Časopis pro pěstování matematiky

Jan Havrda

Gram-Schmidt's orthogonalization based on the concept of generalized orthogonality

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 4, 335--346

Persistent URL: `http://dml.cz/dmlcz/108489`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 106 * PRAHA 23.11.1981 * ČÍSLO 4

GRAM-SCHMIDT'S ORTHOGONALIZATION BASED ON THE CONCEPT OF GENERALIZED ORTHOGONALITY

JAN HAVRDA, Praha

(Received Juny 30, 1977)

1. Introduction. The lattice of all closed subspace of a (complex) Hilbert space H is introduced via the orthogonality relation $\perp$ in H. The relation is given by the scalar product – we have $x \perp y$ iff $\langle x, y\rangle = 0$. If we put $x \perp y$ iff $\operatorname{Re}\langle x, y\rangle \leqq 0$ and follow the very procedure of the example we began with, we obtain the lattice of all closed convex cones with the vertices at the origin (see [4]). Of course, we can analogously produce many other examples in accordance with the choice of the definition of the orthogonality relation in H. It seems therefore useful to adopt the axiomatic approach, that is, to start with an orthogonality relation on a set, introduce the lattice "of all closed subspaces" and investigate the question of how many properties of "concrete" examples are still preserved in general. We have started the effort along this line in the paper [4]. The present paper carries on the investigations of [4] by introducing and discussing some new phenomenae. It should be mentioned that there has been a few papers published with the similar intention to generalize the notion of orthogonality – the motivations coming mostly from quantum physics. We may refer the reader e.g. to [1]–[5].

Let us recall that we call a relation $\perp \subset \Omega \times \Omega$ an orthogonality relation if 1. $\perp$ is symetric, 2. there is a distinguished element o such that $\{o\} \times \Omega \subset \perp$ and the intersection of $\perp$ with the diagonal is exactly (o, o). The presence of an orthogonality relation on the set Ω gives rise to a complete lattice $\mathscr{S}$ of all subsets A of Ω satisfying $A = (A^{\perp})^{\perp}$ (see [4]). The present note brings the analog of the standard Gram-Schmidt orthogonalization in the lattice $\mathscr{S}$. Obviously, the technique used in the procedure is entirely lattice theoretic. We obtain a result on the orthogonalization of arbitrary many independent elements, generalizing thus the result of the paper [5].

2. Throughout the whole paper, Ω denotes a given set endowed with an orthogonality relation $\perp$. The induced complete lattice $\mathscr{S} = (S, \subset, \Omega, \perp)$ with the corresponding orthogonality is an orthomodular lattice satisfying axioms A and V (see [4] for the definitions).

2.1. Lemma. *If $A \in S$, $y \in \Omega$, $y \notin A$, then there exists an element $x \in \Omega$, $x \perp A$, such that $A \vee \{y\}^{\perp\perp} = A \vee \{x\}^{\perp\perp}$.*

Proof. Clearly $A \neq \Omega$ follows from $y \notin A$; hence $A^\perp \neq \{o\}$. If $y \in A^\perp$ we put $x = y$, and the lemma is evidently valid. Suppose now that $y \notin A^\perp$. We have $\{y\}^{\perp\perp} \not\subset A$, $\{y\}^{\perp\perp} \not\subset A^\perp$, and according to Corollary 2.9 in [4], the elements $(\{y\}^{\perp\perp} \vee A^\perp) \cap A$ and $(\{y\}^{\perp\perp} \vee A) \cap A^\perp$ are atoms in $\mathscr{S}$. There exists an element $x \in (\{y\}^{\perp\perp} \vee A) \cap A^\perp, x \neq o$ such that $x \perp A$. We now have $\{x\}^{\perp\perp} = (\{y\}^{\perp\perp} \vee A) \cap A^\perp$ as a consequence of Theorem 3.8 in [4]. Since $A \subset A \vee \{y\}^{\perp\perp}$ and the lattice $\mathscr{S}$ is orthomodular, we get $A \vee \{y\}^{\perp\perp} = A \vee [A^\perp \cap (A \vee \{y\}^{\perp\perp})] = A \vee \{x\}^{\perp\perp}$, which completes the proof.

2.2. Remark. The atom $\{x\}^{\perp\perp}$ from Lemma 2.1 is uniquely determined. Indeed, suppose $A \vee \{y\}^{\perp\perp} = A \vee \{x\}^{\perp\perp} = A \vee A_1$, where $A_1 \subset A^\perp$ is an atom in $\mathscr{S}$. According to 4) of Theorem 2.1 in [4], we have $\{x\}^{\perp\perp} = (A \vee \{x\}^{\perp\perp}) \cap A^\perp = (A \vee A_1) \cap A^\perp = A_1$. Thus $\{x\}^{\perp\perp} = A_1$.

2.3. Lemma. *Suppose the lattice $\mathscr{S} = (S, \subset, \Omega, \perp)$ satisfies axiom A. If the statement of Lemma 2.1 is true then $\mathscr{S}$ is an orthomodular lattice satisfying axiom V.*

Proof. We shall first prove that the lattice $\mathscr{S}$ is orthomodular. To do this, let us consider two sets, $A, B \in S$ such that $A \subset B$, $A^\perp \cap B = \{o\}$. We claim that $A = B$, thus $\mathscr{S}$ is an orthomodular lattice in accordance with the statement 3) of Theorem 2.1 in [4]. Suppose $A \neq B$. Then there exists an element $y \in B$, $y \notin A$. By the assumption, there is $x \in \Omega$, $x \in A^\perp$ such that $A \vee \{y\}^{\perp\perp} = A \vee \{x\}^{\perp\perp}$. Since $A \vee \{y\}^{\perp\perp} \subset B$, we obtain $A \vee \{x\}^{\perp\perp} \subset B$. Then $x \in B$ and, remembering that $x \in A^\perp$, $x \in A^\perp \cap B = \{o\}$, we have $A \vee \{y\}^{\perp\perp} = A$. Hence $y \in A$ – a contradiction.

