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ON QUASISTARS IN n-CUBES

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If $m \geqq 3$ is an integer, then a graph (in the sense of [1]) which is homeomorphic to the star $K(1, m)$ will be refered to as an m-quasistar. Let T be an m-quasistar ($m \geqq 3$) of order p; obviously, T is a tree and $p \geqq m + 1$; we say that T is balanced if p is even and there exists a 2-coloring of T with $p/2$ red vertices and $p/2$ green ones.

The present note was inspired by I. Havel's paper [2]. Let m and n be integers, $3 \leqq m \leqq n$, and let T be a balanced m-quasistar of order 2^n. Havel conjectured that T can be embedded into the n-cube; he proved this conjecture for the case when $m = 3$. In the present note we shall prove this conjecture for the cases when $m = 4$ and 5. Moreover, we shall give an alternative proof of the case $m = 3$.

Let G be an n-cube, $n \geqq 1$. Then there exist vertex-disjoint $(n - 1)$-cubes G' and G'' such that $V(G) = V(G') \cup V(G'')$; we shall say that G can be partitioned into G' and G''. Let $u' \in V(G')$; the only vertex $u'' \in V(G'')$ with the property that $u'u'' \in E(G)$ will be denoted by $u'(G'')$.

Let P be a nontrivial path. Then P is homeomorphic to K_2. If u is a vertex of degree one in P, then P will be refered to as a u-path. Assume that P is a u-path. Then the only vertex of degree one in P which is different from u will be denoted by $\varepsilon(P, u)$.

Lemma 1. *Let G be an n-cube, $n \geqq 3$, and let $u_1, u_2, \bar{u}_1, \bar{u}_2 \in V(G)$, $u_1 \neq u_2$. Assume that a_1 and a_2 are even positive integers such that $a_1 + a_2 = 2^n$. Then there exist vertex-disjoint paths P_1 and P_2 in G with the property that for $i \in \{1, 2\}$, P_i is a u_i-path of order a_i such that $\varepsilon(P_i, u_i) \neq \bar{u}_i$.*

Proof. The case of $n = 3$ is easy. Let $n = n_0 \geqq 4$; assume that for $n = n_0 - 1$, the lemma was proved. Clearly, G can be partitioned into two vertex-disjoint $(n - 1)$-cubes G_1 and G_2 in such a way that $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$. Without loss of generality we shall assume that $a_1 \geqq a_2$. If $a_1 = a_2$, then there exists a hamiltonian u_i-path in G_i such that $\bar{u}_i \neq \varepsilon(P_i, u_i)$ for $i = 1, 2$, and thus the lemma is proved.

We shall assume that $a_1 > a_2$. Then there exists a hamiltonian u_1-path P' in G_1 such that $u_2(G_1) \neq \varepsilon(P', u_1)$. Denote $w = \varepsilon(P', u_1)$. It follows from the induction assumption that there exist vertex-disjoint paths P'' and P_2 in G_2 with the properties that P'' is a $w(G_2)$-path of order $a_1 - 2^{n-1}, \bar{u}_1 \neq \varepsilon(P'', w(G_2))$, P_2 is a u_2-path of order a_2, and $\bar{u}_2 \neq \varepsilon(P_2, u_2)$. We denote by P_1 the path induced by the edges $E(P') \cup \cup \{ww(G_2)\} \cup E(P'')$. It is clear that the paths P_1 and P_2 have the desired properties.

Lemma 2. *Let $k \in \{1, 2, 3\}$, let G be an n-cube, where $n \geqq k$, let $u_1, \ldots, u_k$ be k distinct vertices of G, and let $a_1, \ldots, a_k$ be even positive integers such that $a_1 + \ldots \ldots + a_k = 2^n$. Then there exist vertex-disjoint paths $P_1, \ldots, P_k$ in G such that P_i is an u_i-path of order a_i for each $i \in \{1, \ldots, k\}$.*

