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5 TYPES OF CONFIGURATIONS OF 9 FLEXES AND 27 SEXTACTIC POINTS OF A CUBIC

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The points of a non-singular plane cubic curve can always be represented by Weierstrass' elliptic (doubly periodic) $p(u)$-function with periods $2w$, $2w'$ dependent on the elliptic integral

$$u = \int_p^\infty \frac{\mathrm{d}x}{(4x^3 - g_2x - g_3)^{1/2}} \quad \text{such that} \quad \frac{\mathrm{d}u}{\mathrm{d}p} = -1(4p^3 - g_2p - g_3)^{1/2} \quad \text{or}$$

$$p'^2 = (\mathrm{d}p/\mathrm{d}u)^2 = 4p^3 - g_2p - g_3\,,$$

the well known differential equation. We shall work in the familiar OCS (Orthogonal Cartesian System) of coordinates to arrive at the following 7 interesting results.

(i) Zwikker (pp. 82–92) puts $z = p + \mathrm{i}p'$ in the Gauss Plane to represent a cubic U in the Weierstrass' canonical form: $y^2 = 4x^3 - g_2x - g_3$ in OCS ($x = p$, $y = p'$), to deduce a good many properties of a cubic as in (ii)–(vii) below.

See also Macrobert, pp. 194–198.

(ii) Newton's Theorem states that any cubic U can always be reduced to the form

$$y^2 = ax^2 + 3bx^2 + 3cx + \mathrm{d}\ (x = x_0,\ y = x_1,\ x_2 = 1)$$

in OCS and further to that in (i) by taking new coordinates $x' = x + b/a$, $y' = y$, which is equivalent to a *translation*.

(iii) The most important *projective properties* of cubics are consequences of the so-called *Addition Theorem* of p-function which says that (see Copson, p. 373)

$$\begin{vmatrix} 1 & 1 & 1 \\ p(u_1) & p(u_2) & p(u_3) \\ p'(u_1) & p'(u_2) & p'(u_3) \end{vmatrix} = 0 \quad \text{if} \quad u_1 + u_2 + u_3 = 0 \ (\text{mod}\ 2w, 2w')$$

or

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix} = 0 \quad (z_j = p(u_j) + \mathrm{i}p'(u_j),\ \ j = 1, 2, 3)$$

showing that the 3 points z_j of the cubic U with such 3 values of u_j are collinear.

This property proves very simply many interesting properties of any cubic, the 9 *flexes* having the values of u equal to 0, $\pm 2w/3$, $\pm 2w'/3$, $\pm 2w/3 \pm 2w'/3$ denoted as $(m, n') = (mw + m'w')/3$ $(m, m' = 0, 2, 4)$ by Feld (1936).

(iv) *If a line a passes through a point L on a cubic U to meet it again in a pair of points A', A" and their joins to another point N on U meet it again in B', B" respectively, then the line b = B'B" passes through a fixed point M on U such that the pencil of lines* (a) *is projectively related to that of lines* (b) *that provides Salmon's invariant* $s = (p_1 - p_2)/(p_1 - p_3)$ as the *biratio of the* 4 *tangents* to U from the *ideal flex B* on the y-axis $x = 0$, the ideal line being one of them as the stationary tangent there, where p_i $(i = 1, 2, 3)$ are the roots of the cubic $4x^3 - g_2x = = g_3$ as the abscissae of the meets of the 3 parallel tangents to U through B in (i).

(v) The 4 kissing points of the 4 tangents from any flex (m, m') of the cubic U in (iii) are easily seen to be $(n, n') = (nw + n'w')/3(n, n' = 0, 1, \ldots, 5)$ with $m + 2n$, $m' + 2n' = 0$ or 6 or 12 so that one of them is (m, m') itself and the other 3 are its *sextactic* points of U, as the kissing points of 3 of the 27 conics, which lie on the *harmonic polar* (h.p.) of (m, m') for U.