We shall now prove that the lattice $\mathscr{S}$ satisfies the axiom V. Let us assume that $A \in S$, $\{o\} \neq A \neq \Omega$, $y \in \Omega$, $y \notin A$, $y \notin A^\perp$. By the assumption, there exist elements $a, b \in \Omega$, $a \perp A$, $b \perp A^\perp$ such that $A \vee \{y\}^{\perp\perp} = A \vee \{a\}^{\perp\perp}$, $A^\perp \vee \{y\}^{\perp\perp} = A^\perp \vee \{b\}^{\perp\perp}$. Since $a \in A^\perp$ and $b \in A$, we have $\{a\}^{\perp\perp} \subset A^\perp$, $\{b\}^{\perp\perp} \subset A$. Thus, we obtain

$$(A \vee \{y\}^{\perp\perp}) \cap A^\perp = (A \vee \{a\}^{\perp\perp}) \cap A^\perp = \{a\}^{\perp\perp},$$
$$(A^\perp \vee \{y\}^{\perp\perp}) \cap A = (A^\perp \vee \{b\}^{\perp\perp}) \cap A = \{b\}^{\perp\perp}$$

as a consequence of the statement 4) of Theorem 2.1 in [4]. By the assumption, the elements $\{a\}^{\perp\perp}$ and $\{b\}^{\perp\perp}$ are atoms in $\mathscr{S}$, therefore $\mathscr{S}$ is a V-lattice in accordance with Corollary 2.9 in [4]. Hence the lattice $\mathscr{S}$ satisfies axiom V. The lemma is proved.

2.4. Lemma. *Let $A_i \in S$, $i \in I$, be atoms such that $A_i \perp A_j$ for all $i, j \in I$, $i \neq j$. Let $B \in S$ be an atom such that*

$$B \subset \bigvee_{i \in I} A_i \quad \textit{and, moreover,} \quad B \not\subset \bigvee_{\substack{i \in I \\ i \neq j}} A_i \quad \textit{holds for some} \quad j \in I\,.$$

Then

$$B \vee \bigvee_{\substack{i\in I\\ i\neq j}} A_i = \bigvee_{i\in I} A_i .$$

Proof. It follows from the assumption that

$$B \vee \bigvee_{\substack{i\in I\\ i\neq j}} A_i \subset \bigvee_{i\in I} A_i .$$

Furthermore, we have

$$(\bigvee_{i\in I} A_i) \cap (B \vee \bigvee_{\substack{i\in I\\ i\neq j}} A_i)^{\perp} = (\bigvee_{i\in I} A_i) \cap (\bigvee_{\substack{i\in I\\ i\neq j}} A_i)^{\perp} \cap B^{\perp} .$$

The right hand side of this identity, however, is equal to $A_j \cap B^{\perp}$ according to the statement 4) of Theorem 2.1 in [4]. We shall now show that $A_j \cap B^{\perp} = \{o\}$. Suppose, on the contrary, that $A_j \subset B^{\perp}$, that is, $B \subset A_j^{\perp}$. Then we have by the assumptions of the lemma and the statement 4) of Theorem 2.1 in [4] that

$$B = B \cap A_j^{\perp} \subset (\bigvee_{i\in I} A_i) \cap A_j^{\perp} = \bigvee_{\substack{i\in I\\ i\neq j}} A_i ,$$

contrary to the hypothesis. Summarizing, we have

$$(\bigvee_{i\in I} A_i) \cap (B \vee \bigvee_{\substack{i\in I\\ i\neq j}} A_i)^{\perp} = \{o\} .$$

Finally, it follows from the statement 3) of Theorem 2.1 in [4] that

$$B \vee \bigvee_{\substack{i\in I\\ i\neq j}} A_i = \bigvee_{i\in I} A_i ,$$

which completes the proof.

2.5. Definition. Let A be a nonempty subset of Ω such that $A \neq \{o\}$. We call the set A *independent* if and only if the relation

$$x \notin \bigvee_{y\in A-\{x\}} \{y\}^{\perp\perp}$$

is satisfied for all $x \in A$.

2.6. Remark. Since $(\bigcup_{i\in I} A_i)^{\perp} = \bigcap_{i\in I} A_i^{\perp}$ holds for all $\emptyset \neq A_i \subset \Omega$, $i \in I$, it is also true that $(\bigcup_{i\in I} A_i)^{\perp\perp} = \bigvee_{i\in I} A_i^{\perp\perp}$. Hence the identity

$$\bigvee_{y\in A-\{x\}} \{y\}^{\perp\perp} = (\bigcup_{y\in A-\{x\}} \{y\})^{\perp\perp} = (A - \{x\})^{\perp\perp}$$

is valid.

2.7. Definition. Let A be a nonvoid subset of Ω such that $A \neq \{o\}$. We say that the set A is *l-independent* if and only if $B^{\perp\perp} \neq A^{\perp\perp}$ for every subset $B \subset A$, $\emptyset \neq B \neq A$.

2.8. Lemma. *Let a set A satisfy the conditions of the preceding definition. Then the set A is independent if and only if it is an l-independent set.*

Proof. Let A be an independent set. Assume that A is not an l-independent set. Then there is a nonempty proper subset B of A such that $B^{\perp\perp} = A^{\perp\perp}$. Since there exists $x \in A$, $x \notin B$, we have $B \subset A - \{x\} \subset A$, hence $B^{\perp\perp} \subset (A - \{x\})^{\perp\perp} \subset A^{\perp\perp}$. Relations $B^{\perp\perp} = A^{\perp\perp}$ and $A \subset A^{\perp\perp}$ imply $x \in A \subset A^{\perp\perp} = (A - \{x\})^{\perp\perp}$. Thus the set A is not independent, contrary to the hypothesis.

