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ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

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A CONTRIBUTION TO THE THEORY OF MODULES OVER FINITE DIMENSIONAL LINEAR ALGEBRAS

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In this paper two special cases of a given commutative unitary ring A described below as well as an A-module M are considered. Our goal is to derive, for the both cases of A, necessary and sufficient conditions for M to be a free A-module. Specializing the obtained results in a suitable way we get a condition under which the module M is a free module over a 1-generated finite dimensional linear algebra A over a given field F.

1. In the first case, let us consider a commutative unitary ring A together with a finite system

$$(\mathfrak{J}_1, \ldots, \mathfrak{J}_m) \quad m \geqq 2$$

whose ideals have the following properties:

(a) $$\forall r, s \in \{1, \ldots, m\}, \quad r \neq s : \mathfrak{J}_r + \mathfrak{J}_s = A,$$

(b) $$\mathfrak{J}_1 \cap \ldots \cap \mathfrak{J}_m = 0.$$

For an A-module $\boldsymbol{M}$ let us denote by $\ker \mathfrak{J}_j$ the annulator of

$$\mathfrak{J}_j, \quad \text{i.e.} \quad \ker \mathfrak{J}_j = \{x \in M \mid \forall \xi \in \mathfrak{J}_j : \xi x = 0\}.$$

Proposition 1.

(1) $$M = \ker \mathfrak{J}_1 \oplus \ldots \oplus \ker \mathfrak{J}_m.$$

Proof. As (1) is trivial for $m = 2$, we will continue by induction supposing that $m \geqq 3$ and that our assertion is true for $m - 1$.

Let us put $\mathfrak{J} = \mathfrak{J}_1 \cap \ldots \cap \mathfrak{J}_{m-1}$. Since

(2) $$\mathfrak{J}_1 + \mathfrak{J}_m = A, \ldots, \mathfrak{J}_{m-1} + \mathfrak{J}_m = A,$$

then multiplying the left as well as the right hand sides of (2) we get

$$\mathfrak{J}_1 \ldots \mathfrak{J}_{m-1} + \text{multiples of the ideal } \mathfrak{J}_m = A,$$

hence $\mathfrak{I} + \mathfrak{I}_m = A$. Obviously $\mathfrak{I} \cap \mathfrak{I}_m = 0$, so that

$$M = \ker \mathfrak{I} \oplus \ker \mathfrak{I}_m . \tag{3}$$

The submodules $\ker \mathfrak{I}$; $\ker \mathfrak{I}_1, \ldots, \ker \mathfrak{I}_{m-1}$ of M are also modules over $A/\mathfrak{I}$, moreover, $\ker \mathfrak{I}_1, \ldots, \ker \mathfrak{I}_{m-1}$ are submodules of $\ker \mathfrak{I}$. Let $\mathfrak{I}_1^*, \ldots, \mathfrak{I}_{m-1}^*$ be the ideals of $A/\mathfrak{I}$ corresponding to the ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_{m-1}$ under the canonical epimorphism $A \to A/\mathfrak{I}$. So for the ring $A/\mathfrak{I}$ and the system of its ideals

$$(\mathfrak{I}_1^*, \ldots, \mathfrak{I}_{m-1}^*)$$

we have

$$\forall r, s \in \{1, \ldots, m-1\}, \quad r \neq s : \mathfrak{I}_r^* + \mathfrak{I}_s^* = A/\mathfrak{I}, \tag{a'}$$

$$\mathfrak{I}_1^* \cap \ldots \cap \mathfrak{I}_{m-1}^* = 0 . \tag{b'}$$

Moreover, $\ker \mathfrak{I}_j^* = \ker \mathfrak{I}_j$ for any $j \in \{1, \ldots, m-1\}$. Then, according to the induction hypothesis we conclude

$$\ker \mathfrak{I} = \ker \mathfrak{I}_1 \oplus \ldots \oplus \ker \mathfrak{I}_{m-1}$$

and substituting this result into (3) we get (1).