Proof. The case of $k = 1$ is obvious. The case of $k = 2$ is obvious for $n = 2$, and follows immediately from Lemma 1 for $n \geqq 3$. Let $k = 3$. The proof of the lemma is very easy for $n = 3$. Assume that $n \geqq 4$. It is clear that G contains four vertex-disjoint $(n - 2)$-cubes G_1, G_2, G_3, G_4 such that $u_i \in V(G_i)$ for $i = 1, 2, 3$. Without loss of generality we may assume that $V(G_1) \cup V(G_2)$ induces an $(n - 1)$-cube in G, and that $V(G_4) \cup V(G_1)$ also induces an $(n - 1)$-cube in G. If $a_1 + a_2 \leqq 2^{n-1}$ and $a_2 + a_3 \leqq 2^{n-1}$, then the fact that $a_1 + a_2 + a_3 = 2^n$ implies that $a_2 \leqq 0$, which is a contradiction. Thus, without loss of generality we shall assume that $a_1 + a_2 > > 2^{n-1}$. We denote by G' or G'' the $(n - 1)$-cube in G which is induced by $V(G_1) \cup \cup V(G_2)$ or by $V(G_3) \cup V(G_4)$, respectively. There exists a permutation π on $\{1, 2\}$ such that $a_{\pi(1)} \geqq a_{\pi(2)}$. It is clear that $a_{\pi(2)} \leqq 2^{n-1} - 2$. It follows from Lemma 1 that there exist vertex-disjoint paths P' and $P_{\pi(2)}$ in G' such that P' is a $u_{\pi(1)}$-path of order $2^{n-1} - a_{\pi(2)}$, $u_3(G') \neq \varepsilon(P', u_{\pi(1)})$, and $P_{\pi(2)}$ is a $u_{\pi(2)}$-path of order $a_{\pi(2)}$. Denote $w = \varepsilon(P', u_{\pi(1)})$. It follows from the case $k = 2$ of the present lemma that there exist vertex-disjoint paths P'' and P_3 in G'' such that P'' is a $w(G'')$-path of order $a_{\pi(1)} + a_{\pi(2)} - 2^{n-1}$, and P_3 is a u_3-path of order a_3. We denote by $P_{\pi(1)}$ the path induced by the edges $E(P') \cup \{ww(G'')\} \cup E(P'')$. It is clear that the paths P_1, P_2, P_3 have the desired properties, which completes the proof.

Let m and n be integers such that $2 \leqq m \leqq n$. We denote by $R(m, n)$ the set of sequences $(r_1, \ldots, r_m)$ of positive integers with the properties that $r_1 + \ldots + r_m = = 2^n - 1$ and that r_i is odd for exactly one $i \in \{1, \ldots, m\}$.

Lemma 3. *Let $n \geqq 3$, and let $(r_1, r_2, r_3) \in R(3, n)$. Then there exist an even integer $s \geqq 0$ and a permutation π on $\{1, 2, 3\}$ such that $r_{\pi(1)}$ is even and $(r_{\pi(2)} - s, r_{\pi(3)}) \in R(2, n - 1)$.*

Proof. Without loss of generality we assume that $r_1 \geqq r_2 \geqq r_3$. If $r_1 + r_2 \leqq \leqq 2^{n-1}$; then $2^n - 1 = r_1 + r_2 + r_3 \leqq 2^{n-1} + 2^{n-2}$ and therefore $n \leqq 2$, which is a contradiction. We shall assume that $r_1 + r_2 \geqq 2^{n-1} + 1$.

Let first $r_1 \geqq 2^{n-1} + 1$. Then there exists a permutation π on $\{1, 2, 3\}$ such that $\pi(2) = 1$ and $r_{\pi(1)}$ is even. It is obvious that $(r_{\pi(2)} - (2^{n-1} - r_{\pi(1)}), r_{\pi(3)})$ belongs to $R(2, n - 1)$.

Let now $r_1 \leqq 2^{n-1}$. There exists a permutation π on $\{1, 2, 3\}$ such that $\pi(3) = 3$ and $r_{\pi(1)}$ is even. It is obvious that $(r_{\pi(2)} - (2^{n-1} - r_{\pi(1)}), r_{\pi(3)}$ belongs to $R(2, n - 1)$.