Conversely, let A be an l-independent set. Choose $B = A - \{x\}$, where $x \in A$. Suppose B is a nonempty set. Then $(A - \{x\})^{\perp\perp} \subsetneq A^{\perp\perp}$ and $\{o\} \neq (A - \{x\})^{\perp} \cap \cap A^{\perp\perp} = (A - \{x\})^{\perp} \cap [(A - \{x\}) \cup \{x\}]^{\perp\perp} = (A - \{x\})^{\perp} \cap [(A - \{x\})^{\perp\perp} \vee \{x\}^{\perp\perp}]$ by 3) of Theorem 2.1 in [4] and Remark 2.6. Suppose A is not an independent set. There exists an element $x \in A$ such that $x \in (A - \{x\})^{\perp\perp}$, hence $\{x\}^{\perp\perp} \subset (A - \{x\})^{\perp\perp}$. It follows that $\{o\} \neq (A - \{x\})^{\perp} \cap [(A - \{x\})^{\perp\perp} \vee \{x\}^{\perp\perp}] = (A - \{x\})^{\perp} \cap \cap (A - \{x\})^{\perp\perp} = \{o\}$, which is an evident contradiction. This completes the proof of our assertion.

2.9. Lemma. *If a set A is independent, then $o \notin A$. Furthermore, if $\emptyset \neq B \subset A$ then B is also an independent set.*

Proof is obvious.

2.10. Theorem. *Let $A \subset \Omega$ be an independent set and let B a nonempty subset of A. Suppose there is an injective mapping $f_B : B \to \Omega - \{o\}$ such that the following two conditions are satisfied:*

a$_1$) *if $x, y \in B$, $x \neq y$, then $f_B(x) \perp f_B(y)$;*
b$_1$) $\bigvee_{x \in B} \{x\}^{\perp\perp} = \bigvee_{x \in B} \{f_B(x)\}^{\perp\perp}$.

Then the mapping f_B can be extended over the entire set A in such a way that the extended mapping f_A ($f_A : A \to \Omega - \{o\}$, $f_B(x) = f_A(x)$ for all $x \in B$) satisfies the conditions:

a$_2$) *if $x, y \in A$, $x \neq y$ then $f_A(x) \perp f_A(y)$;*
b$_2$) $\bigvee_{x \in A} \{x\}^{\perp\perp} = \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}$.

Proof. Denote by F the family of all extensions f_C of the mapping f_B ($B \subset C \subset A$) having the following properties:

a$_3$) if $x, y \in C$, $x \neq y$ then $f_C(x) \perp f_C(y)$;
b$_3$) $\bigvee_{x \in C} \{x\}^{\perp\perp} = \bigvee_{x \in C} \{f_C(x)\}^{\perp\perp}$.

Since $f_B \in F$, the family F is nonempty. Now we define a relation "$\leqq$" on $F \times F$ as follows: $f_D \leqq f_E$ if $D \subset E$ and $f_D(x) = f_E(x)$ for all $x \in D$. It can be easily verified that the relation $\leqq$ is an ordering on F. For $\mathscr{D} \neq \emptyset$, let us consider a chain $\{f_D\}_{D\in\mathscr{D}}$ in the ordered set $(F; \leqq)$. Putting $E = \bigcup_{D\in\mathscr{D}} D$, we have clearly $B \subset E \subset A$. According to Lemma 2.9, the set E is independent. If $x \in E$ then there exists $D_1 \in \mathscr{D}$ such that $x \in D_1$. Define $f_E(x) = f_{D_1}(x)$ for such x's. It is to be noticed that the preceding definition is correct. For, if $x_2 \in D_2 \in \mathscr{D}$, then either $D_1 \subset D_2$ or $D_2 \subset D_1$, hence $f_{D_1}(x) = f_{D_2}(x)$. If $x, y \in E$, $x \neq y$, then there exist $D', D'' \in \mathscr{D}$ such that $x \in D'$, $y \in D''$. We may assume that $D' \subset D''$, therefore $x \in D''$. We obtain $f_E(x) = f_{D''}(x) \perp \perp f_{D''}(y) = f_E(y)$. The mapping $f_E : E \to \Omega - \{o\}$ is injective and it satisfies a_3), where $C = E$. Moreover, we shall prove that the mapping f_E satisfies b_3), where, again, $C = E$. We have

$$\bigvee_{x\in D} \{x\}^{\perp\perp} = \bigvee_{D\in D} \{f_D(x)\}^{\perp\perp} = \bigvee_{x\in D} \{f_E(x)\}^{\perp\perp}$$

for all $D \in \mathscr{D}$. Hence we obtain the equality

$$(*) \qquad \bigvee_{D\in\mathscr{D}} \bigvee_{x\in D} \{x\}^{\perp\perp} = \bigvee_{D\in\mathscr{D}} \bigvee_{x\in D} \{f_E(x)\}^{\perp\perp} .$$

According to the definition of the supremum, we have $\bigvee_{D\in\mathscr{D}} \bigvee_{x\in D} \{x\}^{\perp\perp} \subset \bigvee_{x\in E} \{x\}^{\perp\perp}$. For every $x \in E$ there is a $D_x \in \mathscr{D}$ such that $x \in D_x$ and hence $\bigvee_{x\in E} \{x\}^{\perp\perp} \subset \bigvee_{D\in\mathscr{D}} \bigvee_{x\in D} \{x\}^{\perp\perp}$ Similarly, we can prove that $\bigvee_{x\in E} \{f_E(x)\}^{\perp\perp} = \bigvee_{D\in\mathscr{D}} \bigvee_{x\in D} \{f_E(x)\}^{\perp\perp}$. In comparison with $(*)$, we see that the condition b_3) is satisfied.