The existence of the decomposition (1) yields

Theorem 1. *The A-module M is a free A-module if and only if any of the $A/\mathfrak{I}_j$-modules of* $\ker \mathfrak{I}_j$ *is a free $A/\mathfrak{I}_j$-module and the dimensions* $\dim_{A/\mathfrak{I}_j} \ker \mathfrak{I}_j$ *are the same.*

2. In the second case let us consider a commutative unitary local ring A together with its (unique) maximal ideal $\mathfrak{m}$. In addition let us suppose that $\mathfrak{m}$ is a principal ideal $A\vartheta$ and ϑ is a nilpotent element of A of order, say, n. Then we have a descending chain of ideals

$$A = A\,1 \supset A\vartheta \supset A\vartheta^2 \supset \ldots \supset A\vartheta^{n-1} \supset A\vartheta^n = 0 .$$

Evidently:

(a) Any non-zero element $\eta \in A$ may be uniquely expressed by

$$\eta = \varepsilon\vartheta^r ,$$

where ε is a unit of A and r is an integer, $0 \leqq r \leqq n$.

(b) For any integer k, $0 \leqq k \leqq n$, we have: $\ker A\vartheta^k = A\vartheta^{n-k}$.

Now, let us investigate a given A-module M possessing an element a such that $\vartheta^{n-1}a \neq 0$. Then the element ϑ may be viewed as a nilpotent linear operator on M. In this way, we will use the notation $\operatorname{im} \vartheta$, $\ker \vartheta$ and similarly. Clearly $\operatorname{im} \vartheta^i \subseteqq \ker \vartheta^{n-i}$, in particular

$$\operatorname{im} \vartheta \subseteqq \ker \vartheta^{n-1} . \tag{4}$$

Theorem 2. *The A-module M is a free A-module if and only if*

$$\operatorname{im} \vartheta = \ker \vartheta^{n-1} . \tag{5}$$

Proof. I. Let us assume that M is free over A. According to (4) it remains to prove the converse inclusion. Let the system $U = (u_\lambda)_{\lambda \in \Lambda}$ form an A-basis for M. Let

$$x = \sum_{\lambda \in \Lambda} \xi_\lambda u_\lambda$$

(almost all ξ_λ equal to zero) belong to $\ker \vartheta^{n-1}$. Then

$$\sum_{\lambda \in \Lambda} (\vartheta^{n-1} \xi_\lambda)\, u_\lambda = \vartheta^{n-1} \sum_{\lambda \in \Lambda} \xi_\lambda u_\lambda = \vartheta^{n-1} x = 0 ,$$

so that for any

$$\lambda \in \Lambda \quad \text{we have} \quad \vartheta^{n-1} \xi_\lambda = 0 .$$

According to (b*) $\xi_\lambda \in A\vartheta$, hence $x \in \operatorname{im} \vartheta$.

II. Let the identity (5) be true. The factor-modules $M/\operatorname{im} \vartheta$, $\operatorname{im} \vartheta / \operatorname{im} \vartheta^2, \ldots$ $\ldots, \operatorname{im} \vartheta^{n-2}/\operatorname{im} \vartheta^{n-1}$ as well as $\operatorname{im} \vartheta^{n-1} = \operatorname{im} \vartheta^{n-1}/\operatorname{im} \vartheta^n$ are vector spaces over the field $A/A\vartheta$.

Let us start with a system $U = (u_\lambda)_{\lambda \in \Lambda}$, $u_\lambda \in M$, forming an $A/A\vartheta$ – basis for M relatively (= modulo) $\operatorname{im} \vartheta$. Let us investigate the system $\vartheta U = (\vartheta u_\lambda)$. Obviously $\vartheta u_\lambda \in \operatorname{im} \vartheta$ for any $\lambda \in \Lambda$.

