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GRAPHS OF SEMIGROUPS

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Analogously to graphs of groups (see e.g. [1]) we shall introduce graphs of semigroups.

Let S be a semigroup, let A be its subset. The graph $G(S, A)$ is a directed graph whose vertices are elements of S and in which there is a directed edge from a vertex u into a vertex v if and only if $v = ua$, where $a \in A$.

Here we shall characterize finite graphs $G(S, A)$, where A is a one-element set. Thus we shall have $A = \{a\}$ and instead of $G(S, \{a\})$ we shall write simply $G(S, a)$. We shall admit loops and consider them as cycles of the length 1.

Every graph $G(S, a)$ has the property that the outdegree of each of its vertices is 1. The structure of such graphs is well-known. If such a graph is finite, then each of its connected components contains exactly one cycle (by a cycle we mean a directed circuit).

After deleting all edges of this cycle a forest is obtained. Each tree of this forest has the property that for each of its vertices there is a directed path going from this vertex to a vertex of the cycle (Fig. 1). If C is a connected component of such a graph, then by $\varkappa(C)$ we denote the length of the cycle contained in C (it may be 1, if this

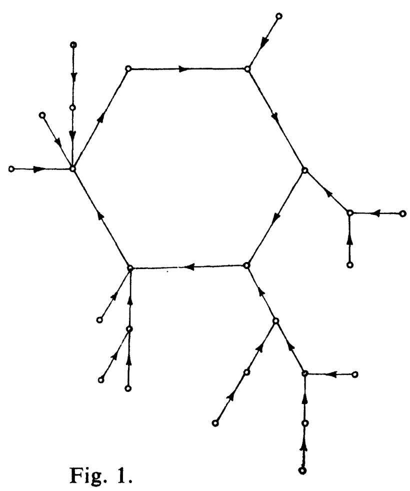

Fig. 1.

cycle is a loop) and by $\lambda(C)$ we denote the maximal length of a directed path in this component which contains no edge of the cycle (it may be 0, if C consists only of a cycle).

Now we shall prove a theorem which gives a characterization of the finite graphs $G(S, a)$.

Theorem. *Let G be a finite directed graph in which each vertex has the outdegree 1. The graph G is isomorphic to the graph $G(S, a)$ for a semigroup S and its element a if and only if it contains a connected component C with the property that for each connected component D of G the number $\varkappa(D)$ divides $\varkappa(C)$ and $\lambda(D) \leqq \lambda(C) + 1$.*

Proof. Suppose that G is isomorphic to $G(S, a)$ for some S and a. Let a have a period h and a pre-period k; this means that the elements $a, a^2, \ldots, a^{h+k-1}$ are pairwise distinct and $a^{h+k} = a^k$. Hence in G there exists a cycle of the length h and a directed path of the length k whose terminal vertex belongs to this cycle; the initial vertex of this path corresponds to the element a. Now let x be an arbitrary vertex of G (i.e. an element of S); let D be the connected component of G containing x. Let p be the length of the directed path outgoing from x, incoming into a vertex of a cycle and containing no edge of this cycle; evidently $p \leqq \lambda(D)$. Let $q = \varkappa(D)$. Then the elements $x, xa, xa^2, \ldots, xa^{p+q-1}$ are pairwise distinct and $xa^{p+q} = xa^p$. If q does not divide h, then k and $h + k$ are not congruent modulo q and thus $xa^k \neq xa^{h+k}$, which is a contradiction with the assumption $a^{h+k} = a^k$. Hence q must divide h and h is $\varkappa(C)$, where C is the connected component of G containing a. Now suppose $p \geqq k + 2$. Then xa^{k+1} is distinct from xa^l for each $l \neq k + 1$. But, as $a^{h+k} = a^k$, we must have $xa^{k+1} = xa^{h+k+1}$, which is a contradiction. Hence $p \leqq k + 1 \leqq \lambda(C) + 1$. Thus the necessity of the condition is proved.

Now suppose that the condition is fulfilled. In C take a directed path containing no edge of a cycle and having the length $\lambda(C)$; its initial vertex will be a. Take all sources of G and if G contains connected components distinct from C which are cycles, choose one vertex in each of them. The set thus obtained will be denoted by B. The vertex a and the vertices of B will be considered elements of a semigroup S. Each remaining vertex will be denoted as a power of a or a product of an element of B with a power of a in the way corresponding to the definition of $G(S, a)$. Further, we introduce the equality $xb = b$ for each $x \in S$ and each $b \in B$. Thus we have defined a semigroup S such that G is isomorphic to $G(S, a)$.

Reference

[1] *Teh, H. H. - Shee, S. C.:* Algebraic Theory of Graphs. Singapore 1976.

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