The mapping f_E is the extension of the mapping f_B. Hence $f_E \in F$. We have proved that the mapping f_E is the upper bound of the chain $\{f_D\}_{D\in\mathscr{D}}$ in the ordered set $(F; \leqq)$. By Zorn's lemma, the ordered set $(F; \leqq)$ has maximal elements and, for every mapping $f_C \in F$, $f_C \leqq f_{C'}$ where $f_{C'}$ is a maximal element in the set $(F; \leqq)$.

We shall prove that the domain in the definition of a maximal element $f_{C'}$ is the entire set A. Assume that $y \in A$ and $f_{C'}(y)$ is not defined. Hence $y \notin C'$. Since $C' \subset A$ then C' is an independent set by Lemma 2.9. Hence we have

$$y \notin \bigvee_{x\in C'} \{x\}^{\perp\perp} = \bigvee_{x\in C'} \{f_{C'}(x)\}^{\perp\perp} .$$

According to Lemma 2.1, there is an element $y' \in \Omega - \{o\}$ such that $y' \perp \bigvee_{x\in C'} \{x\}^{\perp\perp} = \bigvee_{x\in C'} \{f_C(x)\}^{\perp\perp}$ so that

$$\{y\}^{\perp\perp} \vee \bigvee_{x\in C'} \{x\}^{\perp\perp} = \{y'\}^{\perp\perp} \vee \bigvee_{x\in C'} \{x\}^{\perp\perp} = \{y'\}^{\perp\perp} \vee \bigvee_{x\in C'} \{f_{C'}(x)\}^{\perp\perp} .$$

Thus, we put $f_{C'\cup\{y\}}(y) = y'$ and $f_{C'\cup\{y\}}(x) = f_{C'}(x)$ for all $x \in C'$. It follows therefore that the mapping $f_{C'}$ is not a maximal element in the ordered set $(F; \leqq)$. This completes the proof.

2.11. Theorem. *Let $\emptyset \neq A \subset \Omega$ and let A be an independent set. Let $a \in \Omega$, $a \neq o$, $a \perp A$. Then $A \cup \{a\}$ is also an independent set.*

Proof. Since $a \perp A$, it follows that $a \perp x$ for all $x \in A$ which means $a \in \{x\}^{\perp}$ for all $x \in A$. Hence $a \in \bigcap_{x \in A} \{x\}^{\perp}$. Using first the statement 2) of Theorem 3.3 in [4], we have $\bigcap_{x \in A} \{x\}^{\perp} = (\bigvee_{x \in A} \{x\}^{\perp\perp})^{\perp}$ and, in view of $a \neq o$, we have $a \notin \bigvee_{x \in A} \{x\}^{\perp\perp}$.

If the set A is a singleton, the assertion of Theorem 2.11 is clear. Thus, we shall assume that the set A contains at least two points. Let $z \in A$. According to Lemma 2.9, the set $A - \{z\}$ is also independent. According to Theorem 2.10, there is a mapping $f_{A-\{z\}} : A - \{z\} \to \Omega - \{o\}$ such that $f_{A-\{z\}}(x) \perp f_{A-\{z\}}(y)$ for all $x, y \in A - \{z\}$, $x \neq y$ and therefore $\bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp} = \bigvee_{x \in A - \{z\}} \{f_{A-\{z\}}(x)\}^{\perp\perp}$. (We may choose an arbitrary one-point subset of the set $A - \{z\}$ as the set B of Theorem 2.10 and the set $A - \{z\}$ as the set A of Theorem 2.10.) Again, according to Theorem 2.10, the mapping $f_{A-\{z\}}$ can be extended over the entire set A and we may denote the extension by f_A. It is true that $f_A(x) \perp f_A(y)$ for all $x, y \in A$, $x \neq y$ and therefore $\bigvee_{x \in A} \{x\}^{\perp\perp} = \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}$.

It follows that

$$\bigvee_{x \in A - \{z\}} \{f_A(x)\}^{\perp\perp} = (\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}) \cap \{f_A(z)\}^{\perp}$$

in accordance with the statement 4) of Theorem 2.1 in [4]. Hence we have

$$\{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp} = \{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{f_A(x)\}^{\perp\perp} = \{a\}^{\perp\perp} \vee$$

$$\vee [(\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}) \cap \{f_A(z)\}^{\perp}] \subset [\{a\}^{\perp\perp} \vee \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}] \cap [\{a\}^{\perp\perp} \perp \{f_A(z)\}^{\perp}] .$$

Since $a \perp A$, it is true that $a \perp \bigvee_{x \in A} \{x\}^{\perp\perp} = \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}$ as well as $\{a\}^{\perp\perp} \subset \{f_A(z)\}^{\perp\perp}$.
Hence we have

$$(**) \qquad \{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp} \subset [\{a\}^{\perp\perp} \vee \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}] \cap \{f_A(z)\}^{\perp} .$$

To complete the proof, we assume that $z \in \{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp}$. In view of (**), it follows that $z \in \{f_A(z)\}^{\perp}$, which is the same as $z \perp f_A(z)$. According to the statement 4) of Theorem 2.1 in [4], we have

$$z \in \{z\}^{\perp\perp} = (\bigvee_{x \in A} \{x\}^{\perp\perp}) \cap \{z\}^{\perp\perp} = (\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}) \cap \{z\}^{\perp\perp} \subset$$

$$\subset (\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}) \cap \{f_A(z)\}^{\perp} = \bigvee_{x \in A - \{z\}} \{f_A(x)\}^{\perp\perp} = \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp}$$

a contradiction. Hence $z \notin \{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp}$ and this completes the proof.

