldots, n+2$. If C is the circumscribed n-sphere of Σ, C corresponds to E_n in $\mathscr{I}$ and thus contains X.

We have then for $i \neq j$, $i, j = 1, \ldots, n+2$

$$\varrho(A_i, X)\,\varrho(B_i, X) = \varrho(A_j, X)\,\varrho(B_j, X) \tag{17}$$

so that the triangles A_iA_jX and B_jB_iX are similar to each other. Thus

$$\varrho(A_i, A_j)/\varrho(A_i, X) = \varrho(B_i, B_j)/\varrho(B_j, X)$$

as well as

$$\varrho(A_i, A_j)/\varrho(A_j, X) = \varrho(B_i, B_j)/\varrho(B_i, X).$$

By multiplication,

$$\begin{aligned}\varrho^2(A_i, A_j) &= \varrho^2(B_i, B_j)\,\varrho(A_i, X)\,\varrho(A_j, X)\,(\varrho(B_i, X)\,\varrho(B_j, X))^{-1} = \\ &= \sigma^2\,\varrho^2(B_i, B_j)/(\varrho^2(B_i, X)\,\varrho^2(B_j, X))\end{aligned}$$

by (17), if the common value is denoted by σ. Since $\varrho^2(B_i, B_j)$ is constant for all pairs i, j, $i \neq j$, 6° follows (where $\hat{E} = E_{n+1}$, $\hat{\varrho} = \varrho$ is taken).

The implications $6° \Rightarrow 7°$ as well as $7° \Rightarrow 1°$ being trivial, the proof is complete.

A well known theorem from plane geometry, sometimes called Pompeiu's theorem, states:

If $A_1A_2A_3$ is an equilateral triangle and X another point of the plane then XA_1, XA_2, XA_3 form lengths of sides of a triangle iff X does not belong to the circumscribed circle of $A_1A_2A_3$.

We shall generalize now this theorem as follows:

Theorem 8. *Let $A_1, \ldots, A_{n+1}$ be vertices of a regular n-simplex Σ in E_n. If X is a point in E_n then there exists an n-simplex with vertices $B_1, \ldots, B_{n+1}$ such that edges B_iB_k $(i \neq k,\ i, k = 1, \ldots, n + 1)$ have lengths proportional to $(\varrho(A_i, X) \cdot \varrho(A_k, X))^{-1}$ iff X does not belong to the circumscribed $(n - 1)$-sphere of Σ.*

Proof. Assume first that X belongs to the circumscribed $(n - 1)$-sphere of Σ. If $X = A_i$ for some i, the n-simplex clearly does not exist. If $X \neq A_i$ for all $i = 1, \ldots, n + 1$, the equivalence of 7° and 1° in Thm. 7 shows that the realization of the points B_i leads to a complete isodynamic system which is linearly dependent.

Assume now that X does not belong to the circumscribed $(n - 1)$-sphere of Σ. Let $\mathcal{I}$ be any inversion with centre X. If B_i are points which correspond to the points A_i in $\mathcal{I}$, we have similarly as in the proof of 5° $\Rightarrow$ 6° in Thm. 7,

$$\varrho(B_i, B_k) = k(\varrho(A_i, X)\, \varrho(A_k, X))^{-1} .$$

Moreover, the points B_i do not belong to a hyperplane since this would correspond in $\mathcal{I}$ to the circumscribed sphere of Σ and this would contain the centre of inversion X, a contradiction. The proof is complete.

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