First, let us assume that

$$\sum_{\lambda \in \Lambda} \xi_\lambda (\vartheta u_\lambda) \in \operatorname{im} \vartheta^2$$

for a certain system $(\xi_\lambda)_{\lambda \in \Lambda}$ of elements of A whose almost all members equal zero. Then

$$\vartheta^{n-1} \Big(\sum_{\lambda \in \Lambda} \xi_\lambda u_\lambda\Big) = \vartheta^{n-2} \sum_{\lambda \in \Lambda} \xi_\lambda (\vartheta u_\lambda) = 0 \Rightarrow \sum_{\lambda \in \Lambda} \xi_\lambda u_\lambda \in \ker \vartheta^{n-1} ,$$

hence

$$\sum_{\lambda \in \Lambda} \xi_\lambda u_\lambda \in \operatorname{im} \vartheta$$

with respect to (5). From the definition of the system U we get $\xi_\lambda \in A\vartheta$ for any $\lambda \in \Lambda$. This means that the system ϑU is linear independent over $A/A\vartheta$ relatively to $\operatorname{im} \vartheta^2$.

Further, let $x \in \operatorname{im} \vartheta$, then we may write $x = \vartheta y$, $y \in M$. This y may be expressed as

$$y = \sum_{\lambda \in \Lambda} \eta_\lambda u_\lambda + v ,$$

where $(\eta_\lambda)_{\lambda \in \Lambda}$ is a system of elements of A (for almost all λ, $\eta_\lambda = 0$) and $v \in \operatorname{im} \vartheta$. Hence

$$x = \sum_{\lambda \in \Lambda} \eta_\lambda (\vartheta u_\lambda) + \vartheta v, \quad \vartheta v \in \operatorname{im} \vartheta^2 ,$$

which means that the system ϑU generates im ϑ over $A/A\vartheta$ relatively to im ϑ^2. We may conclude: If the system U forms an $A/A\vartheta$-basis of M relatively to im ϑ, then ϑU forms an $A/A\vartheta$-basis of im ϑ relatively to im ϑ^2. Continuing this proces we find that $\vartheta^2 U = (\vartheta^2 u_\lambda)_{\lambda\in\Lambda}$ forms an $A/A\vartheta$-basis for im ϑ^2 relatively to im $\vartheta^3, \ldots, \vartheta^{n-1}U = (\vartheta^{n-1}u_\lambda)_{\lambda\in\Lambda}$ forms an $A/A\vartheta$-basis for the vector space im ϑ^{n-1}.

Let us consider again the system U and let us assume that for a system $(\alpha_\lambda)_{\lambda\in\Lambda}$ of elements of A whose almost all members equal zero, the relation $\sum_{\lambda\in\Lambda} \alpha_\lambda u_\lambda = 0$ is true. Then a fortiori $\sum_{\lambda\in\Lambda} \alpha_\lambda u_\lambda \in \text{im } \vartheta$. By the definition of U, we get that for any $\lambda \in \Lambda : \alpha_\lambda \in A\vartheta$. Thus, we may write $\alpha_\lambda = \vartheta\beta_\lambda$, $\beta_\lambda \in A$ (for any $\lambda \in \Lambda$). Hence

$$\vartheta^{n-1}\left(\sum_{\lambda\in\Lambda} \beta_\lambda u_\lambda\right) = \sum_{\lambda\in\Lambda} (\vartheta^{n-1}\beta_\lambda)\, u_\lambda = \sum_{\lambda\in\Lambda} (\vartheta^{n-2}\alpha_2)\, u_\lambda =$$
$$= \vartheta^{n-2} \sum_{\lambda\in\Lambda} \alpha_\lambda u_\lambda = 0 \Rightarrow \sum_{\lambda\in\Lambda} \beta_\lambda u_\lambda \in \ker \vartheta^{n-1} = \text{im } \vartheta\,.$$

Again, with respect to the definition of U we obtain $\beta_\lambda \in A\vartheta \Rightarrow \alpha_\lambda \in A\vartheta^2$. In a similar way we derive that $\alpha_\lambda \in A\vartheta^3, \ldots, \alpha_\lambda \in A\vartheta^n = 0$. Therefore the system U is linearly independent over A.