2.12. Theorem. *There are maximal independent sets $A \subset \Omega$ with respect to the set inclusion. It is true for such an A that*

$$\bigvee_{x \in A} \{x\}^{\perp\perp} = \Omega .$$

Proof. Let $\mathscr{A} \neq \emptyset$ and let $\{A_\alpha\}_{\alpha \in \mathscr{A}}$ be a chain (with respect to the set inclusion) of independent sets $A_\alpha \subset \Omega$, $\alpha \in \mathscr{A}$. Let us put $A = \bigcup_{\alpha \in \mathscr{A}} A_\alpha$. We shall show that A is an independent set as well. Suppose $y \in A$. There is $\beta \in \mathscr{A}$ such that $y \in A_\beta$. The set $A_\beta - \{y\}$ is independent by Lemma 2.9. In accordance with Theorem 2.10, there is a mapping $f_{A_\beta - \{y\}} : A_\beta - \{y\} \to \Omega - \{o\}$ such that $f_{A_\beta - \{y\}}(u) \perp f_{A_\beta - \{y\}}(v)$ for all $u, v \in A_\beta - \{y\}$, $u \neq v$, and $\bigvee_{x \in A_\beta - \{y\}} \{x\}^{\perp\perp} = \bigvee_{x \in A_\beta - \{y\}} \{f_{A_\beta - \{y\}}(x)\}^{\perp\perp}$. Let us choose an arbitrary element $s \in A - \{y\} = \bigcup_{\alpha \in \mathscr{A}} A_\alpha - \{y\} = \bigcup_{\alpha \in \mathscr{A}} (A_\alpha - \{y\})$. There is $\gamma \in \mathscr{A}$ such that $s \in A_\gamma - \{y\}$. If $A_\gamma - \{y\} \subset A_\beta - \{y\}$ we put $f_{A - \{y\}}(s) = f_{A_\beta - \{y\}}(s)$. If $A_\beta - \{y\} \subset A_\gamma - \{y\}$ then, in accordance with Theorem 2.10, we extend the mapping $f_{A_\beta - \{y\}}$ to the mapping $f_{A_\gamma - \{y\}}$ and then put $f_{A - \{y\}}(s)\, f_{A\gamma - \{y\}}(s)$. It can be shown in a similar way as in the proof of Theorem 2.10 that

$$\bigvee_{x \in A - \{y\}} \{x\}^{\perp\perp} = \bigvee_{\alpha \in \mathscr{A}} \bigvee_{x \in A_\alpha - \{y\}} \{x\}^{\perp\perp} = \bigvee_{\alpha \in \mathscr{A}} \bigvee_{x \in A_\alpha - \{y\}} \{f_{A - \{y\}}(x)\}^{\perp\perp} = \bigvee_{x \in A - \{y\}} \{f_{A - \{y\}}(x)\}^{\perp\perp} .$$

According to Theorem 2.10, we define further $f_{A - \{y\}}(y)$ and we write f_A for the new extended mapping. By the assertion 4) of Theorem 2.1 in [4], we have

$$\bigvee_{x \in A - \{y\}} \{x\}^{\perp\perp} = \bigvee_{x \in A - \{y\}} \{f_A(x)\}^{\perp\perp} = \left(\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp}\right) \cap \{f_A(y)\}^{\perp} .$$

If $y \in \bigvee_{x \in A - \{y\}} \{x\}^{\perp\perp}$:hen $y \in \{f_A(y)\}^{\perp}$. On the other hand,

$$y \notin \bigvee_{x \in A_\beta - \{y\}} \{x\}^{\perp\perp} = \bigvee_{x \in A_\beta - \{y\}} \{f_A(x)\}^{\perp\perp} = \left(\bigvee_{x \in A_\beta} \{f_A(x)\}^{\perp\perp}\right) \cap \{f_A(y)\}^{\perp} .$$

It follows that $y \notin \bigvee_{x \in A} \{f_A(x)\}^{\perp\perp} = \bigvee_{x \in A_\beta} \{x\}^{\perp\perp}$, contrary to $y \in A_\beta$. Hence $y \notin \bigvee_{x \in A - \{y\}} \{x\}^{\perp\perp}$ We have shown that A is an independent set and it is the upper bound of the chain $\{A_\alpha\}_{\alpha \in \mathscr{A}}$. According to Zorn's Lemma, there are maximal independent sets $B \subset \Omega$ and, for every independent set C, it is true that $C \subset B$, where B is a maximal independent set.

If $B \subset \Omega$, B is maximal independent set, then $\bigvee_{z \in B} \{x\}^{\perp\perp} = \Omega$. Indeed, if this is not the case then, according to the statement 3) of Theorem 2.1 in [4] and the statement 2) of Theorem 3.3 in [4], it follows that $\bigcap_{x \in B} \{x\}^{\perp} \cap \Omega \neq \{o\}$. Then there is $y \in \Omega$, $y \neq o$, such that $y \in \{x\}^{\perp}$, i.e. $y \perp x$ for all $x \in B$. By Theorem 2.11, the set $B \cup \{y\}$ is independent as well and hence the set B is not maximal, contrary to the hypothesis. The theorem is proved.

3. Let card A stand for the cardinality of a set A. Theorem 2.12 suggests the question: Is it true for every two maximal independent sets A, B that card A = card B? We shall deal with the problem in this section.

3.1. Theorem. *Let m be a positive integer, $2 \leqq m \leqq \operatorname{card} I$. Let $A_1, \ldots, A_m$ be pairwise orthogonal atoms, let B_j, $j \in I$, be pairwise orthogonal atoms as well and let $\bigvee_{i=1}^{m} A_i = \bigvee_{j \in I} B_j$. Then $m = \operatorname{card} I$.*

Proof. We shall prove the statement of this lemma by induction. Let us suppose $m = 2$. By the hypothesis, it is true that $A_1 \vee A_2 = \bigvee_{j \in I} B_j$. If, for instance, $B_k = A_1$, $k \in I$, then, according to the statement of Theorem 2.1 in [4], we get

$$A_2 = (A_1 \vee A_2) \cap A_1^{\perp} = (\bigvee_{j \in I} B_j) \cap B_k^{\perp} = \bigvee_{\substack{j \in I \\ j \neq k}} B_j .$$

Since

$$A_2 = \bigvee_{\substack{j \in I \\ j \neq k}} B_j \supset B_j$$

for $j \in I$, $j \neq k$, we have $A_2 = B_j$, $j \in I$, $j \neq k$. Since the atoms B_j, $j \in I$, $j \neq k$, are pairwise orthogonal, it follows that card $I = 2$. Let us now suppose $A_1 \neq B_k \neq A_2$. Since $A_1 \vee A_2 = \bigvee_{j \in I} B_j \supset B_k$, Theorem 2.7 in [4] implies that there is an atom $B \in S$, $B \perp B_k$, such that $A_1 \vee A_2 = B_k \vee B$. It follows that

$$B_k \vee \bigvee_{\substack{j \in I \\ j \neq k}} B_j = B_k \vee B .$$

According to the statement 4) of Theorem 2.1 in [4], we get

$$\bigvee_{\substack{j \in I \\ j \neq k}} B_j = (\bigvee_{j \in I} B_j) \cap B_k^{\perp} = (B_k \vee B) \cap B_k^{\perp} = B .$$

It follows as well that card $I = 2$. Theorem 3.1 is proved for $m = 2$.

Let us suppose that the statement of Theorem 3.1 is true for positive integers $2, 3, \ldots, m - 1$. We shall prove it for the positive integer m. Suppose

$$A_1 \vee \ldots \vee A_m = \bigvee_{j \in I} B_j , \quad m \leqq \operatorname{card} I .$$

Let us first suppose that $B_k \subset A_1 \vee \ldots \vee A_{m-1}$, $k \in I$. We shall prove that there are pairwise orthogonal atoms $B'_2, \ldots, B'_{m-1}$, each of them orthogonal to B_k, such that $B_k \vee B'_2 \vee \ldots \vee B'_{m-1} = A_1 \vee \ldots \vee A_{m-1}$. If $m - 1 = 2$ the statement is true by Theorem 2.7 in [4]. Let the statement be true for the positive integers $2, 3, \ldots$ $\ldots, m - 2$; we shall prove it for $m - 1$. If, for instance, $B_k \subset A_1 \vee \ldots \vee A_{m-2}$, then by the induction hypothesis, there are pairwise orthogonal atoms $B'_2, \ldots, B'_{m-2}$, each of them orthogonal to B_k, such that $B_k \vee B'_2 \vee \ldots \vee B'_{m-2} = A_1 \vee \ldots \vee A_{m-2}$. We put $B'_{m-1} = A_{m-1}$. It is obvious that $B_k \vee B'_2 \vee \ldots \vee B'_{m-1} = A_1 \vee \ldots \vee A_{m-1}$, where the atom B'_{m-1} is orthogonal to the atoms $B_k, B'_2, \ldots, B'_{m-2}$. Let now

$$B_k \not\subset \bigvee_{\substack{i=1 \\ i \neq j}}^{m-1} A_i\,, \quad j = 1.2, \ldots, m-1\,.$$

We evidently have $\{o\} \neq B_k \neq \Omega$. It holds that $A_i \not\subset B_k$ for $i = 1, 2, \ldots, m-1$ and, by the proof of Lemma 2.4, it is true that $A_i \not\subset B_k^\perp$. According to Definition 5 in [4], there is an atom $C_i \subset B_k^\perp$ such that $A_i \subset B_k \vee C_i$. It is evident that $A_i \not\subset C_i$. According to Lemma 2.4, we get

$$A_i \vee B_k = B_k \vee C_i = A_i \vee C_i\,, \quad i = 1, 2, \ldots, m-1\,.$$

Hence

$$B_k = \bigvee_{\substack{i=1 \\ i \neq j}}^{m-1} A_i = B_k \vee \bigvee_{\substack{i=1 \\ i \neq j}}^{m-1} C_i\,, \quad j = 1, 2, \ldots, m-1\,.$$

Again by Lemma 2.4,

$$\bigvee_{i=1}^{m-1} A_i = B_k \vee \bigvee_{\substack{i=1 \\ i \neq j}}^{m-1} A_i = B_k \vee \bigvee_{\substack{i=1 \\ i \neq j}}^{m-1} C_i\,, \quad j = 1, 2, \ldots, m-1\,.$$

If the atoms C_i, $i = 1, \ldots, m-1$, $i \neq j$, are pairwise orthogonal, the statement is evident. If this is not the case we apply the generalized Gram-Schmidt orthogonalization to the atoms C_i, $i = 1, 2, \ldots, m-1$, $i \neq j$. Let us suppose here that, for instance, $j = 1$. We put $B'_2 = C_2$. Let $C_3 \neq C_2$. If $C_3 \subset C_2^\perp$ we put $B'_3 = C_3$. If $C_3 \not\subset C_2^\perp$ then, in accordance with Corollary 2.9 in [4], $B'_3 = (C_2 \vee C_3) \cap C_2^\perp$ is an atom. In accordance with the statement 2) of Theorem 2.1 in [4], $C_2 \subset C_2 \vee C_3$ satisfies $C_2 \vee C_3 = C_2 \vee [C_2^\perp \cap (C_2 \vee C_3] = B'_2 \vee B'_3$. Suppose the atoms $B'_2, B'_3, \ldots, B'_{m-2}$ have already been obtained and let $C_2 \vee \ldots \vee C_{m-2} = B'_2 \vee \ldots \vee B'_{m-2}$. Let $C_{m-1} \not\subset C_2 \vee \ldots \vee C_{m-2}$. If $C_{m-1} \subset (C_2 \vee \ldots \vee C_{m-2})^\perp$ we put $B'_{m-1} = C_{m-1}$. If, moreover, $C_{m-1} \not\subset (C_2 \vee \ldots \vee C_{m-2})^\perp$, then in accordance with Corollary 2.9 in [4], $B'_{m-1} = (C_2 \vee \ldots \vee C_{m-2} \vee C_{m-1}) \cap (C_2 \vee \ldots \vee C_{m-2})^\perp$ is an atom. Since $C_2 \vee \ldots \vee C_{m-2} \subset C_2 \vee \ldots \vee C_{m-2} \vee C_{m-1}$, we have according to the statement 2) of Theorem 2.1 in [4] that $C_2 \vee \ldots \vee C_{m-2} \vee C_{m-1} = (C_2 \vee \ldots \vee C_{m-2}) \vee [(C_2 \vee \ldots \vee C_{m-2})^\perp \cap (C_2 \vee \ldots \vee C_{m-2} \vee C_{m-1})] = B'_2 \vee \ldots \vee B'_{m-2} \vee B'_{m-1}$. If it holds, for instance, that $C_{m-1} \subset C_2 \vee \ldots \vee C_{m-2}$, then $A_1 \vee \ldots \vee A_{m-1} = B_k \vee C_2 \vee \ldots \vee C_{m-2} = B_k \vee B'_2 \vee \ldots \vee B'_{m-2}$. Then the induction hypothesis shows that $m-1 = m-2$ (here the positive integers $m-2$ and $m-1$ play the role of m and card I, respectively) – a contradiction. Thus, we have proved the partial statement on the existence of atoms $B'_2, \ldots, B'_{m-1}$. We have $A_1 \vee \ldots \vee A_{m-1} = B_k \vee B'_2 \vee \ldots \vee B'_{m-1}$, which implies that $A_1 \vee \ldots \vee A_m = B_k \vee B'_2 \vee \ldots \vee B'_{m-1} \vee B'_m$, where $B'_m = A_m$, and this atom is orthogonal to the atoms $B_k, B'_2, \ldots, B'_{m-1}$. Since, however, $A_1 \vee \ldots \vee A_m = \bigvee_{j \in I} B_j$ we have $B_k \vee B'_2 \vee \ldots \vee B'_m = \bigvee_{j \in I} B_j$ as well. By the statement 4) of Theorem 2.1 in [4], the last equality implies

$$B'_2 \vee \ldots \vee B'_m = (B_k \vee B'_2 \vee \ldots \vee B'_m) \cap B_k^{\perp} = (\bigvee_{j \in I} B_j) \cap B_k^{\perp} = \bigvee_{\substack{j \in I \\ j \neq k}} B_j .$$

By the induction assumption, we get $m - 1 = \operatorname{card} I - 1$ and therefore $m = \operatorname{card} I$.

Let now

$$B_k \not\subset \bigvee_{\substack{i=1 \\ i \neq p}}^{m} A_i , \quad p = 1, \ldots, m .$$

Similarly as before, we prove that there are pairwise orthogonal atoms $B'_2, \ldots, B'_m$, which are orthogonal to the atom B_k, such that $B_k \vee B'_2 \vee \ldots B'_m = A_1 \vee \ldots \vee A_m$. Thus, we obtain $B_k \vee B'_2 \vee \ldots \vee B'_m = \bigvee_{j \in I} B_j$ which, as above, implies

$$B'_2 \vee \ldots \vee B'_m = \bigvee_{\substack{j \in I \\ j \neq k}} B_j .$$

Hence, by the induction hypothesis, we have $m = \operatorname{card} I$ and the proof is complete.

3.2. Corollary. *Let $A_i \in S$, $i = 1, 2, \ldots, n$, be pairwise orthogonal atoms. Let $B_1 \in S$ be an atom such that $B_1 \subset \bigvee_{i=1}^{n} A_i$. Then there are pairwise orthogonal atoms $B_2, \ldots, B_n$ such that*

$$\bigvee_{j=1}^{n} B_j = \bigvee_{i=1}^{n} A_i .$$

3.3. Remark. Let A, B be two maximal independent sets, $\operatorname{card} A = m$. In accordance with Theorem 2.12, it holds that

$$\bigvee_{x \in A} \{x\}^{\perp\perp} = \Omega = \bigvee_{y \in B} \{y\}^{\perp\perp} .$$

By Theorem 2.10 there are mappings $f_A : A \to \Omega - \{o\}$, $f_B : B \to \Omega - \{o\}$ such that

$$\bigvee_{x \in A} \{f_A(x)\}^{\perp\perp} = \bigvee_{y \in B} \{f_B(y)\}^{\perp\perp}$$

and for $x_1, x_2 \in A$, $x_1 \neq x_2$, the sets $\{f_A(x_1)\}^{\perp\perp}$, $\{f_A(x_2)\}^{\perp\perp}$ are pairwise orthogonal atoms. A similar assertion holds for the mapping f_B. By Theorem 3.1, it holds that $\operatorname{card} B = m = \operatorname{card} A$. Thus, if one maximal independent set is finite, every maximal independent set is finite and they all have the same cardinality. Hence we obtain that if one maximal independent set is infinite, every maximal independent set is infinite. The problem whether, in this case, they have the same cardinality or not remains open. A partial solution of this problem follows.