(vi) It is interesting to observe that the 9 sextactic points of any one of the 12 collinear triads of flexes form a P.C., and H.C. with this triad minus their 3 h.p.'s., as may be noticed by writing them down for the 3 triads:

t: (4, 0), (0, 0), (2, 0); t': (4, 2), (0, 2), (2, 2); t": (4, 4), (0, 4), (2, 4) in the *matrix* form as the 3 P.C.'s:

$$M\colon \begin{bmatrix} (1,0) & (1,3) & (4,3) \\ (3,0) & (3,3) & (0,3) \\ (5,0) & (5,3) & (2,3) \end{bmatrix}, \quad M'\colon \begin{bmatrix} (1,2) & (1,5) & (4,5) \\ (3,2) & (3,5) & (0,5) \\ (5,2) & (5,5) & (2,5) \end{bmatrix}, \quad M''\colon \begin{bmatrix} (1,4) & (1,1) & (4,1) \\ (3,4) & (3,1) & (0,1) \\ (5,4) & (5,1) & (2,1) \end{bmatrix},$$

respectively, as may be easily verified, so that (t, M), (t', M'), (t'', M'') form 3 H.C.'s: *H, H', H" ignoring the* 9 *h.p.'s* of the 9 flexes while t, t', t'' form an M.C. that provides 3 more sets of 3 triads like (t, t', t'') leading to 3 more triads of P.C.'s and the corresponding H.C.'s. Thus *the* 9 *flexes and the* 27 *sextactic points of any cubic form* 12 *P.C.'s and* 12 *H.C.'s inscribed in it.*

Feld further observes that if the elements of the matrices M, M', M'' be denoted as $m_{ij}, m'_{ij}, m''_{ij}$ $(i, j = 1, 2, 3)$, respectively, the 18 triads of points $m_{ij}, m'_{ik}, m''_{ih}$ $(h, k = 1, 2, 3; j \neq h \neq k \neq j)$ are collinear so that *the* 9 *flexes and the* 27 *sextactic points of any cubic lie by* 3's *on* 84 *lines to form a configuration* $(36_7, 84_3)$, including the 9 lines of triads of flexes (other than those of t, t', t'') and their 9 h.p.'s, besides the 48 lines of the 3 H.C.'s: H, H', H''.

It is simply surprising that Feld just missed to observe the most interesting result that *H, H', H" are mutually* 12*-fold perspective so that the join of any point of one to any point of an other passes through a point of the third giving rise to a new configuration* $(36_{16}, 192_3)$, containing the 18 lines of the F.C. (Feld configuration) but not the 9 h.p.'s, *formed of the* 9 *flexes and* 27 *sextactic points of any cubic.*

Or, the 12 *points of any one of the* 3 *H.C.'s are c.p. of the other two* as may be easily verified (c.p. = centres of perspectivity).

Our configuration, in fact, consists of 4 triads of H.C.'s like H, H', H''.

(vii) Taking the points A_i, B_j, C_k $(i, j, k = 1, 2, \ldots, n)$ in *Berman* (1951) *configuration* (B.C.) K_n as points on the cubic U with parameters u_i, v_j, w_k so that such triads are collinear iff $i + j + k = 0 \pmod n$ and $u_i + v_j = w_k = 0$ (mod $2w$, $2w'$), we arrive at the B.C.: $(3n_n, n_3^2)$ inscribed in U as a generalisation of P.C. and H.C.

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*) Referee's remark: The following papers of Czech authors also concern the problems studied in the present paper:

(a) *B. Bydžovský:* Über eine ebene Konfiguration $(12_4, 16_3)$. Věstník Královské české společnosti nauk. 1939, II.

(b) *J. Metelka:* On certain configurations (12, 16) in a plane. (Czech). Ibid, XXI, pp. 1—8.

(c) *J. Metelka:* On planar configurations $(12_4, 16_3)$ (Czech). Čas. pěst. mat. *80* (1955), p. 133.

(d) *V. Metelka:* Über ebene Konfigurationen $(12_4, 16_3)$. die mit einer irreduziblen Kurve dritter Ordnung inzidieren. Ibid *91* (1966), p. 261.

(e) *V. Metelka:* Über gewisse ebene Konfigurationen $(12_4, 16_3)$ die auf der irreduziblen Kurven dritter Ordnung endliche Gruppoide bilden. Ibid *95* (1970), p. 23.

(f) *J. Krys:* Configuration of points of a planar cubic I—III (Czech). Ibid *102* (1977), p. 186.