Finally, let $x \in M$. Then, by the above result, we have the following identities:

$$\begin{aligned} x &= \sum_{\lambda\in\Lambda} \xi_\lambda^{(0)} u_\lambda + v_1\,, \\ v_1 &= \sum_{\lambda\in\Lambda} \xi_\lambda^{(1)} \vartheta u_\lambda + v_2\,, \\ v_2 &= \sum_{\lambda\in\Lambda} \xi_\lambda^{(2)} \vartheta^2 u_\lambda + v_3\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ v_{n-2} &= \sum_{\lambda\in\Lambda} \xi_\lambda^{(n-2)} \vartheta^{n-2} u_\lambda + v_{n-1}\,, \\ v_{n-1} &= \sum_{\lambda\in\Lambda} \xi_\lambda^{(n-1)} \vartheta^{n-1} u_\lambda\,, \end{aligned} \tag{6}$$

where $\xi_\lambda^{(0)}, \xi_\lambda^{(1)}, \ldots, \xi_\lambda^{(n-1)} \in A$ and $v_j \in \text{im } \vartheta^j$ $(j = 1, \ldots, n-1)$. Summing the left as well as the right hand sides of (6), we obtain

$$x = \sum_{\lambda\in\Lambda} \left(\sum_{0\leq i\leq n-1} \xi_\lambda^{(i)} \vartheta^i \right) u_\lambda\,;$$

thus the system U generates M over A.

Altogether we have proved that U forms an A-basis for M.

Remark. It follows from the just finished proof of Theorem 2 that if M is a free A-module, then the vector spaces $M/\text{im } \vartheta$, $\text{im } \vartheta/\text{im } \vartheta^2, \ldots, \text{im } \vartheta^{n-2}/\text{im } \vartheta^{n-1}$, $\text{im } \vartheta^{n-1}$ have a common dimension over $A/A\vartheta$ and this dimension is the same as $\dim_A M$.

3. **Examples**: A. Let M be a vector space over a given field F and let ε be a linear operator on M. Let A be the linear algebra generated over F by ε. Suppose that there

exists a non-zero polynomial $g \in F[X]$ such that $g(\varepsilon) = 0$. Such a polynomial exists always if M has a finite dimension over F. Then there exists a minimal polynomial, say f, of ε over F. Let

$$f(X) = f_1^{r_1}(X) \dots f_m^{r_m}(X)$$

be the canonical decomposition of $f(X)$ over F into the irreducible factors. Now, we may regard M as an A-module. Combining both Theorems 1 and 2 we state that M is a free A-module if and only if

(i) Any the submodules $\ker f_j^{r_j}(\varepsilon)$ is a free $A/A f_j^{r_j}(\varepsilon)$-module ($j \in \{1, \dots, m\}$);

(ii) the dimensions $\dim \ker f_j^{r_j}(\varepsilon)$ over $A/A f_j^{r_j}(\varepsilon)$ are the same;

(iii) $\forall j \in \{1, \dots, m\} : f_j(\varepsilon) \ker f_j^{r_j}(\varepsilon) = \ker f_j^{r_j - 1}(\varepsilon)$.

B. Let again M be a vector space over a given field F. Let us assume that M has a countable F-basis

$$(u_1, u_2, \dots) .$$

Then there exists a unique endomorphism ϑ on M for which $\vartheta u_1 = 0$ and for any natural $n : \vartheta u_{2n} = u_{2n+1}$, $\vartheta u_{2n+1} = 0$. Let A be the linear algebra generated over F by ϑ. Then A is the localning with the maximal ideal $A\vartheta$. By Theorem 2, M is not a free A-module, as $\ker \vartheta = \operatorname{im} \vartheta + Au_1$.

Nonetheless, if we replace the operator ϑ by another one with the properties $\vartheta u_{2n-1} = u_{2n}$, $\vartheta u_{2n} = 0$ for any natural n, then $\ker \vartheta = \operatorname{im} \vartheta$ and M is a free A-module.

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