3.4. Lemma. *Let $A, B \subset \Omega$ be orthogonal sets (this means that $x \perp y$ for all different $x, y \in A$, similarly for the set B), $o \notin A$, $o \notin B$. Let A be a finite set, for instance $A = \{a_1. a_2, \ldots, a_n\}$. Then* $\operatorname{card}(B \cap A^{\perp\perp}) \leqq \operatorname{card} A = n$.

Proof. If $b_1, b_2, \ldots, b_{n+1} \in B \cap A^{\perp\perp}$, where $b_i \perp b_j$ for $i \neq j$, $i, j = 1, 2, \ldots$ $\ldots, n + 1$, then $b_j \in \{b_j\}^{\perp\perp} \subset A^{\perp\perp} = \{a_1\}^{\perp\perp} \vee \ldots \vee \{a_n\}^{\perp\perp}$, $j = 1, 2, \ldots, n + 1$. Hence

$$\bigvee_{j=1}^{n+1} \{b_j\}^{\perp\perp} \subset \bigvee_{j=1}^{n} \{a_j\}^{\perp\perp} .$$

We shall prove by induction that this inclusion is not valid. For $n = 1$, we have $\{b_1\}^{\perp\perp} \vee \{b_2\}^{\perp\perp} \subset \{a_1\}^{\perp\perp}$ so that $\{b_1\}^{\perp\perp} = \{a_1\}^{\perp\perp} = \{b_2\}^{\perp\perp}$, contrary to $b_1 \perp b_2$. Let us suppose that our assertion is proved for positive integers $1, 2, \ldots, n - 1$; we shall now prove it for the positive integer n. For

$$\{b_1\}^{\perp\perp} \subset \bigvee_{j=1}^{n+1} \{b_j\}^{\perp\perp} \subset \bigvee_{i=1}^{n} \{a_i\}^{\perp\perp} ,$$

in accordance with Corollary 3.2, there are atoms $B_2, \ldots, B_n$ such that $\{b_1\}^{\perp\perp}$, $B_2, \ldots, B_n$ are pairwise orthogonal atoms and

$$\{b_1\}^{\perp\perp} \vee \bigvee_{j=2}^{n} B_j = \bigvee_{i=1}^{n} \{a_i\}^{\perp\perp} \supset \bigvee_{j=1}^{n+1} \{b_j\}^{\perp\perp} .$$

Since the atoms $\{b_j\}^{\perp\perp}$, $j = 1, 2, \ldots, n + 1$, are pairwise orthogonal as well, according to the statement 4) of Theorem 2.1 in [4], we get $B_2 \vee \ldots \vee B_n \supset$ $\supset \{b_2\}^{\perp\perp} \vee \ldots \vee \{b_{n+1}\}^{\perp\perp}$, contrary to the induction hypothesis. This completes the proof.

3.5. Lemma. *If the inequality*

$$\operatorname{card} \mathscr{B} \leqq \sum_{\substack{\emptyset \neq A \subset \mathscr{A} \\ A \text{ finite}}} \operatorname{card} (\mathscr{B} \cap A^{\perp\perp})$$

holds for every two infinite maximal orthogonal sets $\mathscr{A}$, $\mathscr{B}$ then $\operatorname{card} \mathscr{A} = \operatorname{card} \mathscr{B}$.

Proof. We have, by the assumption of Lemma 3.5 and by Lemma 3.4,

$$\operatorname{card} \mathscr{B} \leqq \sum_{\substack{\emptyset \neq A \subset \mathscr{A} \\ A \text{ finite}}} \operatorname{card} (\mathscr{B} \cap A^{\perp\perp}) \leqq \sum_{\substack{\emptyset \neq A \subset \mathscr{A} \\ A \text{ finite}}} \operatorname{card} A = \operatorname{card} \mathscr{A} .$$

Similarly, $\operatorname{card} \mathscr{A} \leqq \operatorname{card} \mathscr{B}$ and Lemma 3.5 is proved.

3.6. Remark. Let A, B be two infinite maximal independent sets. By Theorem 2.10, there are pairwise orthogonal atoms A_x, $x \in A$, and pairwise orthogonal atoms B_y, $y \in B$, such that

$$\bigvee_{x\in A} \{x\}^{\perp\perp} = A^{\perp\perp} = \bigvee_{x\in A} A_x = \Omega = \bigvee_{y\in B} B_y = B^{\perp\perp} = \bigvee_{y\in B} \{y\}^{\perp\perp}.$$

We consider the sets $\mathscr{A}$ and $\mathscr{B}$ which consist of just one element different from o of every set A_x, $x \in A$, and just one element different from o of every set B_y, $y \in B$, respectively (see Lemma 3.5). The sets $\mathscr{A}$, $\mathscr{B}$ are infinite maximal orthogonal sets, hence, according to Lemma 3.5 and under its assumptions, it holds that card $\mathscr{B}$ = = card $\mathscr{A}$. In accordance with Theorem 2.10, we have card $\mathscr{A}$ = card A, card $\mathscr{B}$ = = card B so that card A = card B.

Literature

[1] *Fischer H. R., Rüttimann G. T.:* Limits of manuals and logics, Math. Found. of Quantum Theory, Academic Press 1978.

[2] *Foulis D. J., Randall C. H.:* Operational statistics I, Basic concepts, J. Math. Phys. *13*, 1667—1675 (1972).

[3] *Foulis D. J., Randall C. H.:* Empirical logic and quantum mechanics, Synthese *29*, 81—111 (1974).

[4] *Havrda J.:* Orthogonality on sets (Czech), Čas. pěst. mat. *100*, 339—354 (1975).

[5] *Rozenský Z.:* Ein Beitrag zur Theorie der Orthogonalität auf Mengen, Čas. pěst. mat. *104*, 255—266 (1978).

Author's address: 166 27 Praha 6 - Dejvice, Suchbátarova 2 (katedra matematiky FEL ČVUT).