Lemma 4. *Let $m \in \{4, 5\}$, let $n \geqq m$, and let $(r_1, \ldots, r_m) \in R(m, n)$. Assume that $r_1 \geqq \ldots \geqq r_m$. Then there exist even integers $s \geqq 0$ and $t \geqq 0$, and a permutation π on $\{1, 2, 3\}$ such that $r_{\pi(1)}$ is even, and*

$$(r_{\pi(2)} - s,\ r_{\pi(3)} - t,\ r_4, \ldots, r_m) \in R(m - 1,\ n - 1)\,.$$

Proof. If $r_1 + r_2 + r_3 \geqq 2^{n-1} + 3$, then the statement of the lemma follows easily.

Let $r_1 + r_2 + r_3 \leqq 2^{n-1} + 2$. Then

$$2^n - 1 = r_1 + \ldots + r_m \leqq m(2^{n-1} + 2)/3\,.$$

This implies that $(6 - m)\, 2^{n-1} < 2m + 3$. Since $n \geqq m$, we get that $m \notin \{4, 5\}$, which is a contradiction. Thus the lemma is proved.

Theorem. *Let $m \in \{3, 4, 5\}$ and let n be an integer such that $n \geqq m$. Then every balanced m-quasistar of order 2^n can be embedded into the n-cube.*

Proof. Let T be a balanced m-quasistar of order 2^n, and let G be an n-cube. Clearly, G can be partitioned into two vertex-disjoint $(n - 1)$-cubes, say G' and G''.

Obviously, $n \geqq 3$. If $n > 3$, assume that the theorem holds for $n - 1$. We denote by w the vertex of degree m in T. Let $w_1, \ldots, w_m$ be distinct vertices of degree one in T. We denote by r_i the distance between w and w_i in T for $1 \leqq i \leqq m$. It is easy to see that $(r_1, \ldots, r_m)$ belongs to $R(m, n)$. It follows from Lemmas 3 and 4 that there exist even integers s and t and a permutation π on $\{1. \ldots, m\}$ with the properties that

$s \geqq t \geqq 0$,

$r_{\pi(1)}$ is even,

if $m = 3$, then $t = 0$ and $(r_{\pi(2)} - s,\ r_{\pi(3)})$ belongs to $R(2,\ n - 1)$,

if $m \geqq 4$, then $(r_{\pi(2)} - s,\ r_{\pi(3)} - t,\ r_{\pi(4)}, \ldots, r_{\pi(m)})$ belongs to $R(m - 1,\ n - 1)$.

Let k be the integer defined as follows: if $s = 0$, then $k = 1$; if $s > 0$ and $t = 0$, then $k = 2$; and if $t > 0$, then $k = 3$. There exist distinct vertices $u_1, v_1, \ldots, u_k, v_k$ of T with the following properties:

$u_i v_i \in E(T)$ and v_i belongs to the $u_i - w_{\pi(i)}$ path in T for every $i \in \{1, \ldots, k\}$;

$u_1 = w$;

if $k \geqq 2$, then the distance between u_2 and $w_{\pi(2)}$ is s;

if $k = 3$, then the distance between u_3 and $w_{\pi(3)}$ is t.

Let T' be the component of $T - u_1 v_1 - \ldots - u_k v_k$ which contains w. Then T' is a tree of order 2^{n-1}. If T' is a path, then T' can be embedded into G'. Assume that T' is not a path. Then $m \geqq 4$. Since $r_{\pi(1)}$, s and t are even, T' is a balanced $(m - 1)$-quasistar. According to the induction assumption, T' can be embedded into G'. Thus, we can assume that T' is a subgraph of G'.

It follows from Lemma 2 that there exist vertex-disjoint paths $P_1, \ldots, P_k$ in G'' with the following properties:

P_1 is a $u_1(G'')$-path of order $r_{\pi(1)}$;
if $k \geqq 2$, then P_2 is a $u_2(G'')$-path of order s; and
if $k = 3$, then P_3 is a $u_3(G'')$-path of order t.

The subgraph of G induced by

$$E(T') \cup E(P_1) \cup \ldots \cup E(P_k) \cup \{u_1 u_1(G''), \ldots, u_k u_k(G'')\}$$

is isomorphic to T, which completes the proof.

References

[1] *M. Behzad, G. Chartrand,* and *L. Lesniak-Foster:* Graphs & Digraphs. Prindle, Weber & Schmidt, Boston 1979.

[2] *I. Havel:* On hamiltonian circuits and spanning trees of hypercubes. Čas. pěst. mat. *109* (1984), 135